Cubic Equations

We will derive an explicit formula for the roots of the cubic polynomial equation

$$x^3 + ax^2 + bx + c = 0. \tag{1}$$

First of all, remove the quadratic term ax^2 by letting $x = y - \frac{1}{3}a$. Then

$$(y - \frac{1}{3}a)^3 + a(y - \frac{1}{3}a)^2 + b(y - \frac{1}{3}a) + c = 0$$

reduces to

$$y^3 + py + q = 0 \tag{2}$$

where $p = -a^2/3 + b$ and $q = 2a^3/27 - ab/3 + c$.

Now let $y = u + v$ in equation (2), producing

$$u^3 + v^3 + (3uv + p)y + q = 0. \tag{3}$$

Choose u and v such that $3uv + p = 0$ (that is, $u^3v^3 = -p^3/27$). Then from equation (3) we have a system of two equations in the two unknowns u^3 and v^3.

$$\begin{aligned} u^3 + v^3 &= -q \\ u^3v^3 &= -p^3/27 \end{aligned}$$

Solving the second equation for u^3 and substituting into the first equation,

$$v^6 + qv^3 - \frac{p^3}{27} = 0. \tag{4}$$

Now let $t = v^3$ in equation (4) to yield the quadratic equation

$$t^2 + qt - \frac{p^3}{27} = 0. \tag{5}$$

Using the quadratic formula,

$$t = \frac{-q \pm \sqrt{q^2 + \dfrac{4p^3}{27}}}{2} = -\frac{q}{2} \pm \sqrt{\left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3}$$

Since $u^3 + v^3 = -q$, we obtain

$$v^3 = -\frac{q}{2} + \sqrt{\left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3}$$
$$u^3 = -\frac{q}{2} - \sqrt{\left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3}.$$

Taking cube roots,

$$v = \sqrt[3]{-\frac{q}{2} + \sqrt{\left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3}} \tag{6}$$
$$u = \sqrt[3]{-\frac{q}{2} - \sqrt{\left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3}}.$$

Finally, the solution x is given by

$$x = y - \frac{1}{3}a = u + v - \frac{1}{3}a. \tag{7}$$

Before we discuss some of the difficulties with this formula, let us illustrate the above development with a simple example.

Example. Solve $x^3 - 2x^2 + 2x - 1 = 0$.

Solution: Let $x = y - \frac{1}{3}a = y + \frac{2}{3}$. Then equation (2) becomes

$$y^3 + py + q = y^3 + \frac{2}{3}y - \frac{7}{27} = 0.$$

Letting $y = u + v$, we obtain the system

$$u^3 + v^3 = -q = \frac{7}{27}$$
$$u^3v^3 = -p^3/27 = -\frac{8}{729}$$

Solving this system, we obtain

$$v^6 + qv^3 - \frac{p^3}{27} = v^6 - \frac{7}{27}v^3 - \frac{8}{729} = 0.$$

Letting $t = v^3$,

$$t = -\frac{q}{2} \pm \sqrt{\left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3} = \frac{7}{54} \pm \sqrt{\left(\frac{-7}{54}\right)^2 + \left(\frac{2}{9}\right)^3} = \frac{7}{54} \pm \frac{1}{6}$$

Thus we have two solutions: $t = \frac{8}{27}$ and $t = -\frac{1}{27}$. Taking $v^3 = \frac{8}{27}$ and $u^3 = -\frac{1}{27}$, we obtain $v = \frac{2}{3}$ and $u = -\frac{1}{3}$. Finally,

$$x = y + \frac{2}{3} = u + v + \frac{2}{3} = -\frac{1}{3} + \frac{2}{3} + \frac{2}{3} = 1$$

Notice that the original polynomial has one real and two complex roots:
$x^3 - 2x^2 + 2x - 1 = (x-1)(x^2 - x + 1) = 0$. ▲

What happens if the roots t of equation (5) are complex? It is surprising to learn that this will happen when the original polynomial has three real roots, as shown in the next example.

Example. Solve $x^3 - 3x = 0$.

Solution: Since there is no quadratic term, $x = y$, $p = -3$ and $q = 0$. Equation (5) reduces to

$$t^2 + 1 = 0,$$

and hence $t = \pm i$, complex numbers! To resolve this dilemma, let

$$v_1 = \sqrt[3]{i} = \left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)^{1/3} = \frac{\sqrt{3}}{2} + \frac{1}{2}i$$
$$u_1 = \sqrt[3]{-i} = \left(\cos\frac{-\pi}{2} + i\sin\frac{-\pi}{2}\right)^{1/3} = \frac{\sqrt{3}}{2} - \frac{1}{2}i$$

Thus, $x_1 = u_1 + v_1 = \sqrt{3}$ is real.

To obtain the other roots, let $w_2 = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$ and $w_3 = -\frac{1}{2} - \frac{\sqrt{3}}{2}i$ be the two complex cube roots of 1. Define

$$v_2 = v_1 w_2 = -\frac{\sqrt{3}}{2} + \frac{1}{2}i$$
$$v_3 = v_1 w_3 = -i$$
$$u_2 = u_1 w_3 = -\frac{\sqrt{3}}{2} - \frac{1}{2}i$$
$$u_3 = u_1 w_2 = i$$

The three real solutions are

$$x_1 = u_1 + v_1 = \sqrt{3}$$
$$x_2 = u_2 + v_2 = -\sqrt{3}$$
$$x_3 = u_3 + v_3 = 0$$

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