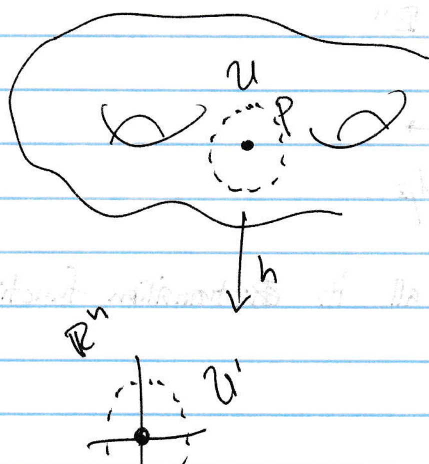
MTG 7396 Differential Topology – Vector bundles

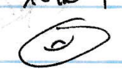
I. Differential Topology

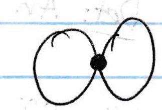
A. Smooth manifolds

Smooth manifolds and their tangent bundles provide some of the oldest and most important examples of vector bundles.

<u>Def</u>: An <u>n-dimensional topological manifold</u> M is a Hausdorff, topological space with a countable basis for its topology, and which is locally homeomorphic to $\mathbb{R}^n$. This means for every $p \in M$, $\exists$ open nbhd $\mathcal{U}$ of p & a homeomorphism $h: \mathcal{U} \to \mathcal{U}'$, where $\mathcal{U}' \subseteq \mathbb{R}^n$ is open.



<u>Ex</u>:
- any open subset of $\mathbb{R}^n$.
- $S^n = \{ x \in \mathbb{R}^n \mid |x| = 1 \}$
- $S^1 \times S^1$ 
- $M \times N$, if M, N are mflds

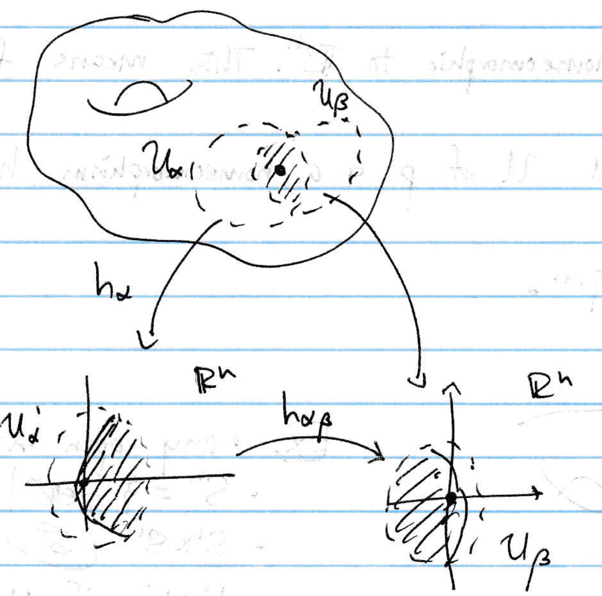
<u>Non-ex</u>: $S^1 \vee S^1$ 

<u>Def</u>: If M is a topological manifold & $h: \mathcal{U} \to \mathcal{U}'$ is a homeo. of an open subset $\mathcal{U} \subseteq M$ onto $\mathcal{U}' \subseteq \mathbb{R}^n$, then h is a <u>chart</u> of M & $\mathcal{U}$ is the <u>chart domain</u>. A collection of charts $\{h_\alpha \mid \alpha \in A\}$ with

domains U_α is called an atlas if $\bigcup_{\alpha \in A} U_\alpha = M$.

From our perspective, manifolds can be described via their transition functions:

Def: Given two charts (h_α, U_α) & (h_β, U_β) with $U_\alpha \cap U_\beta \neq \emptyset$, the transition function $h_{\alpha\beta} = h_\beta \circ h_\alpha^{-1} : h_\alpha(U_\alpha \cap U_\beta) \longrightarrow h_\beta(U_\alpha \cap U_\beta)$



* The transition functions specify how to glue the manifold together from Euclidean space.

Def: An atlas is differentiable if all its ~~fu~~ transition functions are differentiable.

By differentiable, we mean smooth, or C^∞, so that all higher derivatives exist & are continuous.

Def: A smooth structure on M is a collection of smoothly equivalent smooth atlases, where two atlases are smoothly equivalent provided their union is again a smooth atlas.

Def: A smooth manifold is a topological manifold M, together with a smooth structure.

Often, smooth ~~manifolds~~ structures are defined ~~with~~ to be a maximal differentiable atlas. ~~That is, if U is a differentiable atl~~ There is a one-to-one correspondence between smooth structures and maximal smooth atlases, so this is equivalent.

Ex: $\mathbb{RP}^n$, real projective space, $\mathbb{RP}^n = S^n / x \sim -x \quad \forall x \in S^n$.

Equip $\mathbb{RP}^n$ w/ the quotient topology. Then any $p \in \mathbb{RP}^n$ is given by

$p = [x] = [x_0, x_1, \ldots, x_n] = [-x_0, -x_1, \ldots, -x_n]$, $\sum x_i^2 = 1$.

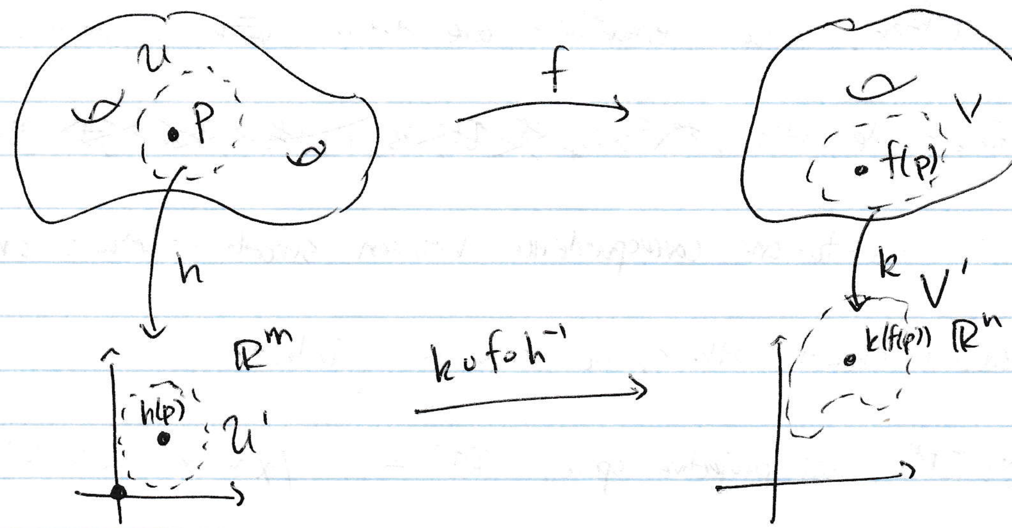
Take $U_k = \{[x_0, x_1, \ldots, x_n] : x_k \neq 0\}$; this is open in $\mathbb{RP}^n$.

Define $h_k : U_k \to \overset{\circ}{D}{}^n$, by $[x_0, \ldots, x_k] \mapsto x_k \cdot |x_k|^{-1} (x_0, x_1, \ldots, x_{k-1}, x_{k+1}, \ldots, x_n)$.

These form a differentiable atlas, since $|x_k|$ is not zero on U_k.

Ex: Any open subset of a differentiable manifold.

Def: A continuous mapping $f : M \to N$ between differentiable manifolds is <u>differentiable at $p \in M$</u> if for some chart $h: \mathcal{U} \to \mathcal{U}'$ of M, $p \in \mathcal{U}$, and $k: V \to V'$ of N, $f(p) \in V$, the composition $k \circ f \circ h^{-1} : h(f^{-1}(V) \cap \mathcal{U}) \to V'$ is differentiable at $h(p) \in \mathcal{U}'$.



<u>Def</u>: A continuous $f: M \to N$ is <u>smooth</u> (or <u>differentiable</u>) if its smooth (or differentiable) at every $p \in M$.

<u>Ex</u>: $Id: M \to M$ is differentiable. Composition preserves smoothness. (These two imply we have the smooth category)

<u>Def</u>: A <u>diffeomorphism</u> is an invertible smooth map.

Note that diffeomorphisms are the isomorphisms in the smooth category: $f: M \to N$ is a diffeo iff

$\exists\ g: N \to M$ a diffeo, with $f \circ g = id_N$, $g \circ f = id_M$.

<u>Notice</u>: A map that is differentiable and a homeomorphism need not be a diffeomorphism! (Why? The inverse may not be smooth.)

<u>Ex</u>: (differentiable structures) Let $\mathcal{U} \subseteq \mathbb{R}^n$ be open + let $h: \mathcal{U} \to \mathcal{U}$ be any diffeomorphism distinct from the identity. Then $(\mathcal{U}, \{id\})$ and $(\mathcal{U}, \{h\})$ are two distinct differentiable ~~manifolds~~ structures, but $h: (\mathcal{U}, \{id\}) \to (\mathcal{U}, \{h\})$ provides a diffeomorphism between them. Their differential topology is exactly the same, so we "identify" them. (This is the strictest or strongest relation most topologists consider.)

On the other hand, S^7 contains 28 distinct differentiable structures, i.e., 28 distinct objects in the smooth category which are not diffeomorphic, but all homeomorphic to S^7.

Often times we will/can work locally and assume

~~there~~

we're working with an open set in Euclidean space:

$f: \mathcal{U} \to N$, consider $f \circ h^{-1}: \mathcal{U}' \to N$; $V \subseteq \mathcal{U}$, consider $h(V) \subseteq \mathcal{U}'$. This allows coordinates to be introduced, since we write $h: \mathcal{U} \to \mathcal{U}'$ as $h = (h_1, \ldots, h_n)$ & each $h_i: \mathcal{U} \to \mathbb{R}$ is differentiable. Therefore, a function on $\mathcal{U}$ is differentiable iff its differentiable in each of its coordinates as in usual calculus.

<u>Def</u>: The <u>disjoint union</u> of two smooth manifolds M_1 & M_2 is denoted $M_1 \amalg M_2$ & is again a smooth n-manifold. There are inclusions $i_k: M_k \to M_1 \amalg M_2$ $k = 1, 2$.

A map $f: M_1 \amalg M_2 \to N$ is differentiable iff both compositions $f \circ i_k$ are differentiable.

$$\begin{tikzcd}
M_1 \arrow[dr] \arrow[drr, bend left] & & \\
& M_1 \amalg M_2 \arrow[r, "f"] & N \\
M_2 \arrow[ur] \arrow[urr, bend right] & &
\end{tikzcd}$$

That is, there's a bijection

$$C^\infty(M_1 \amalg M_2, N) \to C^\infty(M_1, N) \times C^\infty(M_2, N)$$
$$f \longmapsto (f \circ i_1, f \circ i_2)$$

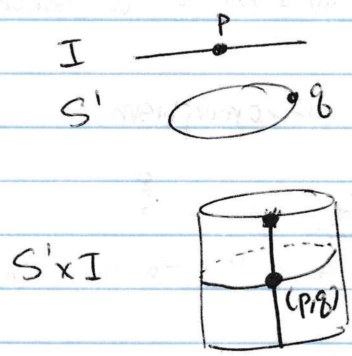
<u>Def</u>: The <u>cartesian product</u> of two smooth manifolds M_1 & M_2 of dimension $m_1 + m_2$ is again a smooth manifold, denoted $M_1 \prod M_2$, or $M_1 \times M_2$. One has canonical projections

$p_k: M_1 \times M_2 \to M_k$, $k = 1, 2$. The smooth structure is given by taking a product of charts $h_1 \times h_2 : U_1 \times U_2 \to U_1' \times U_2' \subseteq \mathbb{R}^{m_1+m_2}$.

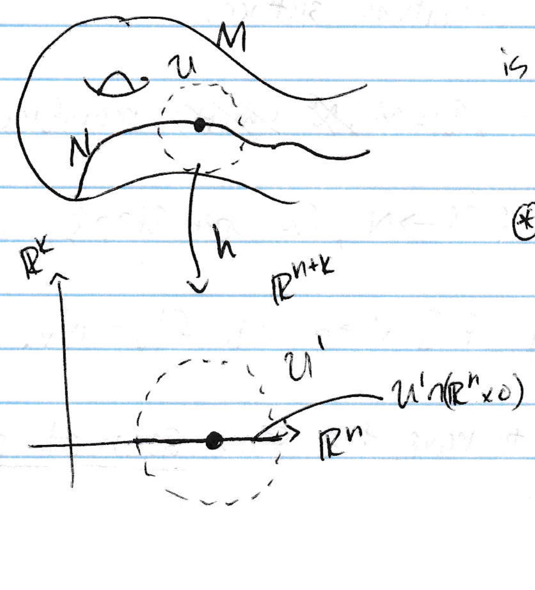
A map $f : N \to M_1 \times M_2$ is differentiable iff both compositions $p_k \circ f$ are differentiable.

$$\begin{tikzcd} & & M_1 \\ N \arrow[rru, bend left] \arrow[rrd, bend right] \arrow[r] & M_1 \times M_2 \arrow[ru, "p_1"] \arrow[rd, "p_2"'] & \\ & & M_2 \end{tikzcd}$$

<u>Ex:</u>



<u>Def:</u> A subset $N \subseteq M^{n+k}$ is an <u>n-dimensional smooth submanifold of M</u> if, $\forall p \in N$, $\exists$ a chart around p, $h : U \to U' \subseteq \mathbb{R}^{n+k} = \mathbb{R}^n \times \mathbb{R}^k$, such that $h(N \cap U) = U' \cap \mathbb{R}^n$, where $\mathbb{R}^n = \mathbb{R}^n \times \{0\} \subseteq \mathbb{R}^{n+k}$. The number $k = \dim M - \dim n$ is the <u>codimension of N</u>.

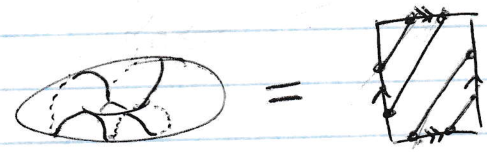


⊛ Locally, N lies in M the same as $\mathbb{R}^n$ lies in $\mathbb{R}^{n+k}$.

⊛ the smooth structure on N comes from restriction of the smooth atlas on M.

Spelled out, for h in the definition of a submanifold, define $h' := h|_{N \cap \mathcal{U}} \to \mathcal{U}' \cap \mathbb{R}^n$, & the set of all these is a smooth atlas.

Def. A smooth map $f: M \to N$ is an embedding if $f(M) \subseteq N$ is a smooth submanifold, and $f: M \to f(M)$ is a diffeomorphism.

Non-Ex: $\mathbb{R} \xrightarrow{f}$ 

This fails to be an embedding w/ rational or irrational slope.

B. Tangent Spaces

There are several equivalent definitions of the tangent space to a manifold, all of which will be useful for us. If the manifold is embedded in some Euclidean space, then the usual, intuitive definitions suffice.

Def: Let M & N be smooth. ~~M~~ ~~smooth~~ manifolds & $p \in M$. ~~Def~~ On the set $\{f \mid f: \mathcal{U} \to N,$ f.s. open $\mathcal{U} \ni p\}$, construct an equiv. relation: $f \sim g$ if $\exists\, V \ni p$ s.t. $f|_V = g|_V$. In this case, f & g are said to have the same germ at $p \in M$.