STA 4321/5325
Introduction to Probability/
Fundamentals of Probability

Solution to Sample Exam 1

Spring 2017

Note: This exam is a sample, and intended to be of approximately the same length and style as the actual exam. However, it is NOT guaranteed to match the content or coverage of the actual exam. DO NOT use this as your primary study tool!

On my honor, I have neither given nor received unauthorized aid on this examination.

Signature: ______________________ ***Date:*** ______________

Print Name: ________________________________ ***UFID:*** ________________

Instructions:

i. This is a 50 minute exam. There are 4 problems, worth a total of 50 points.

ii. The exam consists of 5 pages. You may write on the back of pages if you need more space.

iii. Remember to show your work. Answers lacking adequate justification may not receive full credit.

iv. You can quote and use (without proving) any result proved in class or given in homework.

v. You may use one letter-sized sheet of your own notes *hand-written* on both sides and a scientific calculator. (You are not required to bring a calculator – you may leave your answers in a form from which the numerical answer could be immediately calculated.)

vi. You may not use any books, other references, or any other electronic devices during the exam.

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1. Let A and B be two (arbitrary) events.

(a) $A - B$ is defined as the collection of all sample points which are in A but not in B, i.e., $A - B = \{x \in A | x \notin B\} = A \cap \bar{B}$. Prove that if $B \subseteq A$ then $P(A - B) = P(A) - P(B)$.

Solution. By definition, $A - B = A \cap \bar{B}$, and note that $B \subseteq A$ implies $A \cap B = B$. Now from homework 1.4-5, we have

$$\begin{aligned} & P(A) = P(A \cap B) + P(A \cap \bar{B}) = P(A \cap B) + P(A - B) \\ \Longrightarrow\ & P(A - B) = P(A) - P(A \cap B) = P(A) - P(B). \end{aligned}$$

□

(b) (Bonferroni inequality.) Prove that $P(A \cap B) \geq P(A) + P(B) - 1$.

Solution. By additive law, $P(A \cup B) = P(A) + P(B) - P(A \cap B)$. Since $P(A \cup B) \leq 1$, therefore,

$$\begin{aligned} & P(A) + P(B) - P(A \cap B) = P(A \cup B) \leq 1 \\ \Longrightarrow\ & P(A \cap B) \geq P(A) + P(B) - 1. \end{aligned}$$

□

2. An elevator in an eight-story building leaves the ground floor with five passengers. If each passenger is equally likely to get off at any of the seven floors above the ground floor, and all the passengers act independently of each other, what is the probability that at least two passengers will get off on the same floor?

Solution. There are 7 possible floors on which each individual can get off, so that the total number of outcomes $= \underbrace{7 \times \cdots \times 7}_{\text{5 of them}} = 7^5$. We will assume each outcome to be equally likely.

Let $A = \{\text{at least two passengers get off on the same floor}\}$. Then $\bar{A} = \{\text{no two passengers get off on the same floor}\}$. So, in $\bar{A}$, the first passenger can get off on any of the 7 floors, the second can get off on any of the remaining 6 floors, and so on. Hence, $\#(\bar{A}) = P_5^7 = 7 \times 6 \times 5 \times 4 \times 3$. Thus, $P(\bar{A}) = P_5^7/7^5 = (6 \times 5 \times 4 \times 3)/7^4 = 360/7^4$, which means $P(A) = 1 - P(\bar{A}) = 1 - \frac{360}{7^4}$. $\square$

3. The population of Cyprus is 70% Greek and 30% Turkish. Twenty percent of the Greeks and 10% of the Turks speak English.

(a) What is the probability that a randomly chosen Cypriot speaks English?

Solution. Let $E = \{\text{chosen Cypriot speaks English}\}$, $G = \{\text{chosen Cypriot is Greek}\}$ and $T = \bar{G} = \{\text{chosen Cypriot is Turkish}\}$. Then, $P(G) = 0.7, P(T) = 0.3, P(A|G) = 0.20, P(A|T) = 0.1$. Therefore, by the theorem of total probability, $P(E) = P(E|T)P(T) + P(E|G)P(G) = 0.1 \times 0.3 + 0.2 \times 0.7 = 0.17$. $\square$

(b) You randomly meet an English-speaking Cypriot. What is the probability that this person is Turkish?

Solution. We need $P(T|E)$. By Bayes theorem,

$$P(T|E) = \frac{P(E|T)P(T)}{P(E)} = \frac{0.1 \times 0.3}{0.17} = \frac{3}{17}.$$

$\square$

4. Let X be a discrete RV with support $\mathscr{X} = \{-n, -n+1, \cdots, 0, \cdots, n-1, n\}$ for some positive integer n.

(a) Suppose that the PMF of X satisfies the symmetry property $P(X = -k) = P(X = k)$ for all integers k. Find $E(X)$.

Solution. Given $\mathscr{X} = \{-n, -n+1, \cdots, 0, \cdots, n-1, n\}$. Therefore, by definition,

$$\begin{aligned}
E(X) &= \sum_{x \in \mathscr{X}} x p_X(x) \\
&= \sum_{k=-n}^{n} kP(X = k) \\
&= \sum_{k=-n}^{-1} kP(X = k) + 0 \cdot P(X = 0) + \sum_{k=1}^{n} kP(X = k)
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{k=-n}^{-1} kP(X=-k) + \sum_{k=1}^{n} kP(X=k) \\
&= -[nP(X=n) + (n-1)P(X=n-1) + \cdots + 1P(X=1)] \\
&\quad + [1P(X=1) + \cdots + (n-1)P(X=n-1) + nP(X=n)] \\
&= 0.
\end{aligned}
$$

□

(b) Now suppose $P(X=k) = c|k|$, for $k \in \mathscr{X}$. Find the value of c.

Solution. Observe that

$$
\begin{aligned}
\sum_{x\in\mathscr{X}} p_X(x) = \sum_{k=-n}^{n} P(x=k) = \sum_{k=-n}^{n} c|k| &= 2c\sum_{k=1}^{n} k \\
= 2c\frac{n(n+1)}{2} &= cn(n+1)
\end{aligned}
$$

Hence, from $\sum_{x\in\mathscr{X}} p_X(x) = 1$ we get $c = \frac{1}{n(n+1)}$. □