Second Mean Value Theorem for Integrals

Theorem: Let f be continuous and g integrable on $[a,b]$. If $g(x) \geq 0$ (or $g(x) \leq 0$) on $[a,b]$, then there exists a point $c, a \leq c \leq b,$ such that

$$\int_a^b f(x)g(x)dx = f(c)\int_a^b g(x)dx$$

Proof: Assume $g(x) \geq 0$. Let $m = \min f(x)$ and $M = \max f(x)$ on $[a,b]$. We have

$$m \leq f(x) \leq M$$

$$mg(x) \leq f(x)g(x) \leq Mg(x)$$

$$m\int_a^b g(x)dx = \int_a^b mg(x)dx \leq \int_a^b f(x)g(x)dx \leq \int_a^b Mg(x)dx = M\int_a^b g(x)dx$$

If $\int_a^b g(x)dx = 0$ we are done, so assume $\int_a^b g(x)dx > 0$. Dividing,

$$m \leq \frac{\int_a^b f(x)g(x)dx}{\int_a^b g(x)dx} \leq M$$

By the Intermediate Value Theorem, there exists $c \in [a,b]$ such that

$$f(c) = \frac{\int_a^b f(x)g(x)dx}{\int_a^b g(x)dx} \Rightarrow f(c)\int_a^b g(x)dx = \int_a^b f(x)g(x)dx$$

That is, f takes on all values between m and M. The proof for $g(x) \leq 0$ is similar.