1. Let $X \sim \text{Gamma}(\alpha, \beta)$ with density

$$f(x) = \frac{1}{\beta^\alpha\,\Gamma(\alpha)} x^{\alpha-1} e^{-x/\beta}, \quad x \geq 0.$$

(a) Prove that X has MGF $M_X(t) = 1/(1-\beta t)^\alpha$ provided $t < \frac{1}{\beta}$.

Solution. By definition, the MGF is given by:

$$\begin{aligned}
M_X(t) = E(e^{tX}) &= \int_{-\infty}^{\infty} e^{tx} f(x)\,dx \\
&= \int_0^{\infty} e^{tx} \frac{1}{\beta^\alpha \Gamma(\alpha)} e^{-x/\beta} x^{\alpha-1}\,dx \\
&= \frac{1}{\beta^\alpha \Gamma(\alpha)} \int_0^{\infty} e^{-x\left(\frac{1}{\beta}-t\right)} x^{\alpha-1}\,dx \\
&= \frac{1}{\beta^\alpha \Gamma(\alpha)} \int_0^{\infty} e^{-x\left(\frac{1}{\beta}-t\right)} x^{\alpha-1}\,dx \\
&= \frac{1}{\beta^\alpha \Gamma(\alpha)} \int_0^{\infty} e^{-x/\left(\frac{\beta}{1-\beta t}\right)} x^{\alpha-1}\,dx \\
&= \frac{1}{\beta^\alpha \Gamma(\alpha)} \left(\frac{\beta}{1-\beta t}\right)^\alpha \Gamma(\alpha) \int_0^{\infty} \underbrace{\frac{1}{\left(\frac{\beta}{1-\beta t}\right)^\alpha \Gamma(\alpha)} e^{-x/\left(\frac{\beta}{1-\beta t}\right)} x^{\alpha-1}}_{\text{PDF of Gamma}\left(\alpha, \frac{\beta}{1-\beta t}\right)}\,dx = \frac{1}{(1-\beta t)^\alpha},
\end{aligned}$$

provided $\frac{1}{\beta} - t > 0 \iff t < \frac{1}{\beta}$. □

(b) Using the above MGF, find $E(X)$ and $E(X^2)$. Then find $V(X)$.

Solution. We have $M_X(t) = 1/(1-\beta t)^\alpha = (1-\beta t)^{-\alpha}$. Therefore,

$$\begin{aligned}
M_X'(t) &= \frac{d}{dt} M_X(t) = -\alpha(1-\beta t)^{-\alpha-1}(-\beta) = \alpha\beta(1-\beta t)^{-(1+\alpha)} \\
M_X''(t) &= \frac{d^2}{dt^2} M_X(t) = \frac{d}{dt} M_X'(t) = \alpha\beta\,\{-(1+\alpha)\}(1-\beta t)^{-(1+\alpha)-1}(-\beta) \\
&= \alpha(1+\alpha)\beta^2(1-\beta t)^{-(2+\alpha)}.
\end{aligned}$$

Therefore, $E(X) = M_X'(t)|_{t=0} = \alpha\beta$ and $E(X^2) = M_X''(t)|_{t=0} = \alpha(1+\alpha)\beta^2$. Hence

$$V(X) = E(X^2) - E^2(X) = \alpha(1+\alpha)\beta^2 - \alpha^2\beta^2 = \alpha\beta^2.$$

□

2. Let $X \sim \chi^2_\nu$ (chi-squared distribution with ν degrees of freedom).

(a) Write down the density of X, using the definition of chi-squared distribution as a Gamma distribution with appropriately chosen parameters α and β.

Solution. Recall that $\chi^2_\nu \equiv \text{Gamma}(\alpha = \nu/2, \beta = 2)$. Therefore, the PDF of X is given by:

$$f(x) = \frac{1}{2^{\nu/2}\Gamma(\nu/2)}\, e^{-x/2}\, x^{\frac{\nu}{2}-1};\ x \geq 0.$$

□

(b) Find $E(X)$ and $V(X)$.

Solution. Using the formulas, $E(X) = \alpha\beta = \frac{\nu}{2} \cdot 2 = \nu$ and $V(X) = \alpha\beta^2 = \frac{\nu}{2} \cdot 2^2 = 2\nu$. $\square$

3. Let X be a gamma random variable with mean If $E(X) = 20$ and $V(X) = 100$.

(a) Find α and β, the two parameters in a Gamma distribution.

Solution. Let $X \sim \text{Gamma}(\alpha, \beta)$. Then $E(X) = \alpha\beta = 20$ and $V(X) = \alpha\beta^2 = 100$. Therefore,

$$\frac{V(X)}{E(X)} = \frac{100}{20} \implies \frac{\alpha\beta^2}{\alpha\beta} = \boxed{\beta = 5.}$$

Which means

$$E(X) = \alpha\beta = \alpha \times 5 = 20 \implies \boxed{\alpha = 4.}$$

$\square$

(b) Using Chebyshev's theorem, provide an interval such that the probability that X lies in this interval is atleast 99%.

Solution. Let μ_X and σ_X^2 respectively denote the mean and the variance of X. Then by Chebyshev's theorem,

$$P(\mu_X - k\sigma_X < X < \mu_X + k\sigma_X) = P(|X - \mu_X| < k\sigma_X) \geq 1 - \frac{1}{k^2}.$$

We need k such that $1 - k^{-2} = 0.99 \implies k^{-2} = 0.01 \implies k = 1/0.1 = 10$. Therefore

$$0.99 \leq P(20 - 10\sqrt{100} < X < 20 + 10\sqrt{100}) = P(-80 < X < 120) = P(0 < X < 120),$$

since X is non-negative. Therefore, required interval is $(0, 120)$. $\square$

4. Let Z follow the standard normal distribution with PDF $f(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}$ for $-\infty < z < \infty$. Verify that $f(z)$ is indeed a PDF. Observe that $f(z) > 0$ for any $z \in \mathbb{R}$ is automatic. So all you have to prove is $\int_{-\infty}^{\infty} f(z)\, dz = 1$. [**Hint:** Note that $\int_0^\infty e^{-\frac{z^2}{2}}\, dz = \int_0^\infty z^{-1} e^{-\frac{z^2}{2}} z\, dz$. Now use the substitution $t = z^2/2$ and modify the proof for $E(Z^2) = 1$.]

Solution. Note that

$$\begin{aligned}
\int_{-\infty}^{\infty} f(z)\, dz &= \int_{-\infty}^{\infty} \underbrace{\frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}}_{\text{even function}}\, dz \\
&= 2\int_0^\infty \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}\, dz \\
&= 2\frac{1}{\sqrt{2\pi}} \int_0^\infty z^{-1} e^{-\frac{z^2}{2}} z\, dz \\
&= \frac{2}{\sqrt{2\pi}} \int_0^\infty \left(\sqrt{2t}\right)^{-1} e^{-t}\, dt \quad (\text{substitute } t = -z^2/2) \\
&= \frac{2}{\sqrt{2\pi}} \cdot \frac{1}{\sqrt{2}} \int_0^\infty t^{-\frac{1}{2}} e^{-t}\, dt \\
&= \frac{1}{\sqrt{\pi}} \int_0^\infty t^{\frac{1}{2}-1} e^{-t}\, dt = \frac{1}{\sqrt{\pi}} \Gamma(1/2) = \frac{1}{\sqrt{\pi}} \sqrt{\pi} = 1.
\end{aligned}$$

$\square$

5. The mode of a continuous RV X is defined to be the point where f_X, the density of X, is maximum. Let $X \sim N(\mu, \sigma^2)$. Show that X has mode μ. Also find the density at the mode.

Solution. The density of X is:

$$f_X(x) = \frac{1}{(\sqrt{2\pi})\,\sigma}\, e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2},\ x \in \mathbb{R}$$

There are two ways to solve this problem: using algebra, and using calculus.

Method 1 (using algebra). Note that, for all $x \in \mathbb{R}$

$$\begin{aligned} \left(\frac{x-\mu}{\sigma}\right)^2 &\geq 0 = \left(\frac{\mu-\mu}{\sigma}\right)^2 \\ \implies -\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2 &\leq -\frac{1}{2}\left(\frac{\mu-\mu}{\sigma}\right)^2 \\ \implies e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} &\leq e^{-\frac{1}{2}\left(\frac{\mu-\mu}{\sigma}\right)^2} \quad (e^x \text{ is increasing in } x) \\ \implies f_X(x) = \frac{1}{(\sqrt{2\pi})\,\sigma} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} &\leq \frac{1}{(\sqrt{2\pi})\,\sigma} e^{-\frac{1}{2}\left(\frac{\mu-\mu}{\sigma}\right)^2} = f_X(\mu) \end{aligned}$$

Thus $f_X(x) \leq f_X(\mu)$ for all $x \in \mathbb{R}$, which means $f_X(x)$ is maximum when $x = \mu$. Hence μ is the mode of X.

Method 2 (using calculus). We need to maximize $f_X(x)$ with respect to x. Note that

$$f'_X(x) = \frac{d}{dx} f_X(x) = \frac{1}{(\sqrt{2\pi})\,\sigma} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}\left(-\frac{1}{2\sigma^2}\, 2(x-\mu)(-1)\right) = \frac{1}{(\sqrt{2\pi})\,\sigma} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}\left(\frac{x-\mu}{\sigma^2}\right).$$

Therefore,

$$f'_X(x) = 0 \implies \frac{1}{(\sqrt{2\pi})\,\sigma}\ \underbrace{e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}}_{>0 \text{ for all } x \in \mathbb{R}}\ \left(\frac{x-\mu}{\sigma^2}\right) = 0 \implies x = \mu.$$

Again,

$$f''_X(x) = \frac{d}{dx} f'_X(x) = \frac{1}{(\sqrt{2\pi})\,\sigma}\, e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}\left[\left(\frac{x-\mu}{\sigma^2}\right)^2 - \frac{1}{\sigma^2}\right]$$

which means

$$f''_X(x)\Big|_{x=\mu} = -\frac{1}{(\sqrt{2\pi})\,\sigma^3} < 0.$$

This proves that $f_X(x)$ is maximum when $x = \mu$. Therefore, the mode of X is μ.

The density at the mode is

$$f_X(\mu) = \frac{1}{(\sqrt{2\pi})\,\sigma}$$

□

6. (WMS, Problem 4.73.) The width of bolts of fabric is normally distributed with mean 950 mm (millimeters) and standard deviation 10 mm. For the following two problems, leave the final answers in terms of the standard normal CDF Φ.

(a) What is the probability that a randomly chosen bolt has a width between 947 and 958 mm?

Solution. Let X denote the width of a randomly selected bolt. Then, by assumption, $X \sim N(\mu = 950, \sigma^2 = 10^2)$. Therefore, required probability

$$\begin{aligned} P(947 \le X \le 958) &= P\left(\frac{947 - 950}{10} \le \frac{X - 950}{10} \le \frac{958 - 950}{10}\right) \\ &= P(-0.3 \le Z \le 0.8) \quad \text{where } Z \sim N(0,1) \\ &= \Phi(0.8) - \Phi(-0.3) \end{aligned}$$

□

(b) What is the appropriate value for C such that a randomly chosen bolt has a width less than C with probability 0.8531?

Solution. We need C such that $P(X \le C) = 0.8531$. Now,

$$P(X \le C) = P\left(\frac{X - 950}{10} \le \frac{C - 950}{10}\right) = \Phi\left(\frac{C - 950}{10}\right)$$

since $Z = \frac{X-950}{10} \sim N(0,1)$. Therefore,

$$\Phi\left(\frac{C - 950}{10}\right) = 0.8531 \implies \frac{C - 950}{10} = \Phi^{-1}(0.8531) \implies \boxed{C = 950 + 10\,\Phi^{-1}(0.8531).}$$

□

7. Let $X \sim N(\mu, \sigma^2)$. Using the values $\Phi(1) = 0.841, \Phi(2) = 0.977$ and $\Phi(3) = 0.998$, calculate the probabilities $P(|X - \mu| \le \sigma), P(|X - \mu| \le 2\sigma)$ and $P(|X - \mu| \le 3\sigma)$. [**Note:** This is almost like the empirical law, except for the fact that here the population quantities μ and σ^2 (which are typically unknown in a practical problem) instead of their sample analogues are used.]

Solution. Note that

$$\begin{aligned} P(|X - \mu| \le \sigma) &= P\left(\left|\frac{X - \mu}{\sigma}\right| \le 1\right) \\ &= P(|Z| \le 1) \quad \text{where } Z \sim N(0,1) \\ &= P(-1 \le Z \le 1) \\ &= \Phi(1) - \Phi(-1) \\ &= \Phi(1) - (1 - \Phi(1)) = 2\Phi(1) - 1 = 0.682 \end{aligned}$$

Similarly, $P(|X - \mu| \le 2\sigma) = 2\Phi(2) - 1 = 0.954$ and $P(|X - \mu| \le 3\sigma) = 2\Phi(3) - 1 = 0.996$. □

8. (WMS, Problem 4.80) Assume that Y is normally distributed with mean μ and standard deviation σ. After observing a value of Y, a mathematician constructs a rectangle with length $L = |Y|$ and width $W = 3|Y|$. Let A denote the area of the resulting rectangle. What is $E(A)$?

Solution. We have $A = L \times W$ (area = length × width). Therefore,

$$E(A) = E(|Y| \times 3|Y|) = E(3Y^2) = 3E(Y^2) = 3[V(Y) + E^2(Y)] = 3(\sigma^2 + \mu^2).$$

□