Stochastic Dynamics on CW complexes

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Outline

Motivation

A stochastic process of cycles

Spanning tree & co-trees

Quantization

Langevin dynamics

- We're interested in stochastic processes on CW complexes. These are motivated by Langevin dynamics on smooth manifolds.

Langevin dynamics

- We're interested in stochastic processes on CW complexes. These are motivated by Langevin dynamics on smooth manifolds.
- Fix a smooth, compact, Riemannian manifold (M, g), a Morse function $f : M \to \mathbb{R}$, and a Markovian, Gaussian, stochastic vector field ξ on M, depending on $\beta = \frac{1}{k_B T}$.
- A particle on M will undergo motion governed by the Langevin equation

  $$\dot{x} = u(x) + \xi(x, t)\,,$$

  where locally, $u(x) = -\nabla f(x)$.

Solution

- A solution to this equation is a stochastic trajectory or process represented by $\eta : [0, \tau] \to M$.

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- A solution to this equation is a stochastic trajectory or process represented by $\eta : [0, \tau] \to M$.
- For long times τ, we may assume the trajectory is closed $\eta : S^1 \to M$, giving rise to

$$Q_{\tau,\beta}(u) = \frac{1}{\tau}[\eta] \in H_1(M; \mathbb{R})$$

known as the *average empirical current density*.

Classical currents

- Consider an electrical circuit, represented by a circular wire ($M = S^1 \times D^2$) attached to a battery.
- The *current at* α is the number of charged particle crossings at an oriented cross-section α of the wire, per unit time.

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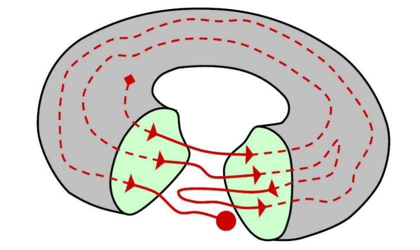
- The *current at* α is the number of charged particle crossings at an oriented cross-section α of the wire, per unit time.
- For a single electron, the contribution to the current is $Q_\alpha = \frac{1}{t}N$, where $N = N_+ - N_-$. If $\eta : S^1 \to M$ is the trajectory, then $Q = [\eta]t^{-1} \in H_1(M;\mathbb{R})$.



On manifolds

- Consider a closed $(d-1)$-cycle $\eta_0 : N \to M$. The process consists of the following two phenomena.
- Initially the cycle will evolve deterministically according to $-\nabla f$, and tend to a neighborhood of $M^{(d-1)}$.
- On longer time scales, ξ can push a segment of η_t against the gradient flow and up to a critical point of dimension d.
- The average current associated to η_0 is

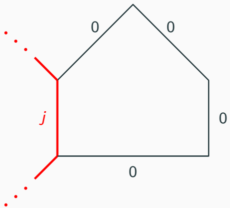
$$Q_{\tau_D,\beta}(u) = \tfrac{1}{\tau}[\eta_\tau] \in H_d(M;\mathbb{R})\,.$$

On CW complexes

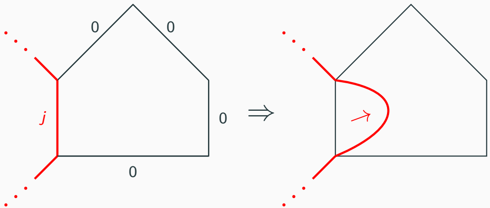
- Consider a closed $(d-1)$-cycle $\hat{x} \in Z_{d-1}(X;\mathbb{Z})$. The process consists of the following two phenomena.
- Initially the cycle will evolve deterministically according to $\mathcal{M}_X$, and evolve within a neighborhood of $X^{(d-1)}$.
- On longer time scales, ξ can push a segment of $\hat{x}$ out of the $(d-1)$-skeleton and across a cell of dimension d.
- The average current associated to $\hat{x}$ is

$$Q_{\tau_D,\beta}(\gamma) = \frac{1}{\tau}[\hat{x}_\tau] \in H_d(X;\mathbb{R})\,.$$

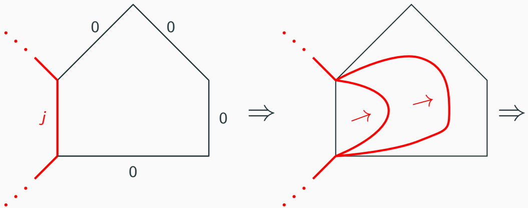
On CW complexes



On CW complexes



On CW complexes



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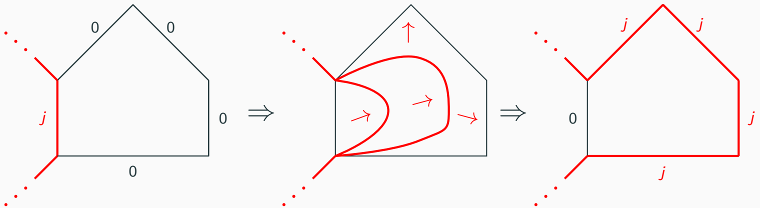
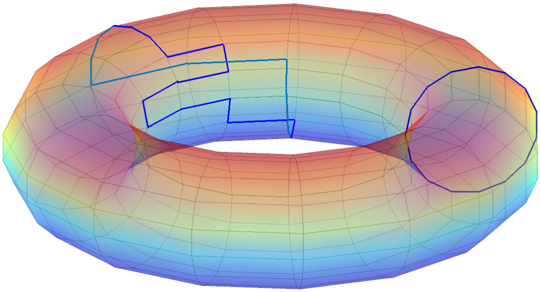


Evolution on CW complexes: the state space

Fix a $(d-1)$-cycle $\hat{x}_0 \in Z_{d-1}(X;\mathbb{Z})$:

1. The state space is $Z_{d-1}^{[\hat{x}]}(X;\mathbb{R})$, which consists of all real $(d-1)$-cycles homologous to $\hat{x}_0$.
2. A transition $z \to z'$ requires a d-cell α and a $(d-1)$-cell i such that

$$z' = z - \langle i, z\rangle \langle \partial\alpha, i, \rangle \partial\alpha \,.$$



Evolution on CW complexes

(movie)

Parameters

- The *space of parameters* is the real vector space

$$\mathcal{M}_X = \{(E, W) |\ E : X_{d-1} \to \mathbb{R}, W : X_d \to \mathbb{R}\}$$

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$$\mathcal{M}_X = \{(E, W)|\ E : X_{d-1} \to \mathbb{R}, W : X_d \to \mathbb{R}\}$$

- We're interested in periodic families of parameters.
- A *periodic driving protocol* is a smooth path

$$\gamma : \mathbb{R} \to \mathcal{M}_X$$

  such that $\gamma(0) = \gamma(\tau_D)$. Equivalently, it is a smooth Moore loop (τ_D, γ), where $\gamma : S^1 \to \mathcal{M}_X$ and τ_D is the period.

Parameters

Extend these to the chain complex:

$$e^{\beta E} : C_{d-1}(X;\mathbb{R}) \to C_{d-1}(X;\mathbb{R}) \qquad e^{\beta W} : C_d(X;\mathbb{R}) \to C_d(X;\mathbb{R})$$
$$x \mapsto e^{\beta E_x} \cdot x \qquad \alpha \mapsto e^{\beta W_\alpha} \cdot \alpha.$$

This allows us to define modified inner products on $C_d(X;\mathbb{R})$ and $C_{d-1}(X;\mathbb{R})$

$$\langle x, y \rangle_E := e^{\beta E_x} \langle x, y \rangle \qquad \langle \alpha, \gamma \rangle_W := e^{\beta W_\alpha} \langle \alpha, \gamma \rangle .$$

Define the adjoint of ∂ with respect to these modified inner products

$$\partial^*_{E,W} = e^{-\beta W} \partial^* e^{\beta E} .$$

The dynamical equation

The *dynamical operator* $\mathcal{H}(t) : C_{d-1}(X;\mathbb{R}) \to C_{d-1}(X;\mathbb{R})$ is

$$\mathcal{H}(t) := \mathcal{H}(\tau_D, \beta, \gamma)(t) = -\partial e^{-\beta W(t)} \partial^* e^{\beta E(t)}$$

Definition

Fix an initial cycle $\hat{x} \in Z_{d-1}(X;\mathbb{Z})$, a periodic driving protocol (τ_D, γ), and $\beta > 0$. The *dynamical equation for* $\hat{x}$ is

$$\frac{d\rho(t)}{dt} = \tau_D \mathcal{H}(t)\rho(t) \qquad \rho(0) = \hat{x}\,.$$

where $\rho : [0, \tau] \to C_{d-1}(X;\mathbb{R})$.

The Adiabatic Theorem

Theorem (C, Chernyak, Klein)

Let (τ_D, γ) be a periodic driving protocol and fix $\hat{x} \in Z_{d-1}(X; \mathbb{Z})$. There exists τ_0 such that for all $\tau_D > \tau_0$, a periodic solution ρ of the dynamical equation for $\hat{x}$ exists and is unique.

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The *current density* is

$$\mathbf{J}(t) := \tau_D \partial^*_{E,W} \rho(t)$$

so that the *average current density* is $Q_{\tau_D,\beta}(\gamma) := \int_0^1 \mathbf{J}(t) dt$.

Current generation on graphs

Theorem (Chernyak, Klein, Sinistyn)

For sufficiently generic γ,

$$\lim_{\tau_D \to \infty} Q_{\tau_D,\beta}(\gamma) = \int_0^1 K(\dot{\rho}^B) dt \,.$$

$$\lim_{\beta \to \infty} \lim_{\tau_D \to \infty} Q_{\tau_D,\beta}(\gamma) \in H_1(X;\mathbb{Z}) \subset H_1(X;\mathbb{R})$$

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- K gives the solution to Kirchhoff's network problem and ρ^B is the Boltzmann distribution.
- K is written as a sum over spanning trees and ρ^B as a sum over vertices.

The network problem

Definition

The network problem *for X is to construct an orthogonal splitting*

$$0 \longrightarrow Z_d(X;\mathbb{R}) \longrightarrow C_d(X;\mathbb{R}) \overset{-\partial}{\underset{K}{\rightleftarrows}} B_{d-1}(X;\mathbb{R}) \longrightarrow 0$$

with respect to the modified inner product $\langle -,-\rangle_W$.

Spanning trees

Definition

*A d-*spanning tree *for X is a subcomplex T such that*

- $H_d(T;\mathbb{Z}) = 0$,
- $\beta_{d-1}(T) = \beta_{d-1}(X)$,
- $X^{(d-1)} \subset T$.

Spanning trees

Definition

A d-spanning tree for X is a subcomplex T such that

- $H_d(T;\mathbb{Z}) = 0$,
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Let θ_T denote the order of the torsion subgroup of $H_{d-1}(T;\mathbb{Z})$ and define the *weight of T* to be the positive real number

$$w_T := \theta_T^2 \prod_{\alpha \in T_d} e^{-\beta W(\alpha)} .$$

Spanning trees

Definition

For a spanning tree T of X, define a linear transformation

$$K^T : B_{d-1}(X;\mathbb{Q}) \to C_d(T;\mathbb{Q}) \to C_d(X;\mathbb{Q}),$$

by setting $K^T(b)$ to be the unique d-chain in T so that $-\partial K^T(b) = b$.

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Theorem (C, Chernyak, Klein)

The orthogonal projection $B_{d-1}(X;\mathbb{R}) \to C_d(X;\mathbb{R})$ is given by

$$K = \frac{1}{\Delta}\sum_T w_T K^T\,,$$

where the sum is over all spanning trees, and $\Delta = \sum_T w_T$.

The Boltzmann distribution

Definition

The combinatorial Hodge problem for X *is to construct an orthogonal splitting*

$$0 \longrightarrow B_{d-1}(X;\mathbb{R}) \longrightarrow Z_{d-1}(X;\mathbb{R}) \underset{\rho^B}{\overset{}{\rightleftarrows}} H_{d-1}(X;\mathbb{R}) \longrightarrow 0\,,$$

with respect to the modified inner product $\langle -,- \rangle_E$.

This is equivalent to constructing a cycle representative that is co-closed, i.e., harmonic.

Spanning co-trees

Definition

A spanning co-tree *for X is a subcomplex L such that*

- $i_* : H_{d-1}(L;\mathbb{Q}) \cong H_{d-1}(X;\mathbb{Q})$,
- $i_* : H_{d-2}(L;\mathbb{Q}) \cong H_{d-2}(X;\mathbb{Q})$,
- $X^{(d-2)} \subset L \subset X^{(d-1)}$.

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By definition we have

$$H_{d-1}(L) \xrightarrow{\cong_{\mathbb{Q}}} H_{d-1}(X) \longrightarrow H_{d-1}(X,L) \longrightarrow H_{d-2}(L) \xrightarrow{\cong_{\mathbb{Q}}} H_{d-2}(X).$$

Define the *weight* of a spanning co-tree to be

$$\tau_L := |H_{d-1}(X,L)|^2 \prod_{b \in T_{d-1}} e^{-\beta E(b)}$$

Spanning co-trees

For a spanning co-tree L of X, define ψ_L by the following diagram

$$
\begin{array}{ccccc}
H_{d-1}(X;\mathbb{Q}) & \xrightarrow{i_*^{-1}} & H_{d-1}(L;\mathbb{Q}) & \xrightarrow{=} & Z_{d-1}(L;\mathbb{Q}) \\
 & \searrow^{\psi_L} & & & \downarrow{i_L} \\
 & & & & Z_{d-1}(X;\mathbb{Q})
\end{array}
$$

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$$H_{d-1}(X;\mathbb{Q}) \xrightarrow{i_*^{-1}} H_{d-1}(L;\mathbb{Q}) \xrightarrow{=} Z_{d-1}(L;\mathbb{Q}) \xrightarrow{i_L} Z_{d-1}(X;\mathbb{Q})$$

$$\psi_L : H_{d-1}(X;\mathbb{Q}) \to Z_{d-1}(X;\mathbb{Q})$$

Theorem (C, Chernyak, Klein)

The splitting $H_{d-1}(X;\mathbb{R}) \to Z_{d-1}(X;\mathbb{R})$ is given by

$$\rho^B = \frac{1}{\tau}\sum_L \tau_L \psi_L$$

where the sum is over all spanning co-trees, and $\tau = \sum_L \tau_L$.

Quantization

In the low temperature, adiabatic limit, we have the following:

Theorem (Chernyak, Klein, Sinitsyn)

For a connected graph X, the image of $Q : L\mathcal{M}_X \to H_1(X;\mathbb{R})$ is contained the integral lattice $H_1(X;\mathbb{Z}) \subset H_1(X;\mathbb{R})$.

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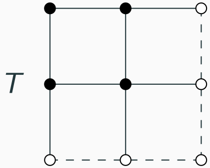
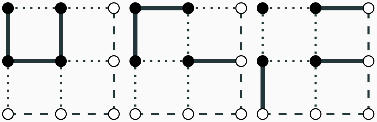
Theorem (C, Chernyak, Klein)

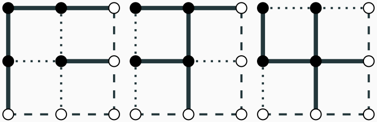
Let X be a d-dimensional connected CW complex.

1. $Q(\gamma) = \int_0^1 K(\dot{\rho}^B)dt$.
2. $Q : L\mathcal{M}_X \to H_d(X;\mathbb{R})$ is contained in $H_d(X;\mathbb{Z}[\frac{1}{D}])$, where D is determined by topological data.





(a) Three distinct spanning trees of T, out of the 32 total.



(b) Three distinct 1-spanning co-trees of T, out of the 20 total.

The Boltzmann distribution on graphs

Definition

The higher Boltzmann distribution *is the real* $(d-1)$*-cycle*

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$$\rho^B := \frac{1}{\tau}\sum_L \tau_L \psi_L \in Z_{d-1}(X;\mathbb{R})\,.$$

When X is a simple graph, the spanning co-trees are given by the vertices, and ϕ_L is an integral isomorphism so that $|H_0(X,L)| = 1$. The weight of a vertex L is then $\tau_L = e^{-\beta E_j}$ and

$$\rho^B = \frac{\sum_j e^{-\beta E_j} j}{\sum_j e^{-\beta E_j}}\,.$$