1. Let A and B be two independent events.

   (a) Recall from Homework exercise 1.5 that $A = (A \cap B) \cup (A \cap \overline{B})$, with $(A \cap B)$ and $(A \cap \overline{B})$ mutually exclusive and $P(A) = P(A \cap B) + P(A \cap \overline{B})$. Using these results or otherwise, prove that A and $\overline{B}$ are independent. [**Note:** Because A and B are arbitrary, we can reverse the roles of A and B to prove that $\overline{A}$ and B are independent.]

   (b) Prove that $\overline{A}$ and $\overline{B}$ are independent. [**Hint:** Start with $P(\overline{A \cup B}) = 1 - P(A \cup B)$, and use additive law.] Thus, from part (a) and (b), it follows that if A, B are independent, then so are $(\overline{A}, B)$, $(A, \overline{B})$ and $(\overline{A}, \overline{B})$.

   Solution. (a) We have $P(A) = P(A \cap B) + P(A \cap \overline{B})$, which means

$$\begin{aligned} P(A \cap \overline{B}) &= P(A) - P(A \cap B) \\ &= P(A) - P(A)P(B) \quad \text{(independence)} \\ &= P(A)[1 - P(B)] = P(A)P(\overline{B}). \end{aligned}$$

   (b) As suggested in the hint,

$$\begin{aligned} P(\overline{A} \cap \overline{B}) = P(\overline{A \cup B}) &= 1 - P(A \cup B) \\ &= 1 - P(A) - P(B) + P(A \cap B) \quad \text{(additive law)} \\ &= 1 - P(A) - P(B) + P(A)P(B) \quad \text{(independence)} \\ &= [1 - P(A)][1 - P(B)] = P(\overline{A})P(\overline{B}). \end{aligned}$$

□

2. Suppose that two balls are selected at random, without replacement, from a box containing r red balls and b blue balls. Find the probability that the first ball is red and the second is blue.

   Solution. Let $A = \{\text{first ball is red}\}$ and $B = \{\text{second ball is blue}\}$. We need $P(A \cap B)$. There are at least two ways to solve the problem.

   Method 1. During the first draw, we have r red balls and b blue balls, so $(r + b)$ balls in total. Hence $P(A) = r/(r+b)$. Now if the first draw yields a red ball, we have $(r-1)$ red balls and b blue balls left in the box. Hence given that the first ball is red, probability of the second ball being blue, i.e., $P(B|A)$ is $b/(r + b - 1)$. Therefore, $P(A \cap B) = P(A)P(B|A) = \frac{r}{r+b} \cdot \frac{b}{r+b-1} = \frac{rb}{(r+b)(r+b-1)}$.

   Method 2. Notice that here we are selecting two balls one by one at random without replacement, and the order is important. So, total number of outcomes for the experiment is $P_2^{(r+b)} = (r + b)(r + b - 1)$. Now $\#(A \cap B) = r \times b$. Therefore, $P(A \cap B) = \frac{rb}{(r+b)(r+b-1)}$.

□

3. (WMS, Problem 2.94.) A smoke detector system uses two devices, A and B. If smoke is present, the probability that it will be detected by device A is 0.95; by device B, 0.90; and by both devices, 0.88.

   (a) If smoke is present, find the probability that the smoke will be detected by either device A or B or both devices.

(b) Find the probability that the smoke will be undetected.

Solution. Define the events $A = \{\text{Device A detects smoke}\}$ and $B = \{\text{Device B detects smoke}\}$. Then $P(A) = 0.95, P(B) = 0.90$ and $P(A \cap B) = 0.88$.

(a) We need $P(A \cup B)$. By additive law, $P(A \cup B) = P(A) + P(B) - P(A \cap B) = 0.95 + 0.90 - 0.88 = 0.97$.

(b) The smoke will be undetected, if none of A and B detects it. Hence we need $P(\overline{A} \cap \overline{B}) = P(\overline{A \cup B}) = 1 - P(A \cup B) = 1 - 0.97 = 0.03$, from part (a).

□

4. Three dice are rolled. If no two show the same face, what is the probability that at least one die shows an ace (i.e., a single dot)? [**Hint:** Consider the complementary event of getting no ace.]

Solution. Let $A = \{\text{The three dice show different faces}\}$ and $B = \{\text{At least one die shows an ace}\}$. We need $P(B|A)$. We'll find $P(\overline{B}|A)$ first. Note that $\overline{B} = \{\text{None of the three dice shows an ace}\}$, so that $(\overline{B} \cap A) = \{\text{The three dice show different faces } \textit{and} \text{ none of them shows an ace}\}$. Now,

$$P(A) = \frac{P_3^6}{6^3} = \frac{6 \times 5 \times 4}{6^3}, \text{ and } P(\overline{B} \cap A) = \frac{P_3^5}{6^3} = \frac{5 \times 4 \times 3}{6^3},$$

since $\#(A) = P_3^6$, as we are considering all possible arrangements of 3 different faces from 6 possible faces (without replacement), and $\#(\overline{B} \cap A) = P_3^5$, as we aren't considering the aces anymore in $(\overline{B} \cap A)$. Therefore,

$$P(\overline{B}|A) = \frac{P(\overline{B} \cap A)}{P(A)} = \frac{5 \times 4 \times 3}{6 \times 5 \times 4} = \frac{3}{6} = \frac{1}{2},$$

which means $P(B|A) = 1 - P(\overline{B}|A) = 1/2$. □

5. (WMS, Problem 2.99.) Suppose that A and B are independent events such that the probability that neither occurs is a and the probability of B is b. Show that $P(A) = \frac{1-b-a}{1-b}$.

Solution. Given that $P(\overline{A} \cap \overline{B}) = P(\overline{A \cup B}) = a$ and $P(B) = b$, and that A and B are independent. Thus $P(A \cup B) = 1 - P(\overline{A \cup B}) = 1 - a$ and $P(A \cap B) = P(A)P(B) = bP(A)$ (independence). Now, from additive law, $P(A \cup B) = P(A) + P(B) - P(A \cap B) = P(A) + b - bP(A)$. Thus,

$$1 - a = P(A) + b - bP(A) = P(A)(1 - b) + b$$
$$\implies P(A) = \frac{1 - a - b}{1 - b}.$$

□

6. (*Boole's* inequality for two events.) Let A and B be two (arbitrary) events. Prove that $P(A \cup B) \leq P(A) + P(B)$.

Solution. From additive law, $P(A \cup B) = P(A) + P(B) - P(A \cap B) \leq P(A) + P(B)$, since $P(A \cap B) \geq 0$. □

7. (WMS, Problem 2.111.) An advertising agency notices that approximately 1 in 50 potential buyers of a product sees a given magazine ad, and 1 in 5 sees a corresponding ad on television. One in 100 sees both. One in 3 actually purchases the product after seeing the ad, 1 in 10 without seeing it. What is the probability that a randomly selected potential customer will purchase the product? [**Hint:** Define the events: $A = \{\text{buyer sees the magazine ad}\}$, $B = \{\text{buyer sees the TV ad}\}$ and $C = \{\text{buyer purchases the product}\}$.]

Solution. We're given that $P(A) = 1/50 = 0.02, P(B) = 1/5 = 0.20$, and $P(A \cap B) = 1/100 = 0.01$. Therefore, $P(\{\text{Buyer sees the add}\}) = P(A \cup B) = P(A) + P(B) - P(A \cap B) = 0.02 + 0.20 - 0.01 = 0.21$. This means, $P(\{\text{Buyer sees no add}\}) = P(\overline{A} \cap \overline{B}) = P(\overline{A \cup B}) = 1 - P(A \cup B) = 0.79$. Now, we're also given that $P(C|A \cup B) = 1/3$ and $P(C|\overline{A \cup B}) = 1/10 = 0.10$. Hence, using the law of total probability,

$$\begin{aligned} P(C) &= P(C|A \cup B)P(A \cup B) + P(C|\overline{A \cup B})P(\overline{A \cup B}) \\ &= \frac{1}{3} \times 0.21 + 0.10 \times 0.79 = 0.070 + 0.079 = 0.149. \end{aligned}$$

□

8. A patient is tested for *aplastic anemia*, a medical condition that afflicts 1% of the population. The test is 95% accurate. Suppose the test result for the patient is positive, i.e., the test claims that the patient does have the disease. What is the probability that the patient is indeed affected? [**Hint:** Define the two events $D = \{\text{Patient is affected}\}$ and $T = \{\text{Test result is positive}\}$. Here *accuracy* means that both $P(T|D) = 0.95$ and $P(\overline{T}|\overline{D}) = 0.95$. The quantity $P(T|D)$ is known as the *sensitivity* or true positive rate of the test, and $P(\overline{T}|\overline{D})$ is known as the *specificity* or true negative rate. Observe that D and $\overline{D}$ are mutually exclusive and exhaustive, and we want $P(D|T)$.]

Solution. Observe that $P(D) = 0.01, P(\overline{D}) = 1 - P(D) = 0.99$ and $P(T|\overline{D}) = 1 - P(\overline{T}|\overline{D}) = 0.05$. Using Bayes' rule (and the law of total probability), we get

$$\begin{aligned} P(D|T) = \frac{P(T|D)P(D)}{P(T)} &= \frac{P(T|D)P(D)}{P(T|D)P(D) + P(T|\overline{D})P(\overline{D})} \\ &= \frac{0.95 \times 0.01}{0.95 \times 0.01 + 0.05 \times 0.99} \approx 0.16. \end{aligned}$$

Thus, there is only a 16% chance that the patient has the disease, given that (s)he tested positive, even though the test seems to be quite reliable! □

9. Two urns, urn 1 and urn 2, contain, respectively, two white and one black balls, and one white and five black balls. One ball is transferred from urn 1 to urn 2 and then one ball is drawn from the latter. Suppose the drawn ball happens to be white. What is the probability that the transferred ball was black? [**Hint:** Define the two events $A = \{\text{Drawn ball is white}\}$ and $B = \{\text{Transferred ball is black}\}$ and note that $P(A|B) = 1/7$ (why?), $P(A|\overline{B}) = 2/7$ (why?). We need $P(B|A)$.]

Solution. Note that $P(A|B) = P(\{\text{one white ball is drawn ball } \textit{given} \text{ that a black ball is transferred to the urn}\}) = P(\{\text{one white ball is drawn from an urn with 1 white and 6 black balls}\})$ $= 1/7$. Similarly, $P(A|\overline{B}) = P(\{$one white ball is drawn from an urn with 2 white and 5 black balls}) = 2/7. Also, $P(B)$ =({one black ball is taken from an urn with 1 black and 2 white balls}) = 1/3, which means $P(\overline{B}) = 1 - 1/3 = 2/3$. Therefore, by Bayes' theorem,

$$P(B|A) = \frac{P(A|B)P(B)}{P(A|B)P(B) + P(A|\overline{B})P(\overline{B})} = \frac{\frac{1}{7} \cdot \frac{1}{3}}{\frac{1}{7} \cdot \frac{1}{3} + \frac{2}{7} \cdot \frac{2}{3}} = \frac{1}{5}.$$

□

10. A box contains three coins with a head on each side, four coins with a tail on each side, and two fair coins. One of these nine coins is selected at random and tossed once.

   (a) What is the probability that a head will be obtained?

   (b) Suppose that the toss does result in a head. What is the probability that the coin used for tossing was fair (i.e., not with a head on each side)?

Solution. Define the events A = {a head is obtained}, B_1 = {a coin with heads on both sides is selected}, B_2 = {a coin with tails on both sides is selected}, and B_3 = {a fair coin is selected}. Then B_1, B_2 and B_3 are mutually exclusive and exhaustive, with $P(B_1) = 3/9$. $P(B_2) = 4/9$, $P(B_3) = 2/9$, $P(A|B_1) = 1$, $P(A|B_2) = 0$ and $P(A|B_3) = 1/2$.

   (a) We need $P(A)$. By the theorem of total probability,

$$P(A) = P(A|B_1)P(B_1) + P(A|B_2)P(B_2) + P(A|B_3)P(B_3) = 1 \times \frac{3}{9} + 0 \times \frac{4}{9} + \frac{1}{2} \times \frac{2}{9} = \frac{4}{9}$$

   (b) It is given that the toss resulted in a head, i.e., A has happened. We need $P(B_3|A)$. Using Bayes theorem,

$$P(B_3|A) = \frac{P(A|B_3)P(B_3)}{P(A)} = \frac{\frac{1}{2} \times \frac{2}{9}}{\frac{4}{9}} = \frac{1}{4}.$$

□

11. (WMS, Problem 2.134) Two methods, A and B, are available for teaching a certain industrial skill. The failure rate is 20% for A and 10% for B. However, B is more expensive and hence is used only 30% of the time. (A is used the other 70%.) A worker was taught the skill by one of the methods but failed to learn it correctly. What is the probability that she was taught by method A?

Solution. Define F = {Worker fails to learn}, A = {Method A used} and B = {Method B used}. Then, $P(F|A) = 0.2$, $P(F|B) = 0.1$, $P(A) = 0.7$, $P(B) = 0.3$. Of course, A and B are ME and exhaustive. Therefore, by Bayes' rule,

$$P(A|F) = \frac{P(F|A)P(A)}{P(F|A)P(A) + P(F|B)P(B)} = \frac{14}{17}.$$

□

12. (WMS, Problem 2.132) Using the law of total probability or otherwise, prove the following:

   (a) If $P(A|B) = P(A|\overline{B})$, then A and B are independent. [**Hint:** $P(A) = P(A|B)P(B) + P(A|\overline{B})P(\overline{B})$].

(b) If $P(A|C) > P(B|C)$ and $P(A|\overline{C}) > P(B|\overline{C})$, then $P(A) > P(B)$.

Solution. (a) Using the theorem of total probability,

$$\begin{aligned} P(A) &= P(A|B)P(B) + P(A|\overline{B})P(\overline{B}) \\ &= P(A|B)P(B) + P(A|B)P(\overline{B}) \\ &= P(A|B)[P(B) + P(\overline{B})] = P(A|B). \end{aligned}$$

Multiplying both sides by $P(B)$ we get $P(A)P(B) = P(A \cap B)$, which means A and B are independent.

(b) Again using theorem of total probability,

$$\begin{aligned} P(A) &= P(A|C)P(C) + P(A|\overline{C})P(\overline{C}) \\ &> P(B|C)P(C) + P(B|\overline{C})P(\overline{C}) \\ &= P(B). \end{aligned}$$

□

13. (WMS, Problem 3.7.) Each of three balls are randomly placed into one of three bowls. Find the PMF and DF of X = the number of empty bowls. [**Hint:** Observe that here $\mathscr{X} = \{0, 1, 2\}$. Now, $p_X(1) = P(X = 1)$ is a bit tricky to calculate, but recall that $\sum_{x \in \mathscr{X}} p_X(x) = 1$.].

Solution. We'll first solve the problem without directly calculating $P(X = 1)$. The first ball can go to any of the three bowls, in 3 ways. Since a bowl can contain more than one ball, (otherwise there won't be any empty bowls), for each such way, the second ball can go to any of the three, again in 3 ways. Similar for the third ball. Thus, there are $3^3 = 27$ ways to place the three balls into the three bowls. Now

$$\begin{aligned} p_X(0) = P(X = 0) &= P(\{\text{No bowl is empty}\}) \\ &= P(\{\text{Three balls go to three different bowls}\}) = \frac{3!}{27} = \frac{6}{27}, \\ \text{and } p_X(2) = P(X = 2) &= P(\{\text{Two bowls are empty}\}) \\ &= P(\{\text{All three balls go to the same bowl}\}) = \frac{3}{27}. \end{aligned}$$

Hence, $p_X(1) = 1 - p_X(0) - p_X(2) = 1 - \frac{9}{27} = \frac{18}{27}$. Therefore, the PMF of X is given by:

$$p_X(x) = \begin{cases} \frac{6}{27}, & x = 0 \\ \frac{18}{27}, & x = 1 \\ \frac{3}{27}, & x = 2 \\ 0, & \text{otherwise} \end{cases}$$

Hence, the DF of X is given by:

$$F_X(b) = \begin{cases} 0, & b < 0 \\ \frac{6}{27}, & 0 \le b < 1 \\ \frac{24}{27}, & 1 \le b < 2 \\ 1, & b \ge 2 \end{cases}$$

Calculating $P(X = 1)$ directly:

Label the three bowls as $1, 2, 3$. First, we need to choose one bowl from three, which will remain empty. This can be done in $\binom{3}{1} = 3$ ways. Say, bowl 1 remains empty. Then all the three balls must be placed into bowl 2 and bowl 3 and none of these two bowls can be empty. That means, one of bowl 2 and bowl 3 will get two balls and the other a single ball. There are 2 ways to choose a bowl (from 2 bowls) that'll get two balls, and for each such way, we can place the balls in $\binom{3}{2}\binom{1}{1}$ ways. Thus, total number of ways of having only bowl 1 empty is $2 \times \binom{3}{2}\binom{1}{1} = 2 \times 3 = 6$. Therefore, total number of ways of having (any) one bowl empty is $3 \times 6 = 18$. Hence, required probability $= 18/27$. □

14. Suppose that a random variable X has a discrete distribution with the following PMF:

$$p_X(x) = \begin{cases} cx & \text{for } x = 0, \ldots, 5 \\ 0 & \text{otherwise.} \end{cases}$$

Determine the value of the constant c and calculate $E(X)$.

Solution. Here $\mathscr{X} = \{0, 1, \cdots, 5\}$. Since any PMF $p_X(\cdot)$ must satisfy $\sum_{x \in \mathscr{X}} p_X(x) = 1$, therefore,

$$\sum_{x=0}^{5} (cx) = c \sum_{x=0}^{5} x = c \sum_{x=1}^{5} x = c \frac{5(5+1)}{2} = 15c \implies c = \frac{1}{15}.$$

Therefore, the PMF is given by

$$p_X(x) = \begin{cases} \frac{x}{15} & \text{for } x = 0, \ldots, 5 \\ 0 & \text{otherwise.} \end{cases}$$

Therefore,

$$E(X) = \sum_{x \in \mathscr{X}} x p_X(x) = \sum_{x=0}^{5} x \cdot \frac{x}{15} = \frac{1}{15} \sum_{x=0}^{5} x^2 = \frac{1}{15} \sum_{x=1}^{5} x^2 = \frac{1}{15} \frac{5(5+1)(2 \cdot 5 + 1)}{6} = \frac{11}{3}.$$

□

15. (WMS, Problem 3.8.) A single cell can either die, with probability 0.1, or split into two cells, with probability 0.9, producing a new generation of cells. Each cell in the new generation dies or splits into two cells independently with the same probabilities as the initial cell. Find the probability distribution for the number of cells in the next generation. (Note that the number of cells cannot be odd.) What is the expected number of cells in the next generation?

Solution. Let X be the number of cells in the next (third) generation. Observe that the support of X is $\mathscr{X} = \{0, 2, 4\}$. Now,

$$\begin{aligned} p_X(0) = P(X = 0) &= P(\{\text{No cell in the third generation}\}) \\ &= P(\{\text{The cell in the first generation dies}\} \\ &\qquad \cup \{\text{The first cell splits and the second two cells die}\}) \\ &= 0.1 + 0.9 \times 0.1 \times 0.1 = 0.109 \end{aligned}$$

$$
\begin{aligned}
p_X(4) &= P(X=4) = P(\{\text{Four cells in the third generation}\}) \\
&= P(\{\text{The first cell splits and both created cells split}\}) \\
&= 0.9 \times 0.9 \times 0.9 = 0.729, \\
\text{and, } p_X(2) &= P(X=2) = 1 - p_X(0) - p_X(4) = 0.162.
\end{aligned}
$$

Therefore the PMF of X is given by:

$$
p_X(x) = \begin{cases} 0.109 & \text{for } x = 0 \\ 0.162 & \text{for } x = 2 \\ 0.729 & \text{for } x = 4 \\ 0 & \text{otherwise.} \end{cases}
$$

Expected number of cells in the next generation $E(X) = 0 \times 0.109 + 2 \times 0.162 + 4 \times 0.729 = 3.24$. □