1. Suppose X is a RV with $E(X) = 2$ and $V(X) = 4$. Find $E(X+2)^2$.

Solution. By the formula, $V(X) = E(X^2) - E^2(X) \implies E(X^2) = V(X) + E^2(X)$. Therefore, in the current setting, $E(X^2) = V(X) + E^2(X) = 4 + 4 = 8$. Therefore,

$$E(X+2)^2 = E(X^2 + 4X + 4) = E(X^2) + 4E(X) + 4 = 8 + 4 \times 2 + 4 = \boxed{20.}$$

□

2. (WMS, Problem 4.28.) The proportion of time per day that all checkout counters in a supermarket are busy is a RV Y with PDF

$$f(y) = \begin{cases} cy^2(1-y)^4, & 0 \le y \le 1 \\ 0, & \text{elsewhere.} \end{cases}$$

(a) Find the value of c that makes $f(y)$ a probability density function.

(b) Find E(Y).

Solution. (a) Observe that

$$\begin{aligned}
\int_{-\infty}^{\infty} f(x)\,dx &= c\int_0^1 y^2(1-y)^4\,dy \\
&= c\int_1^0 (1-z)^2 z^4 (-1)\,dz \qquad \text{(substitute } z = 1-y\text{)} \\
&= c\int_0^1 (1-z)^2 z^4\,dz \qquad \left(\int_a^b g(x)\,dx = -\int_b^a g(x)\,dx\right) \\
&= c\int_0^1 (1-2z+z^2)z^4\,dz \\
&= c\int_0^1 (z^4 - 2z^5 + z^6)\,dz \\
&= c\left(\frac{z^5}{5}\Big|_0^1 - \frac{2z^6}{6}\Big|_0^1 + \frac{z^7}{7}\Big|_0^1\right) = c\left(\frac{1}{5} - \frac{1}{3} + \frac{1}{7}\right) = \frac{c}{105}.
\end{aligned}$$

Hence, from $\int_{-\infty}^{\infty} f(x)\,dx = 1$, we get $\frac{c}{105} = 1 \implies \boxed{c = 105.}$

(b) Note that

$$\begin{aligned}
E(Y) = \int_{-\infty}^{\infty} yf(y)\,dy &= 105\int_0^1 y\,y^2(1-y)^4\,dy \\
&= 105\int_0^1 (1-z)^3 z^4\,dz \qquad \text{(substitute } z = 1-y\text{)} \\
&= 105\int_0^1 (1-3z+3z^2-z^3)z^4\,dz \\
&= 105\int_0^1 (z^4 - 3z^5 + 3z^6 - z^7)\,dz
\end{aligned}$$

$$= 105 \left(\frac{z^5}{5}\Big|_0^1 - \frac{3z^6}{6}\Big|_0^1 + \frac{3z^7}{7}\Big|_0^1 - \frac{z^8}{8}\Big|_0^1 \right)$$

$$= 105 \left(\frac{1}{5} - \frac{1}{2} + \frac{3}{7} - \frac{1}{8} \right) = \frac{105}{280} = \boxed{\frac{3}{8}.}$$

□

3. An absolutely continuous RV X with PDF f is called symmetric (about 0), if for every $x \in \mathbb{R}$, $f(x) = f(-x)$.

   (a) Assuming it exists, show that $E(X) = 0$.

   (b) If F denotes the DF of X, show that $F(0) = \frac{1}{2}$. Hence find the median of X. (You can assume F to be strictly increasing on a neighborhood of 0.)

   [**Hint:** Use the properties of odd and even functions.]

   Solution. (a) **Method 1** (Direct proof): Note that

$$\begin{aligned}
E(X) &= \int_{-\infty}^{\infty} x f(x)\, dx \\
&= \int_{\infty}^{-\infty} (-y) f(-y)(-1)\, dy && \text{substitue } y = -x \implies dy = -dx \\
&= \int_{-\infty}^{\infty} (-y) f(-y)\, dy && \left(\int_a^b g(x)\, dx = -\int_b^a g(x)\, dx \right) \\
&= \int_{-\infty}^{\infty} (-y) f(y)\, dy && \text{since } f(y) = f(-y) \text{ for all } y \\
&= -\int_{-\infty}^{\infty} y f(y)\, dy = -E(X)
\end{aligned}$$

   Hence, $2E(X) = 0 \implies E(X) = 0$.

   Method 2 (Using property of odd functions): By definition, $E(X) = \int_{-\infty}^{\infty} x f(x)\, dx$. Note that the integrand $g(x) = x f(x)$ is odd, since $g(-x) = -x f(-x) = -x f(x) = -g(x)$. Hence, $E(X)$, an (definite) integral of an odd function over the symmetric interval $(-\infty, \infty)$, is zero.

   (b) We have, $\int_{-\infty}^{\infty} f(y)\, dy = 1$ (integral of a PDF over the entire real line). Note that the integrand $f(y)$ is an even function since $f(y) = f(-y)$. Therefore,

$$1 = \int_{-\infty}^{\infty} f(y)\, dy = 2 \int_0^{\infty} f(y)\, dy = 2P(Y > 0) \implies P(Y > 0) = \frac{1}{2}.$$

   Thus, $F(0) = P(Y \le 0) = 1 - P(Y > 0) = 1 - \frac{1}{2} = \frac{1}{2}$.

   To find the median $\phi_{1/2}$, first note that F is strictly increasing on a neighborhood of 0. So, F^{-1} exists as a function on that neighborhood. Therefore, $\phi_{1/2}$ is obtained by solving

$$F(\phi_{1/2}) = \frac{1}{2} \iff \phi_{1/2} = F^{-1}\left(\frac{1}{2}\right) = F^{-1}(F(0)) = 0.$$

□

4. (WMS, Problem 4.30.) The proportion of time Y that an industrial robot is in operation during a 40-hour week is a random variable with probability density function

$$f(y) = \begin{cases} 2y, & 0 \le y \le 1 \\ 0, & \text{elsewhere.} \end{cases}$$

(a) Find $E(Y)$ and $V(Y)$.

(b) For the robot under study, the profit X for a week is given by $X = 200Y - 60$. Find $E(X)$ and $V(X)$.

(c) Find an interval in which the profit should lie for at least 75% of the weeks that the robot is in use.

Solution. (a) We have

$$E(Y) = \int_{-\infty}^{\infty} y f(y)\, dy = \int_0^1 y\, 2y\, dy = \left.\frac{2y^3}{3}\right|_0^1 = \boxed{\frac{2}{3}}$$

and

$$E(Y^2) = \int_{-\infty}^{\infty} y^2 f(y)\, dy = \int_0^1 y^2\, 2y\, dy = \left.\frac{2y^4}{4}\right|_0^1 = \frac{1}{2}$$

Therefore, $V(Y) = E(Y^2) - E^2(Y) = \frac{1}{2} - \frac{4}{9} = \boxed{\frac{1}{18}}.$

(b) By the formula, $E(X) = E(200Y - 60) = 200E(Y) - 6 = 200(2/3) - 60 = \boxed{220/3}$ and $V(X) = V(200Y - 60) = 200^2 V(Y) = \boxed{20000/9.}$

(c) Recall from Chebyshev's theorem, the two standard deviation ($k = 2$) interval about mean has probability greater than or equal to $1 - \frac{1}{k^2} = 1 - \frac{1}{4} = \frac{3}{4} = 75\%$. Hence, required interval in the current setting $= [220/3 - 2\sqrt{20000/9}, 220/3 + 2\sqrt{20000/9}] = \boxed{[-20.9476, 167.6142].}$

□

5. (WMS, Problem 4.47.) The failure of a circuit board interrupts work that utilizes a computing system until a new board is delivered. The delivery time, Y, is uniformly distributed on the interval one to five days. The cost of a board failure and interruption includes the fixed cost c_0 of a new board and a cost that increases proportionally to Y^2. If C is the cost incurred, $C = c_0 + c_1 Y^2$. (c_1 is a constant.)

(a) Find the probability that the delivery time exceeds two days.

(b) In terms of c_0 and c_1, find the expected cost associated with a single failed circuit board.

Solution. (a) By assumption, $Y \sim U(\theta_1 = 1, \theta_2 = 5)$. Let $F_Y(\cdot)$ denote the DF of Y. Therefore, required probability

$$P(Y > 2) = 1 - P(Y \le 2) = 1 - F_Y(2) = 1 - \frac{2 - \theta_1}{\theta_2 - \theta_1} = 1 - \frac{1}{4} = \frac{3}{4} = \boxed{0.75.}$$

(b) Recall, if $Y \sim U(\theta_1, \theta_2)$ then $E(Y) = \frac{\theta_1 + \theta_2}{2} = 3$ and $V(Y) = \frac{(\theta_2 - \theta_1)^2}{12} = \frac{4}{3}$. Therefore, $E(Y^2) = V(Y) + E^2(Y) = \frac{4}{3} + 9 = \frac{31}{3}$. (See solution to problem 1). Hence,

$$E(C) = E(c_0 + c_1 Y^2) = c_0 + c_1 E(Y^2) = \boxed{c_0 + c_1 \frac{31}{3}.}$$

□

6. Recall that the MGF of a RV X is defined by $M_X(t) := E(e^{tX})$. Hence, for continuous X, $M_X(t) = \int_{-\infty}^{\infty} e^{tx} f(x)\,dx$, f being the PDF of X. From this definition, find the MGF of X when $X \sim \text{Exp}(\beta)$. Hence (by differentiating) find $E(X)$ and $E(X^2)$. Then find $V(X)$.

Solution. By definition,

$$\begin{aligned} M_X(t) = \int_{-\infty}^{\infty} e^{tx} f(x)\,dx &= \int_0^{\infty} e^{tx} \frac{1}{\beta} e^{-\frac{x}{\beta}}\,dx \\ &= \frac{1}{\beta} \int_0^{\infty} e^{-(\frac{1}{\beta}-t)x}\,dx \\ &= \frac{1}{\beta} \int_0^{\infty} e^{-\left(\frac{1-\beta t}{\beta}\right)x}\,dx \\ &= \frac{1}{\beta} \left[-\frac{e^{-\left(\frac{1-\beta t}{\beta}\right)x}}{\left(\frac{1-\beta t}{\beta}\right)} \right]_0^{\infty} = \frac{1}{1-\beta t}, \end{aligned}$$

provided $\frac{1}{\beta} - t > 0 \iff t < \frac{1}{\beta}$. Note that

$$\begin{aligned} M'_X(t) &= \frac{d}{dt} M_X(t) = \frac{-1}{(1-\beta t)^2}(-\beta) = \frac{\beta}{(1-\beta t)^2} \\ M''_X(t) &= \frac{d^2}{dt^2} M_X(t) = \frac{d}{dt} M'_X(t) = \beta \frac{-2}{(1-\beta t)^3}(-\beta) = \frac{2\beta^2}{(1-\beta t)^3}. \end{aligned}$$

Therefore, $E(X) = M'_X(t)|_{t=0} = \beta$ and $E(X^2) = M''_X(t)|_{t=0} = 2\beta^2$. Hence

$$V(X) = E(X^2) - E^2(X) = 2\beta^2 - \beta^2 = \beta^2.$$

□

7. (WMS, Problem 4.97 -4.98.) A manufacturing plant uses a specific bulk product. The amount of product used in one day can be modeled by an exponential distribution with $\beta = 4$ (measurements in tons).

   (a) Find the probability that the plant will use more than 4 tons on a given day.

   (b) How much of the bulk product should be stocked so that the plants chance of running out of the product is only 0.05?

Solution. Let X denote the amount (in tons) of product used in one day. Then by assumption $X \sim \text{Exp}(\beta = 4)$. So, the DF of X is $F_X(x) = 1 - e^{-x/\beta} = 1 - e^{-x/4}$ for $x \geq 0$.

   (a) Required probability $= P(X > 4) = 1 - P(X \leq 4) = 1 - F_X(4) = 1 - (1 - e^{-4/4}) = e^{-1} = \boxed{0.3679.}$

   (b) We need x (in tons) such that $P(X > x) = 0.05$. Now $P(X > x) = 1 - F_X(x) = e^{-x/4}$. Therefore, $e^{-x/4} = 0.05 \implies -x/4 = \log(0.05) \implies x = -4\log(0.05) = \boxed{11.9829 \text{ tons.}}$

□

8. (WMS, Problem 4.91.) Let Y have an exponential distribution with $P(Y > 2) = .0821$. Find $E(Y)$ and $P(Y \leq 1.7)$.

Solution. Let $Y \sim \text{Exp}(\beta)$. We need to find β. Recall Y has DF $F_Y(y) = 1 - e^{-y/\beta}$ for $y \geq 0$. Therefore,

$$P(Y > 2) = 1 - F_Y(2) = e^{-2/\beta} = 0.0821 \implies -\frac{2}{\beta} = \log(0.0821) \implies \beta = \frac{2}{-\log(0.0821)} \approx 0.8.$$

Hence, $\boxed{E(Y) = \beta = 0.8}$ and $P(Y \leq 1.7) = 1 - e^{-1.7/0.8} \approx \boxed{0.8806.}$ □