1. First, a result: For any two RVs X and Y,

$$E^2(XY) \le E(X^2)E(Y^2) \quad \text{(Cauchy-Schwarz inequality)}.$$

Using the above inequality, prove that $-1 \le \rho_{X,Y} \le 1$.

Solution. Define $U = X - E(X)$ and $V = Y - E(Y)$. Then $E(U^2) = E(X - E(X))^2 = V(X)$, $E(V^2) = E(Y - E(Y))^2 = V(Y)$ and $E(UV) = E[(X - E(X))(Y - E(Y))] = \text{Cov}(X, Y)$. Therefore, by Cauchy Schwarz inequality

$$E^2(UV) \le E(U^2)E(V^2) \implies 1 \ge \frac{E^2(UV)}{E(U^2)E(V^2)} = \frac{\text{Cov}^2(X,Y)}{V(X)V(Y)} = \rho^2_{X,Y} \implies -1 \le \rho_{X,Y} \le 1.$$

□

2. Let Y denote the number of heads obtained in a sequence of n tosses of a coin with probability of a head p. Note that Y can be represented as $Y = \sum_{i=1}^n X_i$, where for $i = 1, \cdots, n$,

$$X_i = \begin{cases} 1, & \text{if the } i\text{-th toss results in a head} \\ 0, & \text{otherwise.} \end{cases}$$

Using the above representation, show that $E(Y) = np$ and $V(Y) = npq$, where $q = 1 - p$.

Solution. First, note that because the trials are independent, X_i's are also independent. Now for all $i = 1, \cdots, n$,

$$E(X_i) = 1 \times P(X_i = 1) + 0 \times P(X_i = 1) = P(X_i = 1) = 1$$
$$\text{and } E(X_i^2) = 1^2 \times P(X_i = 1) + 0^2 \times P(X_i = 1) = 1.$$

Therefore $V(X_i) = E(X_i^2) - E^2(X_i) = p - p^2 = p(1 - p)$. Also, $\text{Cov}(X_i, X_j) = 0$ for $i \ne j$, since X_i's are independent. Therefore, from $Y = \sum_{i=1}^n X_i$ we have

$$E(Y) = \sum_{i=1}^n E(X_i) = np$$
$$\text{and } V(Y) = \sum_{i=1}^n V(X_i) + 2 \sum\sum_{1\le i<j\le n} \text{Cov}(X_i, X_j) = np(1 - p).$$

□

3. Let $T_1 \sim \text{Exp}(\beta_1)$ and $T_2 \sim \text{Exp}(\beta_2)$ be independent RVs.

   (a) Find the joint density of T_1 and T_2.

   Solution. Because T_1 and T_2 are independent, therefore, their joint PDF is given by

$$f_{T_1,T_2}(t_1, t_2) = f_{T_1}(t_1)\, f_{T_1}(t_1) = \begin{cases} \frac{1}{\beta_1\beta_2}\, e^{-t_1/\beta_1}\, e^{-t_2/\beta_2}, & t_1 > 0,\ t_2 > 0 \\ 0 & \text{otherwise.} \end{cases}$$

□

(b) Show that $P(T_1 \leq T_2) = \beta_2/(\beta_1 + \beta_2)$.

Solution. We have

$$
\begin{aligned}
P(T_1 \leq T_2) &= \int_{t_1=0}^{\infty} \int_{t_2=t_1}^{\infty} \frac{1}{\beta_1 \beta_2} e^{-t_1/\beta_1} e^{-t_2/\beta_2} \, dt_2 \, dt_1 \\
&= \frac{1}{\beta_1 \beta_2} \int_{t_1=0}^{\infty} \left(\int_{t_2=t_1}^{\infty} e^{-t_2/\beta_2} \, dt_2 \right) e^{-t_1/\beta_1} \, dt_1 \\
&= \frac{1}{\beta_1 \beta_2} \int_{t_1=0}^{\infty} \left[-\beta_2 \, e^{-t_2/\beta_2} \right]_{t_1}^{\infty} e^{-t_1/\beta_1} \, dt_1 \\
&= \frac{1}{\beta_1} \int_{t_1=0}^{\infty} e^{-t_1/\beta_2} \, e^{-t_1/\beta_1} \, dt_1 \\
&= \frac{1}{\beta_1} \int_{t_1=0}^{\infty} e^{-t_1\left(\frac{1}{\beta_2}+\frac{1}{\beta_2}\right)} \, dt_1 \\
&= \frac{1}{\beta_1} \left[\frac{e^{-t_1\left(\frac{1}{\beta_2}+\frac{1}{\beta_2}\right)}}{1\Big/\left(\frac{1}{\beta_2}+\frac{1}{\beta_2}\right)} \right]_0^{\infty} = \frac{1}{\beta_1} \cdot \frac{\beta_1 \beta_2}{\beta_1 + \beta_2} = \frac{\beta_2}{\beta_1 + \beta_2}.
\end{aligned}
$$

□

(c) Let $X = T_1 - 2T_2$. Find $E(X)$ and $V(X)$.

Solution. Note that $E(X) = E(T_1) - 2E(T_2) = \beta_1 - 2\beta_2$. Since T_1 and T_2 are independent, therefore, $\text{Cov}(T_1, T_2) = 0$. Hence,

$$
V(X) = V(T_1) + (-2)^2 V(T_2) = \beta_1^2 + 4\beta_2^2.
$$

□

4. (WMS, Problem 5.100.) Let Z be a standard normal random variable and let $Y_1 = Z$ and $Y_2 = Z^2$.

(a) What are $E(Y_1)$ and $E(Y_2)$?

Solution. We have $E(Y_1) = E(Z) = 0$ and $E(Y_2) = E(Z^2) = V(Z) + E^2(Z) = 1 + 0 = 1$. □

(b) Find $\text{Cov}(Y_1, Y_2)$.

Solution. Let $f(z)$ denote the density of Z. Then,

$$
E(Z^3) = \int_{-\infty}^{\infty} \underbrace{z^3 \, f(z)}_{\text{odd function}} \, dz = 0.
$$

Hence, $E(Y_1 Y_2) = E(Z^3) = 0$, and therefore, $\text{Cov}(Y_1, Y_2) = E(Y_1 Y_2) - E(Y_1)E(Y_2) = 0 - 0 \times 1 = 0$. □

5. (WMS, Problem 5.92.) Let Y_1 and Y_2 be RVs with joint PDF

$$
f(y_1, y_2) = \begin{cases} 6(1 - y_2), & 0 \leq y_1 \leq y_2 \leq 1 \\ 0, & \text{elsewhere.} \end{cases}
$$

Find $\text{Cov}(Y_1, Y_2)$. Are Y_1 and Y_2 independent?

Solution. We have,

$$\begin{aligned}
E(Y_1Y_2) &= \int_{y_2=0}^{1}\int_{y_1=0}^{y_2} y_1y_2\,6(1-y_2)\,dy_1\,dy_2 \\
&= 6\int_{y_2=0}^{1}\left(y_2-y_2^2\right)\left(\int_{y_1=0}^{y_2} y_1\,dy_1\right)\,dy_2 \\
&= 6\int_{y_2=0}^{1}\left(y_2-y_2^2\right)\left[\frac{y_1^2}{2}\,dy_1\right]_0^{y_2}\,dy_2 \\
&= 3\int_{y_2=0}^{1}\left(y_2^3-y_2^4\right)\,dy_2 = 3\left[\frac{1}{4}-\frac{1}{5}\right] = \frac{3}{20},
\end{aligned}$$

$$\begin{aligned}
E(Y_1) &= \int_{y_2=0}^{1}\int_{y_1=0}^{y_2} y_1\,6(1-y_2)\,dy_1\,dy_2 \\
&= 6\int_{y_2=0}^{1}(1-y_2)\left(\int_{y_1=0}^{y_2} y_1\,dy_1\right)\,dy_2 \\
&= 3\int_{y_2=0}^{1}\left(y_2^2-y_2^3\right)\,dy_2 = 3\left[\frac{1}{3}-\frac{1}{4}\right] = \frac{1}{4},
\end{aligned}$$

$$\begin{aligned}
\text{and } E(Y_2) &= \int_{y_2=0}^{1}\int_{y_1=0}^{y_2} y_2\,6(1-y_2)\,dy_1\,dy_2 \\
&= 6\int_{y_2=0}^{1}\left(y_2-y_2^2\right)\left(\int_{y_1=0}^{y_2} dy_1\right)\,dy_2 \\
&= 6\int_{y_2=0}^{1}\left(y_2^2-y_2^3\right)\,dy_2 = 6\left[\frac{1}{3}-\frac{1}{4}\right] = \frac{1}{2}.
\end{aligned}$$

Therefore, $\mathrm{Cov}(Y_1,Y_2) = E(Y_1Y_2) - E(Y_1)E(Y_2) = \frac{3}{20} - \frac{1}{4}\cdot\frac{1}{2} = 1/40$. Since $\mathrm{Cov}(Y_1,Y_2) \neq 0$, therefore Y_1 and Y_2 cannot be independent. (Also follows from dependent ranges.) □

6. (WMS, Problem 5.139.) Suppose that a company has determined that the the number of jobs per week, N, varies from week to week and has a Poisson distribution with mean λ. The number of hours to complete each job, Y_i, is Gamma distributed with parameters α and β. The total time to complete all jobs in a week is $T = \sum_{i=1}^{N} Y_i$. Note that T is the sum of a random number of random variables. What is

   (a) $E(T|N=n)$?

   Solution. We have,

$$E(T|N=n) = E\left(\sum_{i=1}^{N} Y_i \,\middle|\, N=n\right) = E\left(\sum_{i=1}^{n} Y_i\right) = \sum_{i=1}^{n} E\left(Y_i\right) = \sum_{i=1}^{n} \alpha\beta = n\alpha\beta$$

□

   (b) $E(T)$, the expected total time to complete all jobs?

Solution. By the tower property,

$$E(T) = E[E(T|N)] = E(N\alpha\beta) = \alpha\beta E(N) = \alpha\beta\lambda.$$

$\square$

7. (WMS, Problem 5.141.) Let Y_1 have an exponential distribution with mean λ and the conditional density of Y_2 given $Y_1 = y_1$ be

$$f(y_2|y_1) = \begin{cases} 1/y_1, & 0 \le y_2 \le y_1 \\ 0, & \text{elsewhere.} \end{cases}$$

Find $E(Y_2)$ and $V(Y_2)$, the unconditional mean and variance of Y_2.

Solution. Note that the conditional distribution of Y_2 given $Y_1 = y_1$ is $U(0, y_1)$. Hence, $E(Y_2|Y_1) = Y_1/2$, and $V(Y_2|Y_1) = (Y_1 - 0)^2/12 = Y_1^2/12$. Also, $Y_1 \sim \text{Exp}(\lambda)$. Hence $V(Y_1) = \lambda^2$. Therefore,

$$\begin{aligned} E(Y_2) &= E[E(Y_2|Y_1)] = E(Y_1/2) = \frac{\lambda}{2}, \\ \text{and } V(Y_2) &= E[V(Y_2|Y_1)] + V[E(Y_2|Y_1)] \\ &= E(Y_1^2/12) + V(Y_1/2) \\ &= \frac{1}{12}E(Y_1^2) + \frac{1}{4}V(Y_1) \\ &= \frac{1}{12}\{V(Y_1) + E^2(Y_1)\} + \frac{1}{4}V(Y_1) \\ &= \frac{1}{12}(\lambda^2 + \lambda^2) + \frac{1}{4}\lambda^2 = \frac{2}{12}\lambda^2 + \frac{3}{12}\lambda^2 = \frac{5\lambda^2}{12}. \end{aligned}$$

$\square$

8. Suppose that X and Y are RVs such that $V(X) = 9$, $V(Y) = 4$, and $\rho_{X,Y} = 1/6$. Determine (a) $V(X + Y)$ and (b) $V(X - 3Y + 4)$.

Solution. Note that

$$\rho_{X,Y} = \frac{\text{Cov}(X,Y)}{\sqrt{V(X)V(Y)}} = \frac{1}{6} \implies \text{Cov}(X,Y) = \frac{1}{6}\sqrt{V(X)V(Y)} = \frac{1}{6} \cdot 6 = 1.$$

Therefore, (a) $V(X+Y) = V(X)+V(Y)+2\text{Cov}(X,Y) = 9+4+2 = 15$, and (b) $V(X-3Y+4) = V(X - 3Y) = V(X) + (-3)^2V(Y) + 2(-3)\text{Cov}(X,Y) = 9 + 9 \times 4 - 6 \times 1 = 39$. $\square$

9. Show that if $E(X|Y)$ is constant for all values of Y, then X and Y are uncorrelated. [**Hint:** Note that $E(Xg(Y)|Y = y) = g(y)E(X|Y = y)$ for any function g and any real number y.]

Solution. It is given that $E(X|Y) = k$, where $k \in \mathbb{R}$ is a constant. This means

$$E(XY) = E[E(XY|Y)] = E[YE(X|Y)] = E(Yk) = kE(Y)$$

and $E(X) = E[E(X|Y)] = E(k) = k$. Therefore,

$$\text{Cov}(X,Y) = E(XY) - E(X)E(Y) = kE(Y) - kE(Y) = 0$$

and hence $\text{Corr}(X,Y) = 0$. $\square$