Implicit Function Theorem

Definition. *Let (a, b) be in the domain of F. We say that F is* **continuous** *at (a, b) if for every $\epsilon > 0$, there is a number $\delta > 0$ such that every point (x, y) in the region $\{(x, y) : |x - a| < \delta \text{ and } |y - b| < \delta\}$ also satisfies $|F(x, y) - F(a, b)| < \epsilon$.*

Definition. *Let (a, b) be in the domain of F. We say that F is* **differentiable** *at (a, b) if F can be approximated by a linear function in a neighborhood of (a, b). That is, if*

$$F(a + h, b + k) = F(a, b) + Ah + Bk + \epsilon_1 h + \epsilon_2 k,$$

where A and B are independent of the variables h and k, and where ϵ_1 and ϵ_2 tend to 0 as h and k do.

If F is differentiable, it follows at once that $A = F_x$ and $B = F_y$. For if we let $k = 0$ and divide by h,

$$\frac{F(a + h, b) - F(a, b)}{h} = A + \epsilon_1.$$

Letting $h \to 0$, we see that the left hand side has a limit, and that limit is A.

Theorem. *If F has continuous partial derivatives at the point (a, b), then F is differentiable at (a, b).*

Proof. We use the Mean Value Theorem twice.

$$\begin{aligned} F(a + h, b + k) - F(a, b) &= \{F(a + h, b + k) - F(a, b + k)\} + \{F(a, b + k) - F(a, b)\} \\ &= hF_x(c_1, b + k) + kF_y(a, c_2), \end{aligned}$$

where c_1 is between a and $a + h$, and c_2 is between b and $b + k$.

Let $\epsilon_1 = F_x(c_1, b + k) - F_x(a, b)$ and $\epsilon_2 = F_y(a, c_2) - F_y(a, b)$. Then

$$\begin{aligned} F(a + h, b + k) - F(a, b) &= h(F_x(a, b) + \epsilon_1) + k(F_y(a, b) + \epsilon_2) \\ &= hF_x(a, b) + kF_y(a, b) + \epsilon_1 h + \epsilon_2 k. \end{aligned}$$

By the continuity of F_x and F_y, ϵ_1 and ϵ_2 tend to 0 as h and k do. Thus, F is differentiable at (a, b).

Theorem. *Let (a,b) be a point on the graph of $F(x,y)=0$. Suppose that both partial derivatives F_x and F_y are continuous in a neighborhood of (a,b), and $F_y(a,b)\neq 0$. Then there exists a rectangle R centered at (a,b),*

$$R=\{(x,y):|x-a|\le\alpha \quad \textit{and} \quad |y-b|\le\beta\}$$

such that for every x in the interval I given by $|x-a|<\alpha$, the equation $F(x,y)=0$ has exactly one solution $y=\phi(x)$ lying in the interval $|y-b|<\beta$. This function ϕ satisfies $b=\phi(a)$ and, for every x in the interval I,

$$F(x,\phi(x))=0$$
$$|f(x)-b|<\beta$$
$$F_y(x,\phi(x))\neq 0$$

Furthermore, ϕ is continuous and has a continuous derivative in I, given by

$$\phi'(x)=-F_x/F_y.$$

Proof. Assume $F_y(a,b)=m>0$. (If $F_y(a,b)<0$, replace F by $-F$.) There exists a rectangle R with center (a,b) such that R is inside the domain of F and $F_y(x,y)>m/2$ throughout R. Let R be given by

$$R=\{(x,y):|x-a|\le\alpha \ \text{ and } \ |y-b|\le\beta\}.$$

F_x is continuous, hence it is bounded on R. There exist positive constants m, M such that $F_y(x,y)>m/2$ and $|F_x(x,y)|\le M$ for all (x,y) in R.

Since $F_y>m/2>0$, $F(x,y)$ is a continuous and monotonically increasing function of y for $|y-b|\le\beta$. By the Mean Value Theorem,

$$F(x,b)=F(x,b)-F(a,b)=F_x(c,b)(x-a)$$

for some c between x and a. If γ is between 0 and α,

$$|F(x,b)|\le|F_x(c,b)||x-a|\le M\gamma$$

for $|x-a|\le\gamma$. In other words, $-M\gamma\le F(x,b)\le M\gamma$.

Since $F_y > m/2$, there exists c' between b and $b+\beta$ such that

$$\begin{aligned}
F(x, b+\beta) &= F(x, b+\beta) - F(x, b) + F(x, b) \\
&= F_y(x, c')(\beta) + F(x, b) \\
&> \frac{m}{2}\beta + F(x, b) \\
&> \frac{m}{2}\beta - M\gamma
\end{aligned}$$

and there exists c'' between $b-\beta$ and b such that

$$\begin{aligned}
F(x, b-\beta) &= -[F(x, b) - F(x, b-\beta)] + F(x, b) \\
&= -[F_y(x, c'')(\beta)] + F(x, b) \\
&< -\left[\frac{m}{2}\beta\right] + M\gamma \\
&= -\left[\frac{m}{2}\beta - M\gamma\right]
\end{aligned}$$

Therefore, for a given x in I, if $F(x, b+\beta) > 0$, then $F(x, b-\beta) < 0$ (assuming $|x-a| \leq \alpha, \gamma \leq \alpha$ and $\gamma < \frac{m\beta}{2M}$).

By the Intermediate Value Theorem, there exists a single value y between $b-\beta$ and $b+\beta$ where $F(x, y)$ vanishes. Thus, for the given x, the equation $F(x, y) = 0$ will have a single solution, $y = \phi(x)$.

See Courant, page 227 for the remainder of the proof.