1. Elections are being held for two offices, with Democratic and Republican candidates (and no other parties or independent candidates) for both offices. Define the events

$$A = \{\text{the first office is won by a Democratic candidate}\}$$
$$B = \{\text{the second office is won by a Democratic candidate}\}$$

Describe the following events using unions, intersections and/or complements of A and B.

(a) The second office is won by a Republican.
(b) Both offices are won by Democrats.
(c) At least one of the offices is won by a Democrat.
(d) Both offices are won by Republicans.
(e) At least one of the offices is won by a Republican.

Solution. (a) $\bar{B}$. (b) $A \cap B$. (c) $A \cup B$. (d) $\bar{A} \cap \bar{B}$. (e) $\bar{A} \cup \bar{B}$. □

2. (WMS, Problem 2.11.) A sample space consists of five simple events, E_1, E_2, E_3, E_4, and E_5.

(a) If $P(E_1) = P(E_2) = 0.15, P(E_3) = 0.4$, and $P(E_4) = 2P(E_5)$, find the probabilities of E_4 and E_5.
(b) If $P(E_1) = 3P(E_2) = 0.3$, find the probabilities of the remaining simple events if you know that the remaining simple events are equally probable.

Solution. (a) Since $P(\mathcal{S}) = P(E_1) + \cdots + P(E_5) = 1$, $1 = 0.15 + 0.15 + 0.40 + 3P(E_5)$. So, $P(E_5) = 0.10$ and $P(E_4) = 0.20$.
(b) Obviously, $P(E_3) + P(E_4) + P(E_5) = 0.6$. Thus, they are all equal to 0.2.

□

3. (WMS, Problem 2.18.) Suppose two balanced coins are tossed and the upper faces are observed.

(a) List the sample points for this experiment.
(b) Assign a reasonable probability to each sample point. (Are the sample points equally likely?)
(c) Let A denote the event that exactly one head is observed and B the event that at least one head is observed. List the sample points in A and B separately.
(d) From your answer to part (c), find $P(A)$, $P(B)$, $P(A \cap B)$, $P(A \cup B)$ and $P(\bar{A} \cup B)$.

Solution. (a) $\mathcal{S} = \{HH, TH, HT, TT\}$.
(b) If the two coins are balanced, each of the four outcomes are equally likely, and hence each simple event has probability $1/4 = 0.25$.
(c) $A = \{HT, TH\}$, $B = \{HT, TH, HH\}$.

(d) $P(A) = 1/2$, $P(B) = 3/4$, $P(A \cap B) = P(A) = 1/2$, $P(A \cup B) = P(B) = 3/4$, $P(\bar{A} \cup B) = P(\mathcal{S}) = 1$.

□

4. (WMS, Problem 2.5.) Let $\mathcal{S}$ denote the sample space (universe) and A, B denote two arbitrary events (sets). Use the identities $A = A \cap \mathcal{S}$ and $\mathcal{S} = B \cup \bar{B}$, and the distributive law to prove that

   (a) $A = (A \cap B) \cup (A \cap \bar{B})$.

   (b) If $B \subseteq A$ then $A = B \cup (A \cap \bar{B})$.

   (c) Further, show that $(A \cap B)$ and $(A \cap \bar{B})$ are mutually exclusive, and therefore, A is the union of two mutually exclusive sets, $(A \cap B)$ and $(A \cap \bar{B})$. [This is called partitioning of A with respect to B.]

   (d) Also show that B and $(A \cap \bar{B})$ are mutually exclusive and if $B \subseteq A$, A is the union of two mutually exclusive sets, B and $(A \cap \bar{B})$.

Solution. (a) Note that $A = A \cap \mathcal{S} = A \cap (B \cup \bar{B})$. Therefore, using the distributive law, we get $A = (A \cap B) \cup (A \cap \bar{B})$.

(b) If $B \subseteq A$, then $A \cap B = B$. Substitute $(A \cap B)$ by B in part (a) to get the result.

(c) $(A \cap B) \cap (A \cap \bar{B}) = A \cap (B \cap \bar{B}) \cap A = \emptyset$ (by associativity of intersection), since $B \cap \bar{B} = \emptyset$. Therefore, using part (a), $A = (A \cap B) \cup (A \cap \bar{B})$, with $(A \cap B)$ and $(A \cap \bar{B})$ mutually exclusive. This completes the proof.

(d) $B \cap (A \cap \bar{B}) = B \cap \bar{B} \cap A = \emptyset$. If $B \subseteq A$, then $A \cap B = B$. Substitute $(A \cap B)$ by B in part (c) to get the result.

□

5. (WMS, Problem 2.21-2.24.) Let A and B be two events.

   (a) Use the results derived in problem 4 and the Axioms from class to prove that $P(A) = P(A \cap B) + P(A \cap \bar{B})$.

   (b) Suppose $B \subseteq A$. Use the result proved in part (a) to show that $P(A) = P(B) + P(A \cap \bar{B})$.

   (c) Finally, using non-negativity of probability show that $P(B) \leq P(A)$, if $B \subseteq A$. [That is, if A contains B then the probability of A cannot be smaller than that of B.]

Solution. (a) Since A is union of two ME events $(A \cap B)$ and $(A \cap \bar{B})$, therefore, by finite additivity (second consequence of the Axioms), we get $P(A) = P((A \cap B) \cup (A \cap \bar{B})) = P(A \cap B) + P(A \cap \bar{B})$.

(b) If $B \subseteq A$, then $A \cap B = B$. Substituting $(A \cap B)$ by B in part (a), we get $P(A) = P(B) + P(A \cap \bar{B})$.

(c) By non-negativity of probability, we get $P(A \cap \bar{B}) \geq 0$. Therefore, from part (b), $P(A) = P(B) + P(A \cap \bar{B}) \geq P(B) + 0 = P(B)$.

□

6. (WMS, Problem 2.28.) Four equally qualified people apply for two identical positions in a company. One and only one applicant is a member of a minority group. The positions are filled by choosing two of the applicants at random.

   (a) List the possible outcomes for this experiment.

   (b) Assign reasonable probabilities to the sample points.

   (c) Find the probability that the applicant from the minority group is selected for a position.

   Solution. (a) Denote the four candidates as A_1, A_2, A_3, and M. Since order is not important (all we need is two people; it does not matter in which order we choose), the sample space is: $\mathcal{S} = \{A_1A_2, A_1A_3, A_1M, A_2A_3, A_2M, A_3M\}$.

   (b) Because the two applicants are chosen at random, we can assume equally likely outcomes, so that each sample point has probability $1/6$.

   (c) Let $C = \{\text{minority hired}\}$. Then $P(C) = P(A_1M) + P(A_2M) + P(A_3M) = 3/6 = 1/2$.

   □