STA 4321/5325 **Solution to Sample Exam 2** Spring 2017
Introduction to Probability/
Fundamentals of Probability

Note: This exam is a sample, and intended to be of approximately the same length and style as the actual exam. However, it is NOT guaranteed to match the content or coverage of the actual exam. DO NOT use this as your primary study tool!

On my honor, I have neither given nor received unauthorized aid on this examination.

Signature: ____________________ ***Date:*** ____________

Print Name: ____________________________ ***UFID:*** ______________

Instructions:

i. This is a 50 minute exam. There are 4 problems, worth a total of 50 points.

ii. The exam consists of 5 pages. You may write on the back of pages if you need more space.

iii. Remember to show your work. Answers lacking adequate justification may not receive full credit.

iv. You may quote and use (without proving) any result proved in class or given in homework.

v. You may use one letter-sized sheet of your own notes *hand-written* on both sides and a scientific calculator. (You are not required to bring a calculator – you may leave your answers in a form from which the numerical answer could be immediately calculated.)

vi. You may not use any books, other references, or any other electronic devices during the exam.

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1. Customer arrivals at a checkout counter in a department store have a Poisson distribution with an average of forty nine per hour.

(a) Find the probability that exactly 25 customers arrive in a given hour.

Solution. Let X denote the number of customers arriving in a specific hour. Then by assumption, $X \sim \text{Poi}(\lambda = 49)$. Hence, the PMF of X is given by:

$$p_X(x) = e^{-49}\frac{(49)^x}{x!},\ x = 0, 1, 2, \cdots .$$

Therefore, required probability:

$$P(X = 25) = p_X(25) = e^{-49}\frac{(49)^{25}}{25!}.$$

□

(b) If it is known that in a given hour the number of customers arriving at the counter is at most one, what is the probability that during that period of time actually no customer arrives?

Solution. Required probability:

$$\begin{aligned} P(X=0|X\le 1) &= \frac{P(\{X=0\}\cap\{X\le 1\})}{P(X\le 1)} \\ &= \frac{P(X=0)}{P(X=0)+P(X=1)} \\ &= \frac{e^{-49}\frac{(49)^0}{0!}}{e^{-49}\frac{(49)^0}{0!}+e^{-49}\frac{(49)^1}{1!}} \\ &= \frac{1}{1+49}=\frac{1}{50}. \end{aligned}$$

□

(c) Use Chebyshev's theorem to find an interval that contains the number of customers arriving in a given hour with at least 75% probability.

Solution. Since $X \sim \text{Poi}(\lambda = 49)$, therefore, $E(X) = V(X) = \lambda = 49$. From Chebyshev's theorem, for $k > 0$

$$P\left(E(X) - k\sqrt{V(X)} < X < E(X) + k\sqrt{V(X)}\right) \ge 1 - \frac{1}{k^2}.$$

We need $1 - \frac{1}{k^2} = 0.75 \implies k = 2$. Hence, required interval:

$$(E(X) - 2\sqrt{V(X)}, E(X) + 2\sqrt{V(X)}) = (49 - 2 \times 7, 49 + 2 \times 7) = (35, 63).$$

□

2. When fishing off the shores of Florida, a spotted sea trout must be between 14 and 30 inches long before it can be kept; otherwise it must be returned to the waters. In a region of the Gulf of Mexico, the lengths of the spotted sea trout that are caught, are normally distributed with a mean of 22 inches, and a standard deviation of 4 inches, i.e, $\mu = 22$ and $\sigma = 4$. Use the well known "empirical law" $[P(-1 \le Z \le 1) \approx 0.68, P(-2 \le Z \le 2) \approx 0.95$, and $P(-3 \le Z \le 3) \approx 1$, where Z is a standard normal variable] to solve the following problems.

(a) What is the probability that a fisherman catches a spotted sea trout within the legal limits?

Solution. Let X be the length of a randomly caught sea trout. Then by assumption $X \sim N(\mu = 22, \sigma^2 = 4^2)$, which means $Z = \frac{X-22}{4} \sim N(0,1)$. Therefore, required probability is:

$$P(14 \le X \le 30) = P\left(\frac{14-22}{4} \le \frac{X-22}{4} \le \frac{30-22}{4}\right) = P(-2 < Z < 2) = 0.95.$$

□

(b) Find x such that the probability that the length of a randomly caught sea trout is less than or equal to x, is 0.5.

Solution. We need x such that

$$\begin{aligned} & P(X \le x) = 0.5 \\ \implies & P\left(\frac{X-22}{4} \le \frac{x-22}{4}\right) = 0.5 = \Phi(0) \end{aligned}$$

$$\Longrightarrow \Phi\left(\frac{x-22}{4}\right) = \Phi(0)$$

$$\Longrightarrow \frac{x-22}{4} = 0 \Longrightarrow \boxed{x = 22.}$$

□

(c) Now suppose σ is unknown and can be varied. Find the maximum allowable value for σ which ensures that a randomly caught sea trout is almost certainly within the legal limits.

Solution. In this case $X \sim N(\mu = 22, \sigma^2)$, so that here $Z = \frac{X-22}{\sigma} \sim N(0,1)$. We need σ such that

$$1 \approx P(14 \le X \le 30).$$

Now,

$$\text{LHS} = 1 \approx P(-3 < Z < 3) = \Phi(3) - \Phi(-3) = \Phi(3) - [1 - \Phi(3)] = 2\Phi(3) - 1,$$

and

$$\text{RHS} = P\left(\frac{14-22}{\sigma} \le \frac{X-22}{\sigma} \le \frac{30-22}{\sigma}\right) = P\left(-\frac{8}{\sigma} < Z < \frac{8}{\sigma}\right) = 2\Phi\left(\frac{8}{\sigma}\right) - 1.$$

Therefore,

$$2\Phi(3) - 1 = 2\Phi\left(\frac{8}{\sigma}\right) - 1 \Longrightarrow \Phi(3) = \Phi\left(\frac{8}{\sigma}\right) \Longrightarrow 3 = \frac{8}{\sigma} \Longrightarrow \sigma = \frac{8}{3}.$$

Clearly, for any $\sigma \le \frac{8}{3}$, $2\Phi\left(\frac{8}{\sigma}\right) - 1 \ge 2\Phi(3) - 1 = 1$ (why?). Hence, the the maximum allowable value for σ is 8/3. □

3. A RV Y is said to have the *standard double exponential* distribution if it has PDF

$$f(y) = Ce^{-|y|}, \; -\infty < y < \infty,$$

(a) Provide an appropriate choice for the constant C that makes f a PDF.

Solution. Note that

$$1 = \int_{-\infty}^{\infty} f(y)\,dy = C\int_{-\infty}^{\infty} \underbrace{e^{-|y|}}_{\text{even function}}\,dy = 2C\underbrace{\int_0^{\infty} \underbrace{e^{-y}}_{\text{PDF of Exp(1)}}\,dy}_{=1} = 1 \Longrightarrow \boxed{C = \frac{1}{2}.}$$

□

(b) Find $E(Y)$.

Solution. We have

$$E(Y) = \frac{1}{2}\int_{-\infty}^{\infty} \underbrace{y\,e^{-|y|}}_{\text{odd function}}\,dy = 0.$$

(Also follows from symmetry and then Homework 5.3.) □

(c) Find $V(Y)$.

Solution. Note that

$$\begin{aligned} V(Y) &= E(Y^2) - \underbrace{E^2(Y)}_{=0} \\ &= E(Y^2) \\ &= \frac{1}{2}\int_{-\infty}^{\infty} \underbrace{y^2\, e^{-|y|}}_{\text{even function}}\; dy \\ &= \frac{1}{2}\, 2\int_0^{\infty} y^2\, e^{-y}\, dy \\ &= \int_0^{\infty} y^{3-1}\, e^{-y}\, dy = \Gamma(3) = (3-1)! = 2 \end{aligned}$$

□

4. (a) Let X be a continuous RV with support $\mathscr{X}$, and CDF F satisfying $F(x) + F(-x) = 1$ for alll $x \in \mathscr{X}$. What can be said about the distribution of X?

Solution. Let f denote the PDF of X. Differentiating both sides of the given condition with respect to x, we get

$$\frac{d}{dx}\{F(x) + F(-x)\} = \frac{d}{dx}\, 1 = 0 \implies f(x) - f(-x) = 0 \implies f(x) = f(-x),$$

for all x. This means that X is symmetric. □

(b) Let $X \sim U(0,1)$. Show that $Y = 1 - X$ has the same distribution as X. [**Hint:** Use MGF.]

Solution. The MGF of X is given by

$$M_X(t) = \int_0^1 e^{tx}\, dx = \left[\frac{e^{tx}}{t}\right]_0^1 = \frac{e^t - 1}{t}.$$

□

Therefore, the MGF of $Y = 1 - X$ is

$$M_Y(t) = E(e^{tY}) = E(e^{t-tX}) = e^t E(e^{-tX}) = e^t M_X(-t) = e^t\, \frac{e^{-t} - 1}{-t} = \frac{e^t - 1}{t} = M_X(t)$$

This is true for all t. Hence X and Y have the same distribution.