LECTURE 4

Agenda:

1. Counting rules.
2. Examples

COUNTING RULES

Recall from Lecture 3 that

- \# of ways of choosing r objects from n distinct objects with replacement (order is important) is n^r
- \# of ways of choosing r objects from n distinct objects without replacement (order is important) is P_r^n, where
  $$P_r^n = \frac{n!}{(n-r)!}. \quad \text{(PERMUTATIONS)}$$

RESULT: \# of ways of choosing r objects from n distinct objects without replacement (order is not important) is
$$C_r^n = \binom{n}{r} = \frac{n!}{r!(n-r)!} \quad \text{(COMBINATIONS)}$$

Proof: Note that now we are only interested in which objects were chosen and not the order in which they are chosen. Since every collection of r objects can be ordered in $r!$ ways, it follows that

1 combination corresponds to $r!$ permutations. Hence, the number of ways of choosing r ~~distinct~~ objects without replacement from n distinct objects (order is not important) is $\frac{1}{r!} P_r^n = \frac{n!}{(n-r)!\,r!}$.

RESULT: # of ways of partitioning n distinct objects into k groups containing $n_1, n_2, \ldots, n_k$ objects, where $\sum_{i=1}^{k} n_i = n$, is

$$\frac{n!}{n_1!\, n_2! \cdots n_k!}$$

PROOF: The n_1 objects for the first group can be chosen in $\binom{n}{n_1}$ ways. The n_2 objects for the second group can be chosen in $\binom{n-n_1}{n_2}$ ways, ------, the n_k objects for the k^{th} group can be chosen in $\binom{n-n_1-n_2-\cdots-n_{k-2}-n_{k-1}}{n_k}$ ways. Hence, the number of ways of partitioning n distinct objects into k groups containing $n_1, n_2, \ldots, n_k$ objects is

$$\binom{n}{n_1}\binom{n-n_1}{n_2}\cdots\binom{n-n_1-n_2-\cdots-n_{k-2}-n_{k-1}}{n_k}$$

$$= \frac{n!}{n_1!\, n_2!\, n_3! \cdots n_{k-1}!\, n_k!}$$

RESULT: # of ways of choosing r objects from n distinct objects with replacement (order is not important) is $\binom{n+r-1}{r}$.

Proof: Since order is not important, every way of choosing r objects uniquely corresponds to a vector of the form $(l_1, l_2, ---, l_n)$ where l_i represents the number of times the i^{th} object is chosen in the r draws. Let us represent the n-tuple $(l_1, l_2, ---, l_n)$ as follows

$$\underbrace{00---0}_{l_1 \text{ times}} \Big| \underbrace{00---0}_{l_2 \text{ times}} \Big| --------- \Big| \underbrace{00---0}_{l_n \text{ times}}$$

Note that $\sum_{i=1}^{n} l_i = r$. It follows that the number of vectors $(l_1, l_2, ---, l_n)$ with $l_i \geq 0$ and $\sum_{i=1}^{n} l_i = r$ is same as the number of ways of arranging $n-1$ "|" symbols and r "0" symbols. But the number of such ways is $\frac{(n-1+r)!}{(n-1)!\ r!}$ (we divide by $(n-1)!$ and $r!$ because the $(n-1)$ "|" symbols are indistinguishable and the r "0" symbols are indistinguishable. Hence the number of ways of choosing r objects with replacement from n distinct objects (order is not important) is $\binom{n+r-1}{r}$.