Theorem. In the low-noise, adiabatic limit, the current quantizes:
\[ j = u(x, t) \cdot \xi(x, t) \]
where \( u = -\nabla f \) for a Morse function \( f : M \rightarrow \mathbb{R} \), and \( \xi \) is a Gaussian stochastic vector field such that:
\[ \langle \xi(x, t), \xi(x', t') \rangle = \beta^{-1} g'(x) \delta(t - t') \].

A solution to this equation is a stochastic trajectory \( \eta : [0, \tau] \rightarrow M \). For large \( \tau \), we may assume \( \eta(0) = \eta(\tau) \), so that \( \eta : S^1 \rightarrow M \), giving rise to the average empirical current density:
\[ Q_{\tau, \beta}(u) = \frac{1}{\tau} \eta T \in H_1(M; \mathbb{R}) \].

Consider an electrical circuit, represented by a circular wire \( M = S^1 \times D^2 \). For a single electron, the contribution to the current is \( \omega_a = \frac{1}{2} N \), where \( N = N_+ - N_- \).

The current generated by such a process is governed by two pieces: the Kirchhoff solution and the Boltzmann distribution.

The Kirchhoff Problem

A network problem for \( X \) consists of constructing an orthogonal splitting according to Kirchhoff’s laws.

A spanning tree for \( X \) is a subcomplex \( T \) such that:
\[ \bullet \ H_d(T; \mathbb{Z}) = 0, \]
\[ \bullet \beta_{d-1}(T) = \beta_{d-1}(X), \]
\[ \bullet X^{(d-1)} \subset T, \] where \( X^{(k)} \) is the \( k \)-skeleton of \( X \).

We discretize the problem to a CW complex, or triangulation, of the manifold. Instead of points, we consider the motion of cycles of higher dimension.

Figure 2: Stochastic motion of a circle on a triangulated torus. The initial cycle (right) evolves over time to the perturbed cycle (back left).

The Boltzmann Distribution

Equip every \((d-1)\)-cell \( b \) with its ‘energy’ by
\[ b \mapsto e^{\beta \Vert b \Vert}. \]

The Hodge problem for \( X \) is to find an explicit formula for an orthogonal splitting of the quotient map in
\[ 0 \rightarrow B_{d-1}(X; \mathbb{R}) \rightarrow Z_{d-1}(X; \mathbb{R}) \rightarrow H_{d-1}(X; \mathbb{R}) \rightarrow 0. \]

A spanning co-tree for \( X \) is a subcomplex \( L \) such that:
\[ \bullet \beta_{d-1}(L; \mathbb{Z}) = \beta_{d-1}(X), \]
\[ \bullet X^{(d-2)} \subset L \subset X^{(d-1)}. \]

Let \( \phi_L : Z_{d-1}(L; \mathbb{Z}) \rightarrow H_{d-1}(X; \mathbb{Q}) \) denote the induced inclusion map. We weight spanning co-trees by
\[ \tau_L = \cok \phi_L^2 \prod b \in L \ e^{\beta \Vert b \Vert}. \]

These two pieces combine to give the main result:

Thermodynamic Fluctuations of Cellular Cycles on CW Complexes

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The Ultimate Goal

Understand stochastic motion of higher dimensional objects on manifolds under the limits of slow driving and low temperature.

Classical Currents

We are interested in stochastic processes on CW complexes. These are motivated by Langevin dynamics on smooth manifolds, governed by the Langevin equation for \( M \):
\[ \dot{x} = u(x, t) + \xi(x, t) \]

where \( u = -\nabla f \) for a Morse function \( f : M \rightarrow \mathbb{R} \), and \( \xi \) is a Gaussian stochastic vector field such that
\[ \langle \xi(x, t), \xi(x', t') \rangle = \beta^{-1} g'(x) \delta(t - t') \].

Figure 1: The stochastic motion of points, e.g., electrons, under the Langevin equation [1]. We are interested in the motion of wires or sheets of electrons, for example. The goal of this work is to generalize the following.

Theorem. In the low-noise, adiabatic limit, the current quantizes:
\[ \lim_{\beta \rightarrow \infty} \lim_{\tau \rightarrow \infty} Q_{\tau, \beta} \in H_1(M; \mathbb{Z}) \subset H_1(M; \mathbb{R}) \]

A Discrete Version

Fix a finite, connected CW complex \( X \) of dimension \( d \). Equip every \((d-1)\)-cell \( \alpha \) with a ‘resistance’ by
\[ \alpha \mapsto e^{\beta \Vert \alpha \Vert}. \]

A network problem for \( X \) consists of constructing an orthogonal splitting.

This splitting is equivalent to Kirchhoff’s laws.

The Kirchhoff Theorem

Theorem. The orthogonal splitting \( K \) is given by
\[ K(b) = \frac{1}{2} \sum_{T} w_T K^T_b, \]
where \( K^T_b \) is the unique \( d \)-chain in \( T \) which bounds \( b \).

Quantization in Higher Dimensions

For a finite, connected CW complex \( X \), in the low-noise, adiabatic limit, the current satisfies:
\[ \lim_{\beta \rightarrow \infty} \lim_{\tau \rightarrow \infty} Q_{\tau, \beta} \in H_1(X; \mathbb{Z}) \subset H_1(X; \mathbb{R}), \]
where
\[ D = \theta_X \prod T \theta_T \nu_T, \]
\[ \theta_X \] is the order of the torsion subgroup of \( H_{d-1}(X; \mathbb{Z}) \),
\[ \mu_T \] is the covection of \( H_{d-1}(T; \mathbb{Z}) \subset H_{d-1}(X; \mathbb{R}), \]
\[ \nu_T \] is the covection of \( H_{d-1}(T; \mathbb{R}) \subset H_{d-1}(X; \mathbb{R}). \]

References