Thermodynamic Fluctuations of Cellular Cycles on CW Complexes

Michael Catanzaro
Department of Mathematics, University of Florida

The Ultimate Goal

Understand stochastic motion of higher dimensional objects on manifolds under the limits of slow driving and low temperature.

Classical Currents

We are interested in stochastic processes on CW complexes. These are motivated by Langevin dynamics on smooth manifolds, governed by the Langevin equation for M:

$$\dot{x} = u(x,t) + \xi(x,t) \tag{1}$$

where,

- $u = -\nabla f$ for a Morse function $f : M \to \mathbb{R}$, and
- ξ is a Gaussian stochastic vector field such that
  - $\langle \xi^j(x,t) \rangle = 0$
  - $\langle \xi^i(x,t), \xi^j(x,t') \rangle = \beta^{-1} g^{ij}(x) \delta(t - t')$.

A solution to this equation is a stochastic trajectory $\eta : [0,\tau] \to M$. For large τ, we may assume $\eta(0) = \eta(\tau)$, so that $\eta : S^1 \to M$, giving rise to the *average empirical current density*:

$$Q_{\tau,\beta}(u) = \frac{1}{\tau}[\eta] \in H_1(M;\mathbb{R}) \,.$$

Consider an electrical circuit, represented by a circular wire $M = S^1 \times D^2$. For a single electron, the contribution to the current is $\omega_\alpha = \frac{1}{t}N$, where $N = N_+ - N_-$.

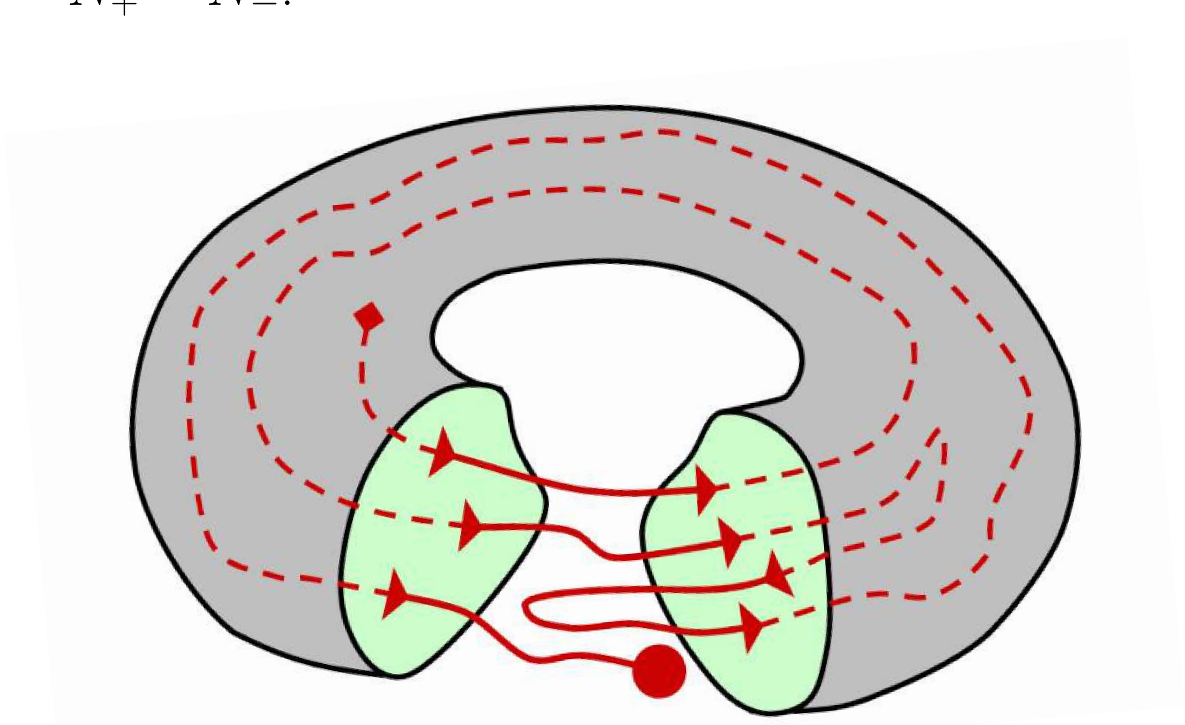

Figure 1: The stochastic motion of points, e.g., electrons, under the Langevin equation [1]. We are interested in the motion of wires or sheets of electrons, for example.

The goal of this work is to generalize the following.

Theorem [2] In the low-noise, adiabatic limit, the current quantizes:

$$\lim_{\beta\to\infty} \lim_{\tau\to\infty} Q_{\tau,\beta} \in H_1(M;\mathbb{Z}) \subset H_1(M;\mathbb{R})$$

A Discrete Version

We discretize the problem to a CW complex, or triangulation, of the manifold. Instead of points, we consider the motion of cycles of higher dimension.

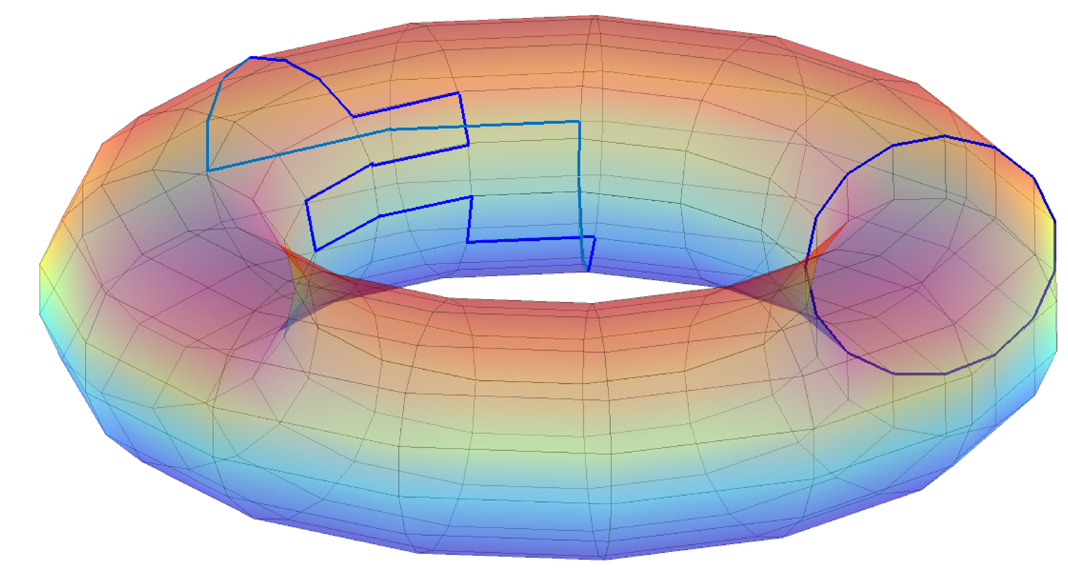

Figure 2: Stochastic motion of a circle on a triangulated torus. The initial cycle (right) evolves over time to the perturbed cycle (back left).

The current generated by such a process is governed by two pieces: the Kirchhoff solution and the Boltzmann distribution.

The Kirchhoff Problem

Fix a finite, connected CW complex X of dimension d. Equip every d-cell α with a 'resistance' by

$$\alpha \mapsto e^{\beta W_\alpha}\alpha \,.$$

A *network problem* for X consists of constructing an orthogonal splitting

$$0 \longrightarrow Z_d(X;\mathbb{R}) \xrightarrow{\ i\ } C_d(X;\mathbb{R}) \underset{K}{\xrightarrow{\ -\partial\ }} B_{d-1}(X;\mathbb{R}) \longrightarrow 0 \,.$$

This splitting is equivalent to Kirchhoff's laws.

A *spanning tree* for X is a subcomplex T such that

- $H_d(T;\mathbb{Z}) = 0$,
- $\beta_{d-1}(T) = \beta_{d-1}(X)$,
- $X^{(d-1)} \subset T$, where $X^{(k)}$ is the k-skeleton of X.

Weight the trees by

$$w_T := \theta_T^2 \prod_{\alpha \in T_d} e^{-\beta W_\alpha} \,.$$

where θ_T is the order of the torsion subgroup of $H_{d-1}(T;\mathbb{Z})$.

The Kirchhoff Theorem

Theorem. The orthogonal splitting K is given by

$$K(b) = \frac{1}{\Delta}\sum_T w_T K_b^T \,,$$

where K_b^T is the unique d-chain in T which bounds b.

The Boltzmann Distribution

Equip every $(d-1)$-cell b with its 'energy' by

$$b \mapsto e^{\beta E_b} b \,.$$

The *Hodge problem* for X is to find an explicit formula for an orthogonal splitting of the quotient map in

$$0 \longrightarrow B_{d-1}(X;\mathbb{R}) \longrightarrow Z_{d-1}(X;\mathbb{R}) \underset{\rho}{\xrightarrow{\ q\ }} H_{d-1}(X;\mathbb{R}) \longrightarrow 0,$$

A *spanning co-tree* for X is a subcomplex L such that

- $H_{d-1}(L;\mathbb{Q}) \cong H_{d-1}(X;\mathbb{Q})$,
- $\beta_{d-2}(L) = \beta_{d-2}(X)$,
- $X^{(d-2)} \subset L \subset X^{(d-1)}$.

Let

$$\phi_L : Z_{d-1}(L;\mathbb{Z}) \longrightarrow H_{d-1}(X;\mathbb{Q})$$

denote the induced inclusion map. We weight spanning co-trees by

$$\tau_L = |\operatorname{cok}\phi_L|^2 \prod_{b \in L_{d-1}} e^{\beta E_b} \,.$$

The Boltzmann Distribution

Theorem. The orthogonal splitting ρ is given by

$$\rho(x) = \frac{1}{\Lambda}\sum_L \tau_L \psi_L(x) \,.$$

where $\psi_L(x)$ is the unique cycle in L representing x.

These two pieces combine to give the main result:

Quantization in Higher Dimensions

For a finite, connected CW complex X, in the low-noise, adiabatic limit, the current satisfies:

$$\lim_{\beta\to\infty} \lim_{\tau\to\infty} Q_{\tau,\beta} \in H_d(X;\mathbb{Z}[\tfrac{1}{D}]) \subset H_d(X;\mathbb{R}) \,,$$

where

$$D = \theta_X \prod_L \mu_L \prod_T \theta_T \nu_T \,,$$

- θ_X is the order of the torsion subgroup of $H_{d-1}(X;\mathbb{Z})$,
- μ_L is the covolume of $H_{d-1}(L;\mathbb{R}) \subset H_{d-1}(X;\mathbb{R})$,
- ν_T is the covolume of $H_{d-1}(T;\mathbb{R}) \subset H_{d-1}(X;\mathbb{R})$.

References

[1] V. Y. Chernyak, M. Chertkov, S. V. Malinin, R. Teodorescu, J. of Stat. Phys. **137**, (2009) 109-147.

[2] V. Y. Chernyak, J. R. Klein, N. A. Sinitsyn, Adv. in Math. **244** (2013), 791-822.