1. (WMS, Problem 4.140.) Identify the distributions of the random variables with the following moment-generating functions:

   (a) $M(t) = (1 - 4t)^{-2}$.

   (b) $M(t) = 1/(1 - 3.2t)$.

   (c) $M(t) = e^{-5t+6t^2}$.

   Solution. (a) Gamma($\alpha = 2, \beta = 4$), (b) Gamma($\alpha = 1, \beta = 3.2$) $\equiv$ Exp($\beta = 3.2$), (c) $N(\mu = -5, \sigma^2 = 2 \times 6 = 12)$. □

2. The life (in hours) of an electronic tube manufactured by a certain process is normally distributed with mean 160 and variance σ^2. What is the maximum allowable value for σ if the life of a tube is to have a probability 0.8 of being between 120 and 200 hours? (Leave the answer in terms of the inverse the standard normal CDF Φ.)

   Solution. Let X be the life (in hours) of a randomly chosen tube. Then by assumption $X \sim N(\mu = 160, \sigma^2)$, so that here $Z = \frac{X-160}{\sigma} \sim N(0, 1)$. Therefore, we need σ such that

$$
\begin{aligned}
& P(120 \leq X \leq 200) = 0.8 \\
\implies & P\left(\frac{120 - 160}{\sigma} \leq \frac{X - 160}{\sigma} \leq \frac{200 - 160}{\sigma}\right) = 0.8 \\
\implies & P\left(-\frac{40}{\sigma} \leq Z \leq \frac{40}{\sigma}\right) = 0.8 \\
\implies & \Phi\left(\frac{40}{\sigma}\right) - \Phi\left(-\frac{40}{\sigma}\right) = 2\Phi\left(\frac{40}{\sigma}\right) - 1 = 0.8 \\
\implies & \Phi\left(\frac{40}{\sigma}\right) = 0.9 \\
\implies & \frac{40}{\sigma} = \Phi^{-1}(0.9) \implies \boxed{\sigma = \frac{40}{\Phi^{-1}(0.9)}\text{(hrs).}}
\end{aligned}
$$

□

3. Suppose that X has the beta distribution with parameters α and β, and let r and s be given positive integers. Determine the value of $E[X^r(1 - X)^s]$.

   Solution. Note that X has PDF

$$f(x) = \frac{1}{B(\alpha, \beta)}\, x^{\alpha-1}(1 - x)^{\beta-1},\ 0 \leq x \leq 1.$$

   Hence,

$$
\begin{aligned}
E[X^r(1 - X)^s] &= \int_0^1 x^{\alpha-1}(1 - x)^{\beta-1}\, \frac{1}{B(\alpha, \beta)}\, x^{\alpha-1}(1 - x)^{\beta-1}\, dx \\
&= \frac{1}{B(\alpha, \beta)} \int_0^1 x^{\alpha+r-1}(1 - x)^{\beta+s-1}\, dx = \frac{B(\alpha + r, \beta + s)}{B(\alpha, \beta)}
\end{aligned}
$$

□

4. (WMS, Problem 4.129.) During an eight-hour shift, the proportion of time Y that a sheet-metal stamping machine is down for maintenance or repairs has a beta distribution with $\alpha = 1$ and $\beta = 2$. That is, $f(y) = 2(1-y), 0 \le y \le 1$. The cost (in hundreds of dollars) of this downtime, due to lost production and cost of maintenance and repair, is given by $C = 10 + 20Y + 4Y^2$. Find the mean of C.

   Solution. Using formulas, $E(Y) = \frac{\alpha}{\alpha+\beta} = \frac{1}{3}$ and $V(Y) = \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)} = \frac{2}{3^2 \times 4} = \frac{1}{18}$. So, $E(Y^2) = V(Y) + E^2(Y) = \frac{1}{18} + \frac{1}{9} = \frac{3}{18}$ and hence,

$$E(C) = E(10 + 20Y + 4Y^2) = 10 + 20E(Y) + 4E(Y^2) = 10 + 20 \times \frac{1}{3} + 4 \times \frac{3}{18} = \frac{52}{3}.$$

□

5. Let X have a beta distribution with parameters $\alpha > 1$ and $\beta > 1$. Recall that the mode of (the distribution of) X is the maximizer of the density of X. Find the mode of X.

   Solution. The PDF of X is given by

$$f(x) = \frac{1}{B(\alpha, \beta)} x^{\alpha-1}(1-x)^{\beta-1},\ 0 < x < 1.$$

   where $\alpha > 1$, $\beta > 1$, and we need to maximize $f(x)$ with respect to x. Since log is an monotonically increasing function, therefore $f(x)$ and $\log f(x)$ have the same maximizer. So, instead of maximizing $f(x)$, we'll maximize

$$l(x) := \log f(x) = -\log B(\alpha, \beta) + (\alpha - 1)\log x + (\beta - 1)\log(1-x).$$

   Note that

$$\begin{aligned} l'(x) &= \frac{\alpha-1}{x} - \frac{\beta-1}{1-x} = 0 \\ \implies \frac{\alpha-1}{x} &= \frac{\beta-1}{1-x} \\ \implies \frac{\beta-1}{\alpha-1} &= \frac{1-x}{x} = \frac{1}{x} - 1 \\ \implies \frac{1}{x} &= \frac{\beta-1}{\alpha-1} + 1 = \frac{\alpha+\beta-2}{\alpha-1} \implies x = \frac{\alpha-1}{\alpha+\beta-2}. \end{aligned}$$

   Also,

$$l''(x) = -\frac{\alpha-1}{x^2} - \frac{\beta-1}{(1-x)^2} < 0$$

   since $\alpha > 1$ and $\beta > 1$. Therefore, $x_0 := \frac{\alpha-1}{\alpha+\beta-2}$ maximizes $l(x)$ and hence $f(x)$. Hence, x_0 is the mode of X.
   Note: Maximizing a log density instead of the actual density is often easier, and has plenty of applications in Statistics (e.g., in *log likelihood* maximization). Try solving Homework 6.5 (mode of the normal distribution) using this method.

□

6. (WMS, Problem 4.145.) A random variable Y has the density function

$$f(y) = \begin{cases} e^y, & y < 0 \\ 0, & \text{elsewhere.} \end{cases}$$

(a) Find $E(e^{3Y/2})$.

Solution. Note that

$$E(e^{3Y/2}) = \int_{-\infty}^{0} e^{3y/2}\, e^{y}\, dy = \int_{-\infty}^{0} e^{5y/2}\, dy = \frac{2}{5}\left[e^{5y/2}\right]_{-\infty}^{0} = \frac{2}{5}.$$

□

(b) Find the moment-generating function for Y.

Solution. By definition,

$$M_Y(t) = E(e^{tY}) = \int_{-\infty}^{0} e^{ty}\, e^{y}\, dy = \int_{-\infty}^{0} e^{(t+1)y}\, dy = \left[\frac{e^{(t+1)y}}{t+1}\right]_{-\infty}^{0} = \frac{1}{t+1},$$

provided $(t+1) > 0 \implies t > -1$. □

(c) Find $V(Y)$.

Solution. We have,

$$M'_X(t) = -\frac{1}{(t+1)^2} \text{ and } M''_X(t) = \frac{2}{(t+1)^3}.$$

Therefore, $E(Y) = M'_X(t)|_{t=0} = -1$ and $E(Y^2) = M''_X(t)|_{t=0} = 2$. Therefore, $V(Y) = E(Y^2) - E^2(Y) = 2 - 1 = 1$. □

7. A RV X is said to have the *Cauchy distribution* with location parameter $\mu \in \mathbb{R}$ and scale parameter $\sigma > 0$ if X has density

$$f(x) = \frac{\sigma}{\pi}\frac{1}{\sigma^2 + (x-\mu)^2}, \quad -\infty < x < \infty$$

(a) Find the CDF F of X. [**Hint:** Use the fact that $\int \frac{dx}{x^2+a^2} = \frac{1}{a}\arctan(\frac{x}{a}) + C$ where C is an arbitrary constant of integration.]

Solution. For $x \in \mathbb{R}$, the CDF F is given by

$$\begin{aligned}
F(x) = \int_{-\infty}^{x} f(u)\, du &= \int_{-\infty}^{x} \frac{\sigma}{\pi}\frac{1}{\sigma^2 + (u-\mu)^2}\, du \\
&= \frac{\sigma}{\pi}\int_{-\infty}^{x-\mu} \frac{dv}{\sigma^2 + v^2} && \text{(substitute } v = u - \mu\text{)} \\
&= \frac{\sigma}{\pi}\cdot\frac{1}{\sigma}\left[\arctan\left(\frac{v}{\sigma}\right)\right]_{-\infty}^{x-\mu} \\
&= \frac{1}{\pi}\left[\arctan\left(\frac{x-\mu}{\sigma}\right) - \underbrace{\arctan(-\infty)}_{=-\pi/2}\right] \\
&= \frac{1}{\pi}\arctan\left(\frac{x-\mu}{\sigma}\right) + \frac{1}{2}
\end{aligned}$$

□

(b) For $0 < p < 1$, find the p-th quantile ϕ_p of (the distribution of) X. [**Hint:** Note that F is continuous and strictly increasing on $\mathbb{R}$.]

Solution. Since F is continuous and strictly increasing on $\mathbb{R}$, therefore, F^{-1} exists and the p-th quantile ϕ_p, for $0 < p < 1$, is given by the equation

$$\begin{aligned} F(\phi_p) = p &\implies \frac{1}{2} + \frac{1}{\pi} \arctan\left(\frac{\phi_p - \mu}{\sigma}\right) + \frac{1}{2} = p \\ &\implies \arctan\left(\frac{\phi_p - \mu}{\sigma}\right) = \pi\left(p - \frac{1}{2}\right) \\ &\implies \frac{\phi_p - \mu}{\sigma} = \tan\left[\pi\left(p - \frac{1}{2}\right)\right] \\ &\implies \boxed{\phi_p = \mu + \sigma \tan\left[\pi\left(p - \frac{1}{2}\right)\right]} \end{aligned}$$

□

(c) Show that μ is the median of X.

Solution. By definition, the median is the 1/2-th quantile $\phi_{1/2}$. Therefore, by plugging in $p = 1/2$ in the expression for ϕ_p in part (b), we get

$$\text{median} = \phi_{1/2} = \mu + \sigma \tan\left[\pi\left(\frac{1}{2} - \frac{1}{2}\right)\right] = \mu + \sigma \underbrace{\tan 0}_{=0} = \mu.$$

□

8. (WMS, Problem 5.3.) Of nine executives in a business firm, four are married, three have never married, and two are divorced. Three of the executives are to be selected for promotion. Let Y_1 denote the number of married executives and Y_2 denote the number of never-married executives among the three selected for promotion. Assuming that the three are randomly selected from the nine available, find the joint probability function of Y_1 and Y_2.

Solution. First, note that $\#(\mathcal{S})$ = number of ways 3 individuals can be chosen from 9 = $\binom{9}{3}$. Now, under the event $\{Y_1 = y_1, Y_2 = y_2\}$ (y_1, y_2 are integers such that $y_1 \geq 0, y_2 \geq 0$ and $y_1 + y_2 \leq 3$), y_1 individuals are chosen form 4 (married), y_2 are chosen form 2 (never married), and $3 - y_1 - y_2$ from 2 (divorced) individuals. Therefore, the joint PMF of (Y_1, Y_2) is given by:

$$p_{Y_1,Y_2}(y_1, y_2) = P(Y_1 = y_1, Y_2 = y_2) = \frac{\binom{4}{y_1}\binom{3}{y_2}\binom{2}{3-y_1-y_2}}{\binom{9}{3}}, \; y_1 \geq 0, y_2 \geq 0, y_1 + y_2 \leq 3$$

□