1. Show that for any RV X, $V(X) \geq 0$. (You can assume X to be discrete, but this result holds in general.) Hence or otherwise show that $E(X^2) \geq E^2(X)$.

   Solution. If X is discrete, say with support $\mathscr{X}$ and PMF p_X and mean $E(X) = \mu_X$, then

   $$V(X) = E[X - E(X)]^2 = \sum_{x \in \mathscr{X}} \underbrace{(x - \mu_X)^2}_{\geq 0} \underbrace{p_X(x)}_{\geq 0} \geq 0.$$

   Thus,

   $$E(X^2) - E^2(X) = V(X) \geq 0 \implies E(X^2) \geq E^2(X).$$

   □

2. (WMS, Problem 3.43.) Many utility companies promote energy conservation by offering discount rates to consumers who keep their energy usage below certain established subsidy standards. A recent EPA report notes that 70% of the island residents of Puerto Rico have reduced their electricity usage sufficiently to qualify for discounted rates. If five residential subscribers are randomly selected from San Juan, Puerto Rico, find the probability of each of the following events:

   (a) All five qualify for the favorable rates.

   (b) At least four qualify for the favorable rates.

   Solution. (To solve this problem we either need to assume that the sample is chosen with replacement, or need to apply the binomial approximation to hypergometric when the population is large). Let $X = \#$ of qualifying subscribers with PMF $X \sim \text{Bin}(n = 5, p = 0.7)$. Then

   $$p(x) = \binom{5}{x}(0.7)^x(0.3)^{5-x}, \text{ for } x = 0, 1, \cdots, 5.$$

   (a) $P(X = 5) = \binom{5}{5}(0.7)^5(0.3)^0 = (0.7)^5 = 0.1681.$

   (b) $P(X \geq 4) = P(X = 4) + P(X = 5) = \binom{5}{4}(0.7)^4(0.3)^1 + 0.1681 = 0.5282.$

   □

3. (WMS, Problem 3.41.) A multiple-choice examination has 15 questions, each with five possible answers, only one of which is correct. Suppose that one of the students who takes the examination answers each of the questions with an independent random guess. What is the probability that he answers at least ten questions correctly?

   Solution. Let $X = \#$ correct answers. Then $X \sim \text{Bin}(n = 15, p = 1/5 = 0.2)$, which means

   $$p(x) = \binom{15}{x}(0.2)^x(0.8)^{15-x}, \text{ for } x = 0, 1, \cdots, 15.$$

   Therefore,

   $$P(\text{at least 10 correct answers}) = P(X \geq 10) = \sum_{x=10}^{15} p(x) = \sum_{x=10}^{15} \binom{15}{x}(0.2)^x(0.8)^{15-x}.$$

   □

4. Suppose a balanced coin is tossed 20 times. Define the two random variables X = total number of heads, and Y = total number of tails in the 20 tosses.

   (a) Show that X and Y have the same distribution, i.e., show that the two random variables X and Y have the same support and the same PMF.

   (b) Find the probability that X and Y are the same, i.e., $P(X = Y)$. [**Note:** This example shows that two different random variables can have the same distributions. In fact, as we can see in this case, $P(X = Y)$ is pretty small, although they have the same distribution.]

   (c) Show that the distribution of X is symmetric about $20/2 = 10$, i.e., show that for any $j \in \mathbb{R}$, $P(X = 10 + j) = P(X = 10 - j)$.

Solution. (a) For X, a success means getting a head. If the coin is balanced, then $P(\text{head}) = 1/2$. Since X counts the number of successes in 20 Bernoulli trials, therefore, $X \sim \text{Bin}(n = 20, p = 1/2)$. Now balanced coin means $P(\text{tail}) = 1/2$ too, which means Y = number of tails in 20 flips $\sim \text{Bin}(20, 1/2)$ as well. Thus, both X and Y have support $\{0, 1, 2, \cdots, 20\}$ and PMF

$$p(z) = \binom{20}{z}\left(\frac{1}{2}\right)^z\left(\frac{1}{2}\right)^{20-z} = \frac{\binom{20}{z}}{2^{20}}; \qquad z = 0, 1, \cdots, 20.$$

(b) Note that

$$P(X = Y) = P(X = 10) = p(10) = \frac{\binom{20}{10}}{2^{20}} \approx 0.1762.$$

Thus, although X and Y have the same distribution, the probability of them being equal is pretty small.

(c) On the outset, note that the support of X is $\mathscr{X} = \{0, 1, \cdots, 20\}$. Now when j is either a non-integer or an integer with absolute value bigger than 10 then $(10 + j) \notin \mathscr{X}$ and $(10 - j) \notin \mathscr{X}$. (How do we get this range? We need to make sure both $10 + j$ and $10 - j$ are integers and they lie in $\mathscr{X}$. If they do, then both $0 \le 10 + j \le 20$ and $0 \le 10 - j \le 20$, which means we must have $-10 \le j \le 10$.) Hence in such cases, trivially, $P(X = 10 - j) = P(X = 10 + j) = 0$. Therefore, the only non-trivial case we need to verify is the case where j is an integer belonging to the set $\{-10, -9, \cdots, 0, \cdots, 9, 10\} = \mathcal{Y}$, say. Let $j \in \mathcal{Y}$. Then

$$P(X = 10 + j) = p(10 + j) = \frac{\binom{20}{10+j}}{2^{20}} = \frac{\binom{20}{20-(10+j)}}{2^{20}} = \frac{\binom{20}{10-j}}{2^{20}} = p(10 - j) = P(X = 10 - j).$$

Thus, for any real number j, $P(X = 10 - j) = P(X = 10 + j)$ holds. This completes the proof.

□

5. (WMS, Problem 3.77.) If Y has a geometric distribution with success probability p, show that $P(Y = \text{an odd integer}) = \frac{p}{1-q^2}$. [**Hint:** Observe that here we are not interested in a particular odd integer, rather, we want the probability that Y is *any* odd integer.]

Solution. Note that Y is an odd (positive) integer means $Y = 1$ OR $Y = 3$ OR $Y = 5, \cdots$. Therefore,

$$\begin{aligned} P(Y = \text{an odd integer}) &= P(Y = 1) + P(Y = 3) + P(Y = 5) + \cdots \\ &= p + q^2 p + q^4 p + \cdots = p\left(1 + q^2 + q^4 + \cdots\right) = \frac{p}{1 - q^2}. \end{aligned}$$

□

6. (WMS, Problem 3.72.) Given that we have already tossed a balanced coin ten times and obtained zero heads, what is the probability that we must toss it at least two more times to obtain the first head? [**Hint:** Save a lot of work by using the memoryless property of the Geometric distribution.]

Solution. Let $X = \#$ tosses required to get the first head. Since the coin is balanced, so $P(\text{head}) = 1/2$. Hence, $X \sim \text{Geo}(p = 1/2)$. It is given that the coin has already been tossed ten times and no heads are obtained, which means $X > 10$. We need to find the probability of at least two more tosses given $X > 10$, i.e., $P(X \geq 10 + 2|X > 10) = P(X > 11|X > 10)$. Now, from memoryless property of Geometric distribution, we have for any two positive integers j and k, $P(X > j + k|X > j) = P(X > k)$. Here $j = 10, j + k = 11$, hence $k = 1$. Therefore,

$$P(X > 11|X > 10) = P(X > 1) = 1 - P(X \leq 1) = 1 - P(X = 1) = 1 - p = 1/2.$$

□

7. (WMS, Problem 3.76.) Of a population of consumers, 60% are reputed to prefer a particular brand, A, of toothpaste. If a group of randomly selected consumers is interviewed, what is the probability that exactly five people have to be interviewed to encounter the first consumer who prefers brand A? At least five people?

Solution. Let $X = \#$ people to be interviewed to get the first brand A admirer. Then $X \sim \text{Geo}(p = 0.6)$, with PMF:

$$p(x) = (0.4)^{x-1}(0.6), \text{ for } x = 1, 2, 3, \cdots.$$

So, $P(X = 5) = (0.4)^4 \times 0.6 = 0.01536$, and $P(X \geq 5) = P(X > 4) = (0.4)^4 = 0.0256$. □

8. Let $X \sim \text{Geo}(p)$. Find the DF F_X of X, and comment on the discontinuity points of F_X.

Solution. Recall, if k is any positive integer, then $P(X > k) = q^k$. Therefore, $F_X(k) = P(X \leq k) = 1 - P(X > k) = 1 - q^k$, for any positive integer k. Hence, for any b in $[k, k + 1)$, $F_X(b) = P(X \leq b) = P(X \leq k) = 1 - q^k$, for a positive integer k. If $b < 1$, then $F_X(b) = P(X \leq b) = 0$. Thus, the DF of X is given by:

$$F_X(b) = \begin{cases} 0, & b < 1 \\ 1 - q^k, & k \leq b < k + 1;\ k \in \{1, 2, 3, \cdots\}. \end{cases}$$

For any discrete RV X, the DF F_X has discontinuity at the points where the PMF is positive. In other words, the discontinuity points of F_X are exactly the points in $\mathscr{X}$. Hence, in this case, the discontinuity points are the positive integers $1, 2, 3. \cdots$. □

9. Suppose that a box contains five red balls and ten blue balls. If seven balls are selected at random without replacement, what is the probability that at least three red balls will be obtained?

Solution. Let $X = \#$ red balls. Then $X \sim \text{HG}(N = 15, r = 5, n = 7)$. Therefore, the PMF of X is:

$$p(x) = \frac{\binom{5}{x}\binom{10}{7-x}}{\binom{15}{7}}, \text{ for } x = 0, 1, \cdots, 5.$$

Hence,

$$P(\text{at least 3 red balls}) = P(X \geq 3) = p(3) + p(4) + p(5) = \sum_{x=3}^{5} \frac{\binom{5}{x}\binom{10}{7-x}}{\binom{15}{7}} \approx 0.4266.$$

□

10. (WMS, Problem 3.117.) In an assembly-line production of industrial robots, gearbox assemblies can be installed in one minute each if holes have been properly drilled in the boxes and in ten minutes if the holes must be redrilled. Twenty gearboxes are in stock, 2 with improperly drilled holes. Five gearboxes must be selected from the 20 that are available for installation in the next five robots.

(a) Find the probability that all 5 gearboxes will fit properly.

(b) Find the mean, variance, and standard deviation of the time it takes to install these 5 gearboxes.

Solution. Let $X = \#$ improperly drilled gearboxes. Then, $X \sim \text{HG}(N = 20, r = 2, n = 5)$ with PMF

$$p(x) = \frac{\binom{2}{x}\binom{18}{5-x}}{\binom{20}{5}}, \text{ for } x = 0, 1, 2.$$

(a) $P(\text{all gearboxes fit properly}) = P(X = 0) = p(0) = \frac{\binom{2}{0}\binom{18}{5}}{\binom{20}{5}} = 0.5526.$

(b) Since X denotes the number of improperly drilled gearboxes, hence time needed to install the gearboxes $= T = (5 - X) \times 1 + X \times 10 = 5 + 9X$ minutes. Now $E(X) = n \times \frac{r}{N} = 5 \times \frac{2}{20} = 0.5$ and $V(X) = n \times \frac{r}{N} \times \frac{N-r}{n} \times \frac{N-n}{N-1} = 5 \times \frac{2}{20} \times \frac{18}{20} \times \frac{15}{19} = 0.3553$. Therefore, $E(T) = E(5 + 9X) = 5 + 9E(X) = 5 + 9 \times 0.5 = 9.5$, and $V(T) = V(5 + 9X) = 9^2\, V(X) = 81 \times 0.3553 = 28.7793$ and $SD(X) = \sqrt{V(X)} = 5.3646$.

□

11. Suppose that a sequence of independent tosses are made with a coin for which the probability of obtaining a head on each given toss is 1/30.

(a) What is the expected number of tosses that will be required in order to obtain five heads?

(b) What is the variance of the number of tosses that will be required in order to obtain five heads?

(c) What is the expected number of tails that will be obtained before five heads have been obtained? [**Hint:** What is the relationship between number of trials and number of failures in this case?]

(d) What is the variance of the number of tails that will be obtained before five heads have been obtained?

Solution. (a) Let $X = \#$ tosses required to get five heads. Then $X \sim \text{NB}(r = 5, p = 1/30)$. Then $E(X) = r/p = 5 \times 30 = 150$.

(b) $V(X) = \frac{rq}{p^2} = 5 \times \frac{29}{30} \times 30^2 = 4350.$

(c) Let $Y = \#$ tails obtained before getting five heads. Observe that $Y+5 = \#$ trials needed to get 5 heads $= X$, which means $Y = X-5$. Therefore, $E(Y) = E(X-5) = E(X)-5 = 145$.

(d) $V(Y) = V(X-5) = V(X) = 4350$.

$\square$

12. (WMS, Problem 3.96.) The telephone lines serving an airline reservation office are all busy about 60% of the time.

(a) If you are calling this office, what is the probability that you will complete your call on the first try? The second try? The third try?

(b) If you and a friend must both complete calls to this office, what is the probability that a total of four tries will be necessary for both of you to get through?

Solution. (a) Let $X = \#$ attempts until the call gets completed. Probability of getting the call completed on a single try is $(100-60)\% = 40\% = 0.4$. Thus, $X \sim \text{Geo}(p = 0.4)$ with PMF

$$p_X(x) = (0.6)^{x-1}(0.4), \text{ for } x = 1, 2, 3, \cdots .$$

Therefore, $P(X = 1) = 0.4, P(X = 2) = (0.6)(0.4) = 0.24, P(X = 3) = (0.6)^2(0.4) = 0.144$.

(b) Let $Y = \#$ of attempts until both calls are completed. Then $Y \sim \text{NB}(r = 2, p = 0.4)$ with PMF:

$$p_Y(y) = \binom{y-1}{1}(0.4)^2(0.6)^{y-2} = (y-1)(0.4)^2(0.6)^{y-2}, \text{ for } y = 2, 3, 4, \cdots .$$

Thus, $P(Y = 4) = 3(0.4)^2(0.6)^2 = 0.1728$.

$\square$