Generalized Rolle's Theorem

Recall Rolle's Theorem: Let f be continuous on $[a,b]$ and differentiable on (a,b). If $f(a)=f(b)$ then there exists $c\in(a,b)$ such that $f'(c)=0$. The Generalized Rolle's Theorem extends this idea to higher order derivatives:

Generalized Rolle's Theorem: Let f be continuous on $[a,b]$ and n times differentiable on (a,b). If f is zero at the $n+1$ distinct points $x_0<x_1<\ldots<x_n$ in $[a,b]$, then there exists a number c in (a,b) such that $f^{(n)}(c)=0$.

Proof: The argument uses mathematical induction. If $n=1$ then we have the original Rolle's Theorem. To see how the induction argument works, consider the next case, $n=2$. Then $f(x)=0$ at three points $x_0<x_1<x_2$. Applying Rolle's Theorem on the interval $[x_0,x_1]$, there exists $c_1\in(x_0,x_1)$ such that $f'(c_1)=0$. Similarly, there exists $c_2\in(x_1,x_2)$ such that $f'(c_2)=0$. Now, using Rolle's Theorem again, $f'(c_1)=f'(c_2)=0$ implies that there exists $c\in(a,b)$ such that $f''(c)=0$. The full mathematical induction argument follows.