1. First, a result: For any two RVs X and Y,

$$E^2(XY) \leq E(X^2)E(Y^2) \quad \text{(Cauchy-Schwarz inequality)}.$$

Using the above inequality, prove that $-1 \leq \rho_{X,Y} \leq 1$.

2. Let Y denote the number of heads obtained in a sequence of n tosses of a coin with probability of a head p. Note that Y can be represented as $Y = \sum_{i=1}^{n} X_i$, where for $i = 1, \cdots, n$,

$$X_i = \begin{cases} 1, & \text{if the } i\text{-th toss results in a head} \\ 0, & \text{otherwise.} \end{cases}$$

Using the above representation, show that $E(Y) = np$ and $V(Y) = npq$, where $q = 1 - p$.

3. Let $T_1 \sim \text{Exp}(\beta_1)$ and $T_2 \sim \text{Exp}(\beta_2)$ be independent RVs.

   (a) Find the joint density of T_1 and T_2.

   (b) Show that $P(T_1 \leq T_2) = \beta_2/(\beta_1 + \beta_2)$.

   (c) Let $X = T_1 - 2T_2$. Find $E(X)$ and $V(X)$. [**Answer:** $E(X) = \beta_1 - 2\beta_2$, $V(X) = \beta_1^2 + 4\beta_2^2$.]

4. (WMS, Problem 5.100.) Let Z be a standard normal random variable and let $Y_1 = Z$ and $Y_2 = Z^2$.

   (a) What are $E(Y_1)$ and $E(Y_2)$?

   (b) Find $\text{Cov}(Y_1, Y_2)$.

5. (WMS, Problem 5.92.) Let Y_1 and Y_2 be RVs with joint PDF

$$f(y_1, y_2) = \begin{cases} 6(1 - y_2), & 0 \leq y_1 \leq y_2 \leq 1 \\ 0, & \text{elsewhere.} \end{cases}$$

Find $\text{Cov}(Y_1, Y_2)$. Are Y_1 and Y_2 independent? [Answer: $\text{Cov}(Y_1, Y_2) = 1/40$.]

6. (WMS, Problem 5.139.) Suppose that a company has determined that the the number of jobs per week, N, varies from week to week and has a Poisson distribution with mean λ. The number of hours to complete each job, Y_i, is Gamma distributed with parameters α and β. The total time to complete all jobs in a week is $T = \sum_{i=1}^{N} Y_i$. Note that T is the sum of a random number of random variables. What is

   (a) $E(T|N = n)$? [**Answer:** $n\alpha\beta$.]

   (b) $E(T)$, the expected total time to complete all jobs? [**Answer:** $\alpha\beta\lambda$.]

7. (WMS, Problem 5.141.) Let Y_1 have an exponential distribution with mean λ and the conditional density of Y_2 given $Y_1 = y_1$ be

$$f(y_2|y_1) = \begin{cases} 1/y_1, & 0 \leq y_2 \leq y_1 \\ 0, & \text{elsewhere.} \end{cases}$$

Find $E(Y_2)$ and $V(Y_2)$, the unconditional mean and variance of Y_2. [**Answer:** $E(Y_2) = \lambda/2, V(Y_2) = 5\lambda^2/12$.]

8. Suppose that X and Y are RVs such that $V(X) = 9$, $V(Y) = 4$, and $\rho_{X,Y} = 1/6$. Determine (a) $V(X+Y)$ and (b) $V(X - 3Y + 4)$.

9. Show that if $E(X|Y)$ is constant for all values of Y, then X and Y are uncorrelated. [**Hint:** Note that $E(Xg(Y)|Y = y) = g(y)E(X|Y = y)$ for any function g and any real number y.]