STA 4321/5325 **Homework 7** **March 27, 2017**

1. Suppose that X and Y have a discrete joint distribution for which the joint PMF is defined as follows:
$$p(x,y) = \begin{cases} c|x+y|, & x=-1,0,1 \text{ and } y=-1,0,1 \\ 0, & \text{otherwise.} \end{cases}$$
Determine:
   (a) the value of the constant c. [**Answer:** $c = 1/8$.]
   (b) $P(X = 0, Y = -1)$.
   (c) $P(X = 1)$.
   (d) $P(|X - Y| < 1)$.

2. (WMS, Problem 5.14, 5.32.) Suppose that the random variables Y_1 and Y_2 have joint probability density function $f(y_1, y_2)$ given by
$$f(y_1,y_2) = \begin{cases} 6y_1^2 y_2, & 0 \le y_1 \le y_2,\ y_1 + y_2 \le 2 \\ 0, & \text{elsewhere.} \end{cases}$$
   (a) Verify that this is a valid joint density function.
   (b) What is the probability that $Y_1 + Y_2$ is less than 1? [**Answer:** 1/32.]
   (c) Show that the marginal density of Y_1 is a beta density with $\alpha = 3$ and $\beta = 2$.
   (d) Derive the conditional density of Y_2 given $Y_1 = y_1$. [**Answer:** $f_{Y_2|Y_1}(y_2|y_1) = \frac{1}{2}y_2/(1-y_1)$, $0 \le y_1 \le y_2 \le 2 - y_1$.]
   (e) Find $P(Y_2 < 1.1|Y_1 = 0.60)$.

3. A fair die is rolled, and then a coin with probability p of Heads is flipped as many times as the die roll says, e.g., if the result of the die roll is a 3, then the coin is flipped 3 times. Let X be the result of the die roll and Y be the number of times the coin lands Heads.
   (a) Find the conditional PMF of Y given $X = x$. [**Answer:** $p_{Y|X}(y|x) = \binom{x}{y} p^y (1-p)^{x-y}, y = 0, \cdots, x;\ x = 1, 2, \cdots, 6$.]
   (b) Find the joint PMF of X and Y. Are they independent? [**Hint:** Note that $p_{X,Y}(x, y) = p_{Y|X}(y|x)\, p_X(x)$.]
   (c) Find the marginal PMFs of X and Y. [**Answer:** $p_X(x) = 1/6$, $x = 1, \cdots, 6$ and $p_Y(y) = \frac{1}{6} p^y \sum_{x=y}^{6} \binom{x}{y} (1-p)^{x-y}$, $y = 0, 1, \cdots, 6$.]

4. (WMS, Problem 5.34.) If Y_1 is uniformly distributed on the interval $(0, 1)$ and, for $0 < y_1 < 1$,
$$f(y_2|y_1) = \begin{cases} 1/y_1, & 0 \le y_2 \le y_1, \\ 0, & \text{elsewhere,} \end{cases}$$
   (a) what is the "name" of the conditional distribution of Y_2 given $Y_1 = y_1$?
   (b) find the joint density function of Y_1 and Y_2. [**Answer:** $f_{Y_1,Y_2}(y_1, y_2) = 1/y_1, 0 \le y_2 \le y_1 \le 1$.]
   (c) find the marginal density function for Y_2. [**Answer:** $f_{Y_2}(y_2) = -\log(y_2), 0 \le y_2 \le 1$.]

5. (WMS, Problem 5.62.) Suppose that the probability that a head appears when a coin is tossed is p and the probability that a tail occurs is $q = 1 - p$. Person A tosses the coin until the first head appears and stops. Person B does likewise. The results obtained by persons A and B are assumed to be independent. What is the probability that A and B stop on exactly the same number toss? [**Answer:** $p^2/[1-(1-p)^2]$.]