Order of Convergence for the Secant Method

Assume that r is a root to $f(x)=0$. The sequence $\{x_n\}$ of the Secant Method is given by $x_{n+1}=x_n - f(x_n)\dfrac{x_n - x_{n-1}}{f(x_n)-f(x_{n-1})}$. We want to find the exponent p such that

$$\lim_{n\to\infty}\frac{|x_{n+1}-r|}{|x_n-r|^p}=\lim_{n\to\infty}\frac{|e_{n+1}|}{|e_n|^p}=\lambda$$

where $e_n = x_n - r$. By Taylor's Theorem,

$$f(x_{n-1}) = f(e_{n-1}+r) = f(r)+e_{n-1}f'(r)+\frac{e_{n-1}^2}{2}f''(r)+\mathrm{O}(e_{n-1}^3)$$
$$f(x_n) = f(e_n+r) = f(r)+e_n f'(r)+\frac{e_n^2}{2}f''(r)+\mathrm{O}(e_n^3)$$

Since $f(r)=0$, we have for $e=e_n$ or $e=e_{n-1}$

$$f(e+r)=ef'(r)+\frac{e^2}{2}f''(r)+\mathrm{O}(e^3)$$
$$f(e+r)\approx ef'(r)+\frac{e^2}{2}f''(r)$$

The Secant Method gives

$$x_{n+1}=x_n - f(x_n)\frac{x_n - x_{n-1}}{f(x_n)-f(x_{n-1})}$$
$$\Rightarrow e_{n+1}+r = e_n + r - f(e_n+r)\frac{(e_n+r)-(e_{n-1}+r)}{f(e_n+r)-f(e_{n-1}+r)}$$
$$\Rightarrow e_{n+1}=e_n-\frac{f(e_n+r)(e_n-e_{n-1})}{f(e_n+r)-f(e_{n-1}+r)} \qquad \text{Equation 1}$$

Let $M=\frac{f''(r)}{2f'(r)} \Leftrightarrow \frac{f''(r)}{2}=Mf'(r)$. Ignoring the higher order terms, the numerator of Equation 1 becomes

$$f(e_n+r)(e_n-e_{n-1})=\left(e_n f'(r)+\frac{e_n^2}{2}f''(r)\right)(e_n-e_{n-1})$$
$$=\left(e_n f'(r)+Me_n^2 f'(r)\right)(e_n-e_{n-1})$$
$$=e_n f'(r)(1+Me_n)(e_n-e_{n-1})$$

The denominator of Equation 1 becomes

$$f(e_n+r)-f(e_{n-1}+r)=\left[e_n f'(r)+\frac{1}{2}f''(r)e_n^2\right]-\left[e_{n-1}f'(r)+\frac{1}{2}f''(r)e_{n-1}^2\right]$$
$$=f'(r)(e_n-e_{n-1})+\frac{1}{2}f''(r)\left(e_n^2-e_{n-1}^2\right)$$
$$=f'(r)(e_n-e_{n-1})+Mf'(r)\left(e_n^2-e_{n-1}^2\right)$$
$$=f'(r)(e_n-e_{n-1})\left(1+M(e_n+e_{n-1})\right)$$

Hence, Equation 1 is now

$$e_{n+1}=e_n-\frac{e_n f'(r)(1+Me_n)(e_n-e_{n-1})}{f'(r)(e_n-e_{n-1})\left(1+M(e_n+e_{n-1})\right)}$$
$$=e_n-\frac{e_n(1+Me_n)}{1+M(e_n+e_{n-1})}$$
$$=\frac{e_n+e_nM(e_n+e_{n-1})-e_n(1+Me_n)}{1+M(e_n+e_{n-1})}$$
$$=\frac{Me_ne_{n-1}}{1+M(e_n+e_{n-1})}$$

This implies that

$$e_{n+1}\approx Me_ne_{n-1}\approx\frac{f''(r)}{2f'(r)}e_ne_{n-1}$$

[Note that Newton's method is $e_{n+1} \approx \frac{f''(r)}{2f'(r)} e_n^2$]

We now calculate the exponent p. We have $|e_{n+1}| = \lambda |e_n|^p$ and $|e_{n+1}| \approx |M||e_n||e_{n-1}|$, so

$$\begin{aligned} &\lambda |e_n|^p = |M||e_n||e_{n-1}| \\ &\Rightarrow |e_n|^{p-1} = \left|M/\lambda\right| |e_{n-1}| \\ &\Rightarrow |e_n| = \left|M/\lambda\right|^{1/p-1} |e_{n-1}|^{1/p-1} = \lambda |e_{n-1}|^p \end{aligned}$$

So $p = \frac{1}{p-1} \Rightarrow p^2 - p - 1 = 0 \Rightarrow p = \frac{1+\sqrt{5}}{2}$