HW1 – Due Friday, January 20, start of class

No electronic submissions, only hard copy

Be sure to justify all your answers completely.

1. Let $g(x) = 2^{-x-1}$.
   (a) Show that $g([0,1]) \subset [0,1]$.
   (b) Show that g is a contraction on $[0,1]$.
   (c) Compute the smallest possible contraction constant k for g on $[0,1]$ and estimate how many iterates n it will take for $g^n(0)$ to be within 10^{-4} of the fixed point.
2. $f(x) = \log(1-x)$ (Note: log is the natural log).
   (a) Estimate the error in using the second order Taylor polynomial T_2 expanded about zero for f to compute $\log(.9)$.
   (b) Compute $T_2(.1)$
   (c) Compute the actual error in using $T_2(.1)$ to compute $\log(.9)$ (You can assume here that your calculator/computer computes $\log(.9)$ exactly).
   (d) Compare the actual error with your error estimate.
3. Assume that $g \in C[a,b]$ and $g(a) > a$ and $g(b) < b$. Show that g has a fixed point in $[a,b]$.
4. Write a computer program that implements the midpoint method and use it to compute *all* the roots of $f(x) = x^3 - 6.1x^2 + 10.8x - 5.8$ to within an accuracy of 10^{-5}. Turn in both your code and the results of your computation.
5. Assume $g \in C^2[a,b]$ with $g([a,b]) \subset [a,b]$ and fixed point $p \in (a,b)$. Assume that $g'(p) = 0$. Show using the Taylor theorem with remainer expanded about p that for any $x \in [a,b]$ with $x \neq p$
$$\frac{|g(x)-p|}{|x-p|^2} \leq M$$
where $M = \max\{|g''(z)| \colon z \in [a,b]\}/2$.

Note: Just so there is no confusion about indices, the nth Taylor poloynomial of f expanded about a is
$$T_n(x) = f(a) + (x-a)f'(a) + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n.$$