Error of the Trapezoidal Rule

**Lemma:** Let $f''$ be continuous on $[0,1]$. Then there exists $c \in (0,1)$ such that

$$\int_0^1 f(x)dx = \frac{1}{2}[f(0) + f(1)] - \frac{1}{12} f''(c)$$

**Proof:** Let $p(x) = f(0) + [f(1) - f(0)]x$. Note that $p(0) = f(0), p(1) = f(1)$.

$$\int_0^1 p(x)dx = \left[ f(0)x + \frac{1}{2}(f(1) - f(0))x^2 \right]_0^1 = f(0) + \frac{1}{2}(f(1) - f(0)) = \frac{1}{2}(f(0) + f(1))$$

Recall the error formula for the Lagrange Interpolating polynomial

$$f(x) - p(x) = \frac{1}{(n+1)!} f^{(n+1)}(c) \prod_{i=0}^n (x-x_i)$$

In our case, $n=1, x_0 = 0, x_1 = 1$, so we have

$$f(x) - p(x) = \frac{1}{2} f''(c(x)) x(x-1)$$

Integrating,

$$\int_0^1 f(x)dx - \int_0^1 p(x)dx = \frac{1}{2} \int_0^1 f''(c(x)) x(x-1)dx$$

Since $x(x-1) \leq 0$ on $[0,1]$, the Second Mean Value Theorem says that there exists $\xi$ such that

$$\frac{1}{2} \int_0^1 f''(c(x)) x(x-1)dx = \frac{1}{2} f''(c(\xi)) \int_0^1 x(x-1)dx$$

$$= \frac{1}{2} f''(c(\xi)) \left( \frac{-1}{6} \right)$$

$$= \frac{-1}{12} f''(c(\xi))$$
Finally, \( \int_0^1 f(x)\,dx = \frac{1}{2} [f(0) + f(1)] - \frac{1}{12} f''(c(\xi)). \)

**Lemma:** (Change of variables) : On the interval \([a,b] \),
\[
\int_a^b f(x)\,dx = \frac{b-a}{2} \left[ f(a) + f(b) \right] - \frac{1}{12} (b-a)^3 f''(\xi), a \leq \xi \leq b
\]

**Proof:** Let \( g(t) = f(a + t(b-a)) \) and \( x = a + (b-a)t \). Then \( t \in [0,1] \Leftrightarrow x \in [a,b] \), \( g(0) = f(a) \), \( g(1) = f(b) \). Also, \( dx = (b-a)\,dt \), \( g'(t) = f'(a + t(b-a))(b-a) \) and \( g''(t) = f''(a + t(b-a))(b-a)^2 \). So we have
\[
\int_a^b f(x)\,dx = \int_0^1 f(a + t(b-a))(b-a)\,dt
= (b-a) \int_0^1 g(t)\,dt
= (b-a) \left[ \frac{1}{2} (g(0) + g(1)) - \frac{1}{12} g''(c) \right], 0 \leq c \leq 1
= \frac{b-a}{2} \left[ f(a) + f(b) \right] - \frac{b-a}{12} f''(a + c(b-a))(b-a)^2
= \frac{b-a}{2} \left[ f(a) + f(b) \right] - \frac{(b-a)^3}{12} f''(\xi), a \leq \xi \leq b
\]

**Theorem:** Assume that \( f'' \) exists and is continuous on \([a,b] \). Let \( T \) be the Composite Trapezoidal Rule with spacing \( h \) \( , \) and let \( I = \int_a^b f(x)\,dx \). Then for some \( c \in (a,b) \),
\[
I - T = -\frac{1}{12} (b-a) h^2 f''(c) \approx O(h^2)
\]

**Proof:** Consider subintervals \( x_0 < x_1 < \cdots < x_n \) with spacing \( h \). Apply the previous lemma to \([x_i, x_{i+1}]\) :
\[
\int_{x_i}^{x_{i+1}} f(x) \, dx = \frac{h}{2} [f(x_i) + f(x_{i+1})] - \frac{1}{12} h^3 f''(c_i), \quad x_i < c_i < x_{i+1}
\]

So on the interval \([a, b]\),

\[
\int_a^b f(x) \, dx = \sum_{i=0}^{n-1} \int_{x_i}^{x_{i+1}} f(x) \, dx = \frac{h}{2} \sum_{i=0}^{n-1} [f(x_i) + f(x_{i+1})] - \frac{1}{12} h^3 \sum_{i=0}^{n-1} f''(c_i)
\]

The first summation is the Trapezoidal Rule, \(T\). The error term is

\[
-\frac{1}{12} h^3 \sum_{i=0}^{n-1} f''(c_i) = -\frac{b-a}{12} h^2 \left[ \frac{1}{n} \sum_{i=0}^{n-1} f''(c_i) \right], \quad h = \frac{b-a}{n}
\]

Note that \(\left[ \frac{1}{n} \sum_{i=0}^{n-1} f''(c_i) \right]\) lies between the largest and smallest value of \(f''(x)\) on \([a, b]\). So by the Intermediate Value Theorem, there exists \(c \in [a, b]\) such that this expression equals \(f''(c)\). Finally,

\[
I = T - \frac{b-a}{12} h^2 f''(c)
\]