Second Mean Value Theorem for Integrals

**Theorem:** Let $f$ be continuous and $g$ integrable on $[a,b]$. If $g(x) \geq 0$ (or $g(x) \leq 0$) on $[a,b]$, then there exists a point $c, a \leq c \leq b$, such that

$$
\int_a^b f(x)g(x)\,dx = f(c)\int_a^b g(x)\,dx
$$

**Proof:** Assume $g(x) \geq 0$. Let $m = \min f(x)$ and $M = \max f(x)$ on $[a,b]$. We have

$$
m \leq f(x) \leq M
$$

$$
mg(x) \leq f(x) g(x) \leq Mg(x)
$$

$$
m\int_a^b g(x)\,dx = \int_a^b mg(x)\,dx \leq \int_a^b f(x) g(x)\,dx \leq \int_a^b Mg(x)\,dx = M\int_a^b g(x)\,dx
$$

If $\int_a^b g(x)\,dx = 0$ we are done, so assume $\int_a^b g(x)\,dx > 0$. Dividing,

$$
m \leq \frac{\int_a^b f(x) g(x)\,dx}{\int_a^b g(x)\,dx} \leq M
$$

By the Intermediate Value Theorem, there exists $c \in [a,b]$ such that

$$
f(c) = \frac{\int_a^b f(x) g(x)\,dx}{\int_a^b g(x)\,dx} \Rightarrow f(c)\int_a^b g(x)\,dx = \int_a^b f(x) g(x)\,dx
$$

That is, $f$ takes on all values between $m$ and $M$. The proof for $g(x) \leq 0$ is similar.