Error Formula for the

Lagrange Interpolating Polynomial

Theorem: Let $f \in C^{n+1}[a,b]$ and $x_0, x_1, \ldots, x_n$ be distinct points in $[a,b]$. If $P(x)$ is the interpolating polynomial, then for each $x \in [a,b]$, there exists a number $c = c(x) \in (a,b)$ such that

$$f(x) = P(x) + \frac{f^{(n+1)}(c)}{(n+1)!} \prod_{i=0}^{n} (x - x_i)$$

Proof: If $x = x_k, k = 0,1,\ldots,n$, the product is zero and the theorem is clearly true. If $x \neq x_k$ then define

$$g(t) = f(t) - P(t) - \left[f(x) - P(x) \right] \prod_{i=0}^{n} \frac{(t - x_i)}{(x - x_i)}$$

Notice that $g \in C^{n+1}[a,b]$. For $t = x_k, k = 0,1,\ldots,n,$

$$g(x_k) = f(x_k) - P(x_k) - \left[f(x) - P(x) \right] \prod_{i=0}^{n} \frac{(x_k - x_i)}{(x - x_i)} = 0$$

Also,

$$g(x) = f(x) - P(x) - \left[f(x) - P(x) \right] \prod_{i=0}^{n} \frac{(x - x_i)}{(x - x_i)} = 0$$

The function g is zero at $n+2$ distinct points. So by the Generalized Rolle's Theorem, there exists $c = c(x) \in (a,b)$ such that $g^{(n+1)}(c) = 0$. We now have

$$(*)\quad 0=g^{(n+1)}(c)=f^{(n+1)}(c)-P^{(n+1)}(c)-\left[f(x)-P(x)\right]\frac{d^{n+1}}{dt^{n+1}}\left[\prod_{i=0}^{n}\frac{(t-x_i)}{(x-x_i)}\right]_{t=c}$$

Since P is a polynomial of degree $\leq n$, $P^{(n+1)}(c)=0$. The product $\prod_{i=0}^{n}\frac{(t-x_i)}{(x-x_i)}$ is a polynomial of degree $n+1$ of the form

$$\prod_{i=0}^{n}\frac{(t-x_i)}{(x-x_i)}=\frac{1}{\prod_{i=0}^{n}(x-x_i)}t^{n+1}+(\text{lower degree terms})$$

Hence,

$$\frac{d^{n+1}}{dt^{n+1}}\left[\prod_{i=0}^{n}\frac{(t-x_i)}{(x-x_i)}\right]_{t=c}=\frac{(n+1)!}{\prod_{i=0}^{n}(x-x_i)}$$

Finally, Equation $(*)$ simplifies to

$$0=f^{(n+1)}(c)-\left[f(x)-P(x)\right]\frac{(n+1)!}{\prod_{i=0}^{n}(x-x_i)}$$

Rearranging, we have arrived at the conclusion,

$$f(x)=P(x)+\frac{f^{(n+1)}(c)}{(n+1)!}\prod_{i=0}^{n}(x-x_i)$$