STA 4321/5325
Introduction to Probability/
Fundamentals of Probability

Sample Exam 3

Spring 2017

Note: This exam is a sample, and intended to be of approximately the same length and style as the actual exam. However, it is NOT guaranteed to match the content or coverage of the actual exam. DO NOT use this as your primary study tool!

On my honor, I have neither given nor received unauthorized aid on this examination.

Signature: ______________________ ***Date:*** ______________

Print Name: ________________________________ ***UFID:*** ____________________

Instructions:

i. This is a 50 minute exam. There are 4 problems, worth a total of 55 points. Maximum score is 50, and whatever point you get above 50 will be considered as extra credit, and will be added to your total score accordingly.

ii. The exam consists of 6 pages, including one formula sheet. You may write on the back of pages if you need more space. Extra papers will be provided, if necessary. Make sure you arrange all the sheets in order before you turn in your exam.

iii. Remember to show your work. Answers lacking adequate justification may not receive full credit.

iv. You may quote and use (without proving) any result proved in class or given as homework (including extra homework).

v. You may use one letter-sized sheet of your own notes *hand-written* on both sides and a scientific calculator. (You are not required to bring a calculator – you may leave your answers in a form from which the numerical answer could be immediately calculated.)

vi. You may not use any books, other references, or any other electronic devices during the exam.

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1. Suppose X and Y are jointly continuous with joint PDF

$$f_{X,Y}(x,y) = \begin{cases} 2, & 0 \le x \le 1, 0 \le y \le 1, 0 \le x+y \le 1, \\ 0, & \text{otherwise.} \end{cases}$$

(a) Provide the marginal PDF of Y at $Y = 0.7$, i.e., provide $f_Y(0.7)$.

Solution. Note that for a given $0 \le y \le 1$, the dependent range of X is obtained by

$$0 \le x \le 1 \text{ and } 0 \le x + y \le 1 \implies 0 \le x \le 1 - y.$$

(The RHS is the intersection of the two ranges $[0,1]$ and $[-y, 1-y]$ for x. Fix a $0 < y < 1$ such as $y = 0.5$ and draw the intervals for a better understanding.) Therefore, when $y = 0.7$, the dependent range of x is $0 \le x \le 1 - 0.7 = 0.3$. Thus,

$$f_{X,Y}(x, 0.7) = \begin{cases} 2, & 0 \le x \le 0.3, \\ 0, & \text{otherwise.} \end{cases}$$

Therefore, the Marginal PDF of Y at $Y = 0.7$ is given by

$$f_Y(0.7) = \int_0^{0.3} 2\, dx = 2 \times 0.3 = 0.6.$$

$\square$

(b) Provide the conditional PDF of X given $Y = 0.7$.

Solution. The conditional PDF of X given $Y = 0.7$ is given by

$$\begin{aligned} f_{X|Y}(x|0.7) &= \frac{f_{X,Y}(x, 0.7)}{f_Y(0.7)} \\ &= \begin{cases} \frac{2}{0.6}, & 0 \le x \le 0.3, \\ \frac{0}{0.6}, & \text{otherwise.} \end{cases} \\ &= \begin{cases} \frac{10}{3}, & 0 \le x \le 0.3, \\ 0, & \text{otherwise.} \end{cases} \end{aligned}$$

$\square$

(c) Find $E(X + Y)$.

Solution. From part (a), we obtained that the dependent range of X is $0 \le x \le 1 - y$. Of course, the independent range of Y is $0 \le y \le 1$. Therefore,

$$\begin{aligned} E(X+Y) &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (x+y)\, f_{X,Y}(x,y)\, dx\, dy \\ &= \int_0^1 \int_0^{1-y} (x+y)\, 2\, dx\, dy \\ &= 2\left(\int_0^1 \int_0^{1-y} x\, dx\, dy + \int_0^1 \int_0^{1-y} y\, dx\, dy \right) \\ &= 2\left(\frac{1}{2} \int_0^1 (1-y)^2\, dy + \int_0^1 y(1-y)\, dy \right) \\ &= \int_0^1 \left(1 - 2y + y^2\right)\, dy + 2 \int_0^1 \left(y - y^2\right)\, dy \\ &= 1 - 2 \times \frac{1}{2} + \frac{1}{3} + 2 \times \frac{1}{2} - 2 \times \frac{1}{3} = 1 - \frac{1}{3} = \boxed{\frac{2}{3}.} \end{aligned}$$

$\square$

2. Let Y_1 and Y_2 be independent Poisson random variables with means λ_1 and λ_2 respectively.

(a) Write down the joint PMF of Y_1 and Y_2.

Solution. The joint PMF of Y_1 and Y_2 is given by

$$P(Y_1 = y_1, Y_2 = y_2) = e^{-\lambda_1} \frac{\lambda_1^{y_1}}{y_1!}\, e^{-\lambda_2} \frac{\lambda_1^{y_2}}{y_2!} = e^{-(\lambda_1+\lambda_2)} \frac{\lambda_1^{y_1} \lambda_2^{y_2}}{y_1!\, y_2!},\ y_1 = 0, 1, \cdots,\ y_2 = 0, 1, \cdots.$$

$\square$

(b) Using the method of MGF or otherwise, show that $Y_1 + Y_2 \sim \text{Poisson}(\lambda_1 + \lambda_2)$.

Solution. For $i = 1, 2$, the MGF of Y_i is $M_{Y_i}(t) = e^{\lambda_i(e^t - 1)}$. Therefore, The MGF of $Y_1 + Y_2$ is given by

$$\begin{aligned}
M_{Y_1+Y_2}(t) &= E\left(e^{t(Y_1+Y_2)}\right) \\
&= E\left(e^{tY_1}e^{tY_2}\right) \\
&= E\left(e^{tY_1}\right) E\left(e^{tY_2}\right) && \text{(independence)} \\
&= M_{Y_1}(t)M_{Y_2}(t) \\
&= e^{\lambda_1(e^t-1)}e^{\lambda_2(e^t-1)} \\
&= e^{(\lambda_1+\lambda_2)(e^t-1)}
\end{aligned}$$

which is the MGF of $\text{Poisson}(\lambda_1 + \lambda_2)$. Hence, $Y_1 + Y_2 \sim \text{Poisson}(\lambda_1 + \lambda_2)$. □

(c) Find the conditional PMF of Y_1, given that $Y_1 + Y_2 = m$.

Solution. We assume m to be a non-negative integer, as from part (b), $Y_1 + Y_2$ has a Poisson distribution, and hence it can take only non-negative integer values with positive probability. Also, if m is a non-negative integer, then Y_1 must be an integer between 0 and m. Therefore, for $m = 0, 1, \cdots$, and $y_1 = 0, 1, \cdots, m$, the conditional PMF of Y_1 given $Y_1 + Y_2 = m$ given by

$$\begin{aligned}
p_{Y_1|Y_1+Y_2}(y_1|m) &= P(Y_1 = y_1 | Y_1 + Y_2 = m) \\
&= \frac{P\left(Y_1 = y_1, Y_1 + Y_2 = m\right)}{P(Y_1 + Y_2 = m)} \\
&= \frac{P\left(Y_1 = y_1, Y_2 = m - y_1\right)}{P(Y_1 + Y_2 = m)} \\
&= \frac{e^{-(\lambda_1+\lambda_2)} \dfrac{\lambda_1^{y_1}\lambda_2^{m-y_1}}{y_1!\,(m-y_1)!}}{e^{-(\lambda_1+\lambda_2)} \dfrac{(\lambda_1+\lambda_2)^m}{m!}} && \text{(numerator from part (a), den. from (b))} \\
&= \frac{m!}{y_1!\,(m-y_1)!} \cdot \frac{\lambda_1^{y_1}\lambda_2^{m-y_1}}{(\lambda_1+\lambda_2)^m} \\
&= \binom{m}{y_1} \left(\frac{\lambda_1}{\lambda_1+\lambda_2}\right)^{y_1} \left(\frac{\lambda_2}{\lambda_1+\lambda_2}\right)^{m-y_1} \\
&= \binom{m}{y_1} \left(\frac{\lambda_1}{\lambda_1+\lambda_2}\right)^{y_1} \left(1 - \frac{\lambda_1}{\lambda_1+\lambda_2}\right)^{m-y_1},
\end{aligned}$$

which is the PMF of $\text{Bin}\left(n = m, p = \frac{\lambda_1}{\lambda_1+\lambda_2}\right)$. □

3. (a) X is said to have Weibull distribution with parameters $\alpha > 0$ and $m > 0$, if X has PDF

$$f_X(x) = \begin{cases} \frac{m}{\alpha}\, x^{m-1}e^{-x^m/\alpha}, & x > 0 \\ 0, & \text{elsewhere.} \end{cases}$$

Find the PDF of $Y = X^m$.

Solution. Note that $x > 0 \implies x^{1/m} > 0$. Hence the support of Y is $\mathscr{Y} = (0, \infty)$. Here $y = g(x) = x^m$ is a strictly increasing function on the support of $X = \mathscr{X} = (0, \infty)$. Hence, we can find the inverse function of g: $y = x^m \implies x = y^{1/m} = h(y)$. Also, $\frac{d}{dy}h(y) = \frac{d}{dy}y^{1/m} = \frac{1}{m}y^{\frac{1}{m}-1}$. Therefore, by method of transformations, the PDF of Y is obtained as

$$f_Y(y) = f_X(h(y))\left|\frac{d}{dy}h(y)\right| = \frac{m}{\alpha}\left(y^{\frac{1}{m}}\right)^{m-1} e^{-\frac{\left(y^{\frac{1}{m}}\right)^m}{\alpha}} \frac{1}{m}y^{\frac{1}{m}-1} = \frac{1}{\alpha}e^{-y/\alpha}y^{1-\frac{1}{m}+\frac{1}{m}-1} = \frac{1}{\alpha}e^{-y/\alpha}, y \geq 0.$$

□

(b) Suppose $Y_1, \cdots, Y_n$ is a random sample from the *standard Pareto distribution* with CDF

$$F(y) = \begin{cases} 0, & y < 1, \\ 1 - \frac{1}{y} & y \geq 1. \end{cases}$$

Find the PDF of $Y_{(1)} = \min\{Y_1, \cdots, Y_n\}$.

Solution. For $y \geq 1$ the CDF of $Y_{(1)}$ is given by

$$\begin{aligned} F_{Y_{(1)}}(y) &= P\left(Y_{(1)} \leq y\right) \\ &= 1 - P\left(Y_{(1)} > y\right) \\ &= 1 - P\left(Y_1 > y, \cdots, Y_n > y\right) \\ &= 1 - P\left(Y_1 > y\right) \cdots P\left(Y_n > y\right) && \text{(independence)} \\ &= 1 - [1 - F(y)]^n = 1 - \frac{1}{y^n}. \end{aligned}$$

Therefore, the CDF of $Y_{(1)}$ is given by

$$f_{Y_{(1)}}(y) = \begin{cases} \frac{d}{dy}\left(1 - \frac{1}{y^n}\right) = \frac{n}{y^{n+1}}, & y \geq 1 \\ 0, & \text{otherwise} \end{cases}$$

□

4. (a) Suppose X is a symmetric RV. Show that if g is an odd function (i.e., if $g(-x) = -g(x)$ for all $x \in \mathbb{R}$), then $Y = g(X)$ is also symmetric.

Solution. By a result proved in class, we have

$$\begin{aligned} & X \text{ is symmetric} \\ \iff & X \text{ and } -X \text{ have the same distribution} \\ \implies & g(X) \text{ and } g(-X) \text{ have the same distribution} && (\text{“} \Longleftarrow \text{” doesn't necessarily hold}) \\ \iff & g(X) \text{ and } -g(X) \text{ have the same distribution} && (g \text{ is odd, so } g(-X) = -g(X)) \\ \iff & Y = g(X) \text{is symmetric.} \end{aligned}$$

□

(b) Suppose X is an exponentially distributed random variable with mean β. Denote the “ceiling” of X by $Y = [X]$. Thus, Y is defined in the following way: $Y = k$ if and only if $k - 1 < X \leq k$ for $k = 1, 2, \cdots$. Show that Y has a geometric distribution with $p = 1 - e^{-1/\beta}$.

Solution. Recall that the PDF of X is given by

$$f_X(x) = \begin{cases} \frac{1}{\beta} e^{-x/\beta}, & x \geq 0 \\ 0, & x < 0. \end{cases}$$

For $k = 1, 2, \cdots$,

$$\begin{aligned} P(Y = k) &= P(k-1 < X \leq k) \\ &= \int_{k-1}^{k} \frac{1}{\beta} e^{-x/\beta}\, dx = \left[e^{-x/\beta} \right]_{k-1}^{k} \\ &= e^{-(k-1)/\beta} - e^{-k/\beta} \\ &= e^{-(k-1)/\beta} - e^{-(k-1+1)/\beta} \\ &= e^{-(k-1)/\beta} - e^{-(k-1)/\beta} \cdot e^{-1/\beta} \\ &= \left(e^{-1/\beta} \right)^{k-1} \left(1 - e^{-1/\beta} \right) \end{aligned}$$

which is the PMF of Geometric($p = 1 - e^{-1/\beta}$). □

Formula Sheet

Discrete Distributions

Distribution	Probability Function	Mean	Variance	Moment-Generating Function
Binomial	$p(y) = \binom{n}{y} p^y (1-p)^{n-y}$; $y = 0, 1, \ldots, n$	np	$np(1-p)$	$[pe^t + (1-p)]^n$
Geometric	$p(y) = p(1-p)^{y-1}$; $y = 1, 2, \ldots$	$\frac{1}{p}$	$\frac{1-p}{p^2}$	$\frac{pe^t}{1-(1-p)e^t}$
Hypergeometric	$p(y) = \frac{\binom{r}{y}\binom{N-r}{n-y}}{\binom{N}{n}}$; $y = 0, 1, \ldots, n$ if $n \le r$, $y = 0, 1, \ldots, r$ if $n > r$	$\frac{nr}{N}$	$n\left(\frac{r}{N}\right)\left(\frac{N-r}{N}\right)\left(\frac{N-n}{N-1}\right)$	
Poisson	$p(y) = \frac{\lambda^y e^{-\lambda}}{y!}$; $y = 0, 1, 2, \ldots$	λ	λ	$\exp[\lambda(e^t - 1)]$
Negative binomial	$p(y) = \binom{y-1}{r-1} p^r (1-p)^{y-r}$; $y = r, r+1, \ldots$	$\frac{r}{p}$	$\frac{r(1-p)}{p^2}$	$\left[\frac{pe^t}{1-(1-p)e^t}\right]^r$

Continuous Distributions

Distribution	Probability Function	Mean	Variance	Moment-Generating Function
Uniform	$f(y) = \frac{1}{\theta_2 - \theta_1}$; $\theta_1 \le y \le \theta_2$	$\frac{\theta_1 + \theta_2}{2}$	$\frac{(\theta_2 - \theta_1)^2}{12}$	$\frac{e^{t\theta_2} - e^{t\theta_1}}{t(\theta_2 - \theta_1)}$
Normal	$f(y) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left[-\left(\frac{1}{2\sigma^2}\right)(y-\mu)^2\right]$; $-\infty < y < +\infty$	μ	σ^2	$\exp\left(\mu t + \frac{t^2\sigma^2}{2}\right)$
Exponential	$f(y) = \frac{1}{\beta} e^{-y/\beta}$; $\beta > 0$; $0 < y < \infty$	β	β^2	$(1 - \beta t)^{-1}$
Gamma	$f(y) = \left[\frac{1}{\Gamma(\alpha)\beta^\alpha}\right] y^{\alpha-1} e^{-y/\beta}$; $0 < y < \infty$	$\alpha\beta$	$\alpha\beta^2$	$(1 - \beta t)^{-\alpha}$
Chi-square	$f(y) = \frac{(y)^{(v/2)-1} e^{-y/2}}{2^{v/2}\Gamma(v/2)}$; $y^2 > 0$	v	$2v$	$(1-2t)^{-v/2}$
Beta	$f(y) = \left[\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\right] y^{\alpha-1}(1-y)^{\beta-1}$; $0 < y < 1$	$\frac{\alpha}{\alpha+\beta}$	$\frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$	does not exist in closed form