Error of the Trapezoidal Rule

Lemma: Let f'' be continuous on $[0,1]$. Then there exists $c \in (0,1)$ such that

$$\int_0^1 f(x)dx = \frac{1}{2}\left[f(0)+f(1)\right] - \frac{1}{12}f''(c)$$

Proof: Let $p(x) = f(0) + \left[f(1) - f(0)\right]x$. Note that $p(0) = f(0), p(1) = f(1)$.

$$\int_0^1 p(x)dx = \left[f(0)x + \frac{1}{2}\left(f(1)-f(0)\right)x^2\right]_0^1 = f(0) + \frac{1}{2}\left(f(1)-f(0)\right) = \frac{1}{2}\left(f(0)+f(1)\right)$$

Recall the error formula for the Lagrange Interpolating polynomial

$$f(x) - p(x) = \frac{1}{(n+1)!}f^{(n+1)}(c)\prod_{i=0}^{n}(x - x_i)$$

In our case, $n = 1, x_0 = 0, x_1 = 1$, so we have

$$f(x) - p(x) = \frac{1}{2}f''(c(x))x(x-1)$$

Integrating,

$$\int_0^1 f(x)dx - \int_0^1 p(x)dx = \frac{1}{2}\int_0^1 f''(c(x))x(x-1)dx$$

Since $x(x-1) \le 0$ on $[0,1]$, the Second Mean Value Theorem says that there exists ξ such that

$$\begin{aligned}\frac{1}{2}\int_0^1 f''(c(x))x(x-1)dx &= \frac{1}{2}f''(c(\xi))\int_0^1 x(x-1)dx \\ &= \frac{1}{2}f''(c(\xi))\left(\frac{-1}{6}\right) \\ &= \frac{-1}{12}f''(c(\xi))\end{aligned}$$

Finally, $\int_0^1 f(x)dx = \frac{1}{2}\left[f(0)+f(1)\right]-\frac{1}{12}f''(c(\xi))$.

Lemma: (Change of variables) : On the interval $[a,b]$,

$$\int_a^b f(x)dx = \frac{b-a}{2}\left[f(a)+f(b)\right]-\frac{1}{12}(b-a)^3 f''(\xi), a\le\xi\le b$$

Proof: Let $g(t)=f(a+t(b-a))$ and $x=a+(b-a)t$. Then $t\in[0,1]\Leftrightarrow x\in[a,b], g(0)=f(a), g(1)=f(b)$. Also, $dx=(b-a)dt$, $g'(t)=f'(a+t(b-a))(b-a)$ and $g''(t)=f''(a+t(b-a))(b-a)^2$. So we have

$$\begin{aligned}\int_a^b f(x)dx &= \int_0^1 f(a+t(b-a))(b-a)dt \\ &= (b-a)\int_0^1 g(t)dt \\ &= (b-a)\left[\frac{1}{2}(g(0)+g(1))-\frac{1}{12}g''(c)\right], 0\le c\le 1 \\ &= \frac{b-a}{2}\left[f(a)+f(b)\right]-\frac{b-a}{12}f''(a+c(b-a))(b-a)^2 \\ &= \frac{b-a}{2}\left[f(a)+f(b)\right]-\frac{(b-a)^3}{12}f''(\xi), a\le\xi\le b\end{aligned}$$

Theorem: Assume that f'' exists and is continuous on $[a,b]$. Let T be the Composite Trapezoidal Rule with spacing h, and let $I=\int_a^b f(x)dx$. Then for some $c\in(a,b)$,

$$I-T=-\frac{1}{12}(b-a)h^2 f''(c)\approx \mathrm{O}(h^2)$$

Proof: Consider subintervals $x_0<x_1<\cdots<x_n$ with spacing h. Apply the previous lemma to $[x_i, x_{i+1}]$:

$$\int_{x_i}^{x_{i+1}} f(x)dx = \frac{h}{2}\left[f(x_i) + f(x_{i+1})\right] - \frac{1}{12}h^3 f''(c_i), x_i < c_i < x_{i+1}$$

So on the interval $[a,b]$,

$$\int_a^b f(x)dx = \sum_{i=0}^{n-1} \int_{x_i}^{x_{i+1}} f(x)dx$$

$$= \frac{h}{2}\sum_{i=0}^{n-1} \left[f(x_i) + f(x_{i+1})\right] - \frac{1}{12}h^3 \sum_{i=0}^{n-1} f''(c_i)$$

The first summation is the Trapezoidal Rule, T. The error term is

$$-\frac{1}{12}h^3 \sum_{i=0}^{n-1} f''(c_i) = -\frac{b-a}{12}h^2\left[\frac{1}{n}\sum_{i=0}^{n-1} f''(c_i)\right], h = \frac{b-a}{n}$$

Note that $\left[\frac{1}{n}\sum_{i=0}^{n-1} f''(c_i)\right]$ lies between the largest and smallest value of $f''(x)$ on $[a,b]$. So by the Intermediate Value Theorem, there exists $c \in [a,b]$ such that this expression equals $f''(c)$. Finally,

$$I = T - \frac{b-a}{12}h^2 f''(c)$$