Generalized Rolle’s Theorem

Recall Rolle’s Theorem: Let $f$ be continuous on $[a,b]$ and differentiable on $(a,b)$. If $f(a) = f(b)$ then there exists $c \in (a,b)$ such that $f'(c) = 0$. The Generalized Rolle’s Theorem extends this idea to higher order derivatives:

**Generalized Rolle’s Theorem:** Let $f$ be continuous on $[a,b]$ and $n$ times differentiable on $(a,b)$. If $f$ is zero at the $n+1$ distinct points $x_0 < x_1 < \ldots < x_n$ in $[a,b]$, then there exists a number $c$ in $(a,b)$ such that $f^{(n)}(c) = 0$.

**Proof:** The argument uses mathematical induction. If $n = 1$ then we have the original Rolle’s Theorem. To see how the induction argument works, consider the next case, $n = 2$. Then $f(x) = 0$ at three points $x_0 < x_1 < x_2$. Applying Rolle’s Theorem on the interval $[x_0, x_1]$, there exists $c_1 \in (x_0, x_1)$ such that $f'(c_1) = 0$. Similarly, there exists $c_2 \in (x_1, x_2)$ such that $f'(c_2) = 0$. Now, using Rolle’s Theorem again, $f'(c_1) = f'(c_2) = 0$ implies that there exists $c \in (a,b)$ such that $f''(c) = 0$. The full mathematical induction argument follows.