1. Suppose that X and Y have a discrete joint distribution for which the joint PMF is defined as follows:

$$f(x,y) = \begin{cases} c|x+y|, & x=-1,0,1 \text{ and } y=-1,0,1 \\ 0, & \text{otherwise.} \end{cases}$$

Determine:

(a) the value of the constant c.

Solution. The joint PMF is tabulated as follows.

X \ Y	-1	0	1	$P_X(x)$
-1	2c	c	0	3c
0	c	0	c	2c
1	0	c	2c	3c
$P_Y(y)$	3c	2c	3c	8c

Therefore, from $\sum_x \sum_y p(x,y) = 1$, we get $8c = 1 \implies c = 1/8$. □

(b) $P(X = 0, Y = -1)$.

Solution. From the table, $P(X = 0, Y = -1) = c = 1/8$. □

(c) $P(X = 1)$.

Solution. Again, from the table, $P(X = 1) = 3/8$. □

(d) $P(|X - Y| < 1)$.

Solution. Note that $P(|X-Y| < 1) = P(|X-Y| = 0) = P(X = Y) = p(-1,-1)+p(0,0)+p(1,1) = 4/8 = 1/2$. □

2. (WMS, Problem 5.14, 5.32.) Suppose that the random variables Y_1 and Y_2 have joint probability density function $f(y_1, y_2)$ given by

$$f(y_1,y_2) = \begin{cases} 6y_1^2 y_2, & 0 \le y_1 \le y_2,\ y_1 + y_2 \le 2 \\ 0, & \text{elsewhere.} \end{cases}$$

(a) Verify that this is a valid joint density function.

Solution. Note that $f(y_1, y_2)$ is non negative everywhere. So we only need to show that it integrates up to 1. We will integrate over the independent range of y_1 and the dependent range of y_2. Note that $1 \le y_1 \le y_2 \implies y_1 + y_2 \ge 2y_1$. Therefore, $0 \le 2y_1 \le y_1 + y_2 \le 2 \implies 0 \le y_1 \le 1$, and of course $y_1 \le y_2 \le 2 - y_1$. Hence,

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(y_1,y_2)\,dy_2\,dy_1 = \int_0^1 \int_{y_1}^{2-y_1} 6y_1^2 y_2\,dy_2\,dy_1$$

$$= \int_0^1 3y_1^2 \left[y_2^2\right]_{y_1}^{2-y_1} \, dy_1$$

$$= \int_0^1 3y_1^2 \, 2 \cdot 2(1-y_1) \, dy_1$$

$$= 12 \, B(3,2) = 12 \, \frac{\Gamma(3)\Gamma(2)}{\Gamma(5)} = 12 \, \frac{2! \cdot 1}{4!} = 1.$$

□

(b) What is the probability that $Y_1 + Y_2$ is less than 1?

Solution. By similar arguments as in part (a), here we have $0 \le y_1 \le 1/2$ and $y_1 \le y_2 \le 1 - y_1$. Therefore,

$$P(X + Y \le 1) = \int_0^{\frac{1}{2}} \int_{y_1}^{1-y_1} 6y_1^2 y_2 \, dy_2 \, dy_1$$

$$= \int_0^{\frac{1}{2}} 3y_1^2 \left[y_2^2\right]_{y_1}^{1-y_1} \, dy_1$$

$$= \int_0^{\frac{1}{2}} 3y_1^2 (1 - 2y_1) \, dy_1$$

$$= \int_0^{\frac{1}{2}} (3y_1^2 - 6y_1^3) \, dy_1 = \left[y^3 - 3\frac{y^4}{2}\right]_0^{\frac{1}{2}} = \frac{1}{8} - \frac{3}{32} = \boxed{\frac{1}{32}}.$$

□

(c) Show that the marginal density of Y_1 is a beta density with $\alpha = 3$ and $\beta = 2$.

Solution. The dependent range for y_2 is $y_1 \le y_2 \le 2 - y_1$. Therefore, marginal PDF of Y_1 is (see part (a)):

$$f_{Y_1}(y_1) = \int_{y_1}^{2-y_1} 6y_1^2 y_2 \, dy_2 = 12 \cdot y_1^{3-1} \, (1-y_1)^{2-1}, \; 0 \le y_1 \le 1,$$

which is the density of Beta($\alpha = 3$, $\beta = 2$). □

(d) Derive the conditional density of Y_2 given $Y_1 = y_1$.

Solution. By definition, the conditional PDF of Y_2 given $Y_1 = y_1$ is

$$f_{Y_2|Y_1}(y_2|y_1) = \frac{f(y_1, y_2)}{f_{Y_1}(y_1)} = \frac{1}{2} \frac{y_2}{(1-y_1)}, \; 0 \le y_1 \le y_2 \le 2 - y_1.$$

□

(e) Find $P(Y_2 < 1.1 | Y_1 = 0.60)$.

Solution. From part (d),

$$f_{Y_2|Y_1}(y_2|0.60) = \frac{1}{2} \frac{y_2}{(1-0.60)} = (1.25) \, y_2, \; 0.60 \le y_2 \le 1.40.$$

Therefore, $P(Y_2 < 1.1 | Y_1 = 0.60) = \int_{0.60}^{1.10} (1.25) y_2 \, dy_2 = 1.25 \times 0.5 [y_2^2]_{0.60}^{1.10} \approx 0.531.$ □

3. A fair die is rolled, and then a coin with probability p of Heads is flipped as many times as the die roll says, e.g., if the result of the die roll is a 3, then the coin is flipped 3 times. Let X be the result of the die roll and Y be the number of times the coin lands Heads.

   (a) Find the conditional PMF of Y given $X = x$.

   Solution. Note that the (marginal) PMF of X is $p_X(x) = 1/6$ for $x = 1, \cdots, 6$. Now, given that the result of the die roll is x, Y is the number of heads in a series of x tosses. So, conditional on $X = x$, $Y \sim \text{Bin}(n = x, p)$. Hence, the conditional PMF of Y given $X = x$ is given by

$$p_{Y|X}(y|x) = \binom{x}{y} p^y (1-p)^{x-y}, \; y = 0, \cdots, x, \; x = 1, 2, \cdots, 6.$$

□

   (b) Find the joint PMF of X and Y. Are they independent? [**Hint:** Note that $p_{X,Y}(x,y) = p_{Y|X}(y|x)\, p_X(x)$.] The joint PMF of X and Y is:

$$p_{X,Y}(x,y) = p_{Y|X}(y|x)\, p_X(x) = \frac{1}{6}\binom{x}{y} p^y (1-p)^{x-y}, \; y = 0, \cdots, x, \; x = 1, 2, \cdots, 6.$$

   X and Y are not independent since the conditional distribution of Y given $X = x$ depends on x.

   (c) Find the marginal PMFs of X and Y.

   Solution. The marginal PMF of X is $p_X(x) = 1/6$ for $x = 1, \cdots, 6$. To find the marginal PMF of Y, we first need to find the dependent range of x. Note that $y = 0, \cdots, x \implies x \geq y$ and of course $x \leq 6$. Since x is an integer, therefore, the dependent range of x is: $x = y, y+1, \cdots, 6$ for $y = 0, 1 \cdots, 6$ (result of a die roll). Therefore, the marginal PMF of Y is given by:

$$p_Y(y) = \sum_{x=y}^{6} p_{X,Y}(x,y) = \frac{1}{6}\, p^y \sum_{x=y}^{6} \binom{x}{y} (1-p)^{x-y}, \; x = y, y+1, \cdots, 6, \; y = 0, 1 \cdots, 6.$$

□

4. (WMS, Problem 5.34.) If Y_1 is uniformly distributed on the interval $(0,1)$ and, for $0 < y_1 < 1$,

$$f(y_2|y_1) = \begin{cases} 1/y_1, & 0 \leq y_2 \leq y_1, \\ 0, & \text{elsewhere,} \end{cases}$$

   (a) what is the "name" of the conditional distribution of Y_2 given $Y_1 = y_1$?

   Solution. Uniform distribution with $\theta_1 = 0, \theta_2 = 1$, i.e., $U(0, y)$. □

   (b) find the joint density function of Y_1 and Y_2.

   Solution. The joint PDF of X and Y is:

$$f_{Y_1,Y_2}(y_1, y_2) = f_{Y_2|Y_1}(y_2|y_1)\, f_{Y_1}(y_1) = \frac{1}{y_1}, 0 \leq y_2 \leq y_1, 0 \leq y_1 \leq 1.$$

□

(c) find the marginal density function for Y_2.

Solution. Note that the dependent range for y_1 is $y_2 \leq y_1 \leq 1$ for $0 \leq y_2 \leq 1$. Therefore, the marginal PDF of Y_2 is:

$$f_{Y_2}(y_2) = \int_{y_2}^{1} \frac{dy_1}{y_1} = [\log y_1]_{y_2}^{1} = -\log y_2,\ 0 \leq y_2 \leq 1.$$

□

5. (WMS, Problem 5.62.) Suppose that the probability that a head appears when a coin is tossed is p and the probability that a tail occurs is $q = 1 - p$. Person A tosses the coin until the first head appears and stops. Person B does likewise. The results obtained by persons A and B are assumed to be independent. What is the probability that A and B stop on exactly the same number toss?

   Solution. Let X and Y respectively denote the number of trials required to get the first head by A and B. Then, $X \sim \text{Geo}(p)$ and $Y \sim \text{Geo}(p)$ independently. Therefore, required probability is:

$$\begin{aligned} P(X = Y) &= \sum_{k=1}^{\infty} P(X = Y = k) \\ &= \sum_{k=1}^{\infty} P(X = k)P(Y = k) \\ &= \sum_{k=1}^{\infty} p^2\,(q^2)^{k-1} = \frac{p^2}{1 - q^2}. \end{aligned}$$

□