Solve the following problems. Be sure to show all work and prove all statements.

Part I. Do any three problems from this section.

1. Let $M$ and $N$ be two smooth, non-empty manifolds of the same dimension, such that $N$ is compact and $M$ is connected.
   (a) If $g : N \to M$ is a submersion, then show $g$ is surjective.
   (b) If $f : N \to M$ is an embedding, then show $f$ is a diffeomorphism.

2. (a) Let $e^k_a \subset \mathbb{R}^k$ be the open ball $e^k_a = \{x \in \mathbb{R}^k | |x|^2 < a\}$.
    Show $e^k_a$ is homeomorphic to $\mathbb{R}^k$.
   (b) If $M$ is a $k$-dimensional manifold, show that every point of $M$ has a neighborhood homeomorphic to all of $\mathbb{R}^k$. Therefore, charts can always be chosen with all of Euclidean space as their co-domains.
   (c) Suppose $N$ is a non-empty, $n$-dimensional manifold, with $k \leq n$. Show there exists an embedding of $\mathbb{R}^k$ into $N$.

3. Prove that a finite dimensional real vector space is a smooth manifold.

4. Show the inclusion $\mathbb{R}^{n+1} \subset \mathbb{R}^{n+2}$ induces an embedding $\mathbb{RP}^n \subset \mathbb{RP}^{n+1}$. Show $\mathbb{RP}^{n+1} \setminus \mathbb{RP}^n \cong \mathbb{R}^{n+1}$.

Part II. Do any two problems from this section.

5. Define complex projective space $\mathbb{CP}^n$ as follows. On the complex vector space $\mathbb{C}^{n+1}$, set $x \sim y$ iff there exists $\lambda \in \mathbb{C}$, $\lambda \neq 0$, such that $\lambda x = y$. The resulting quotient space is $\mathbb{CP}^n := (\mathbb{C}^{n+1} \setminus \{0\}) / \sim$. Just as for real projective spaces, denote the class of $x = (x_0, x_1, \ldots, x_n) \in \mathbb{C}^{n+1}$ under $\sim$ by $[x] = [x_0, x_1, \ldots, x_n]$.
   (a) Show that $\mathbb{CP}^n$ is a $2n$-dimensional smooth manifold.
   (b) Show that the mapping $f : \mathbb{CP}^n \times \mathbb{CP}^m \to \mathbb{CP}^{n+m+nm}$, given by
      $$(x, y) \mapsto [x_0y_0, x_0y_1, \ldots, x_iy_k, \ldots, x_ny_m],$$
      is an embedding.
   (c) Write down the analogous map for real projective spaces and show the same is true.

6. Let $M_2(\mathbb{R})$ be the set of all 2 by 2 matrices with real entries.
   (a) Show that $M_2(\mathbb{R})$ is a manifold of dimension 4.
(b) Let \( SL_2(\mathbb{R}) \subset M_2(\mathbb{R}) \) denote those matrices with determinant +1. Show \( SL_2(\mathbb{R}) \) is a submanifold of dimension 3.

(c) Let \( R \subset M_2(\mathbb{R}) \) denote those matrices of rank 1. Show \( R \) is a submanifold of dimension 3.

7. Let \( \omega \) be an irrational number and let \( T = \mathbb{R}^2/\mathbb{Z}^2 \) denote the torus. Define \( \alpha : \mathbb{R} \to T \) by \( \alpha(t) = \pi(t, \omega t) \), where \( \pi : \mathbb{R}^2 \to T^2 \) is the standard projection.

(a) Show that \( \alpha \) is an injective immersion.

(b) Show that the image of \( \alpha \) is everywhere dense in \( T \). (Hint: \( \mathbb{Z} + \mathbb{Z}\omega \subset \mathbb{R} \) is dense in \( \mathbb{R} \).)

(c) Is \( \alpha \) a smooth embedding?

Part III. Do any one problem.

7. For a smooth manifold \( M \), \( C^\infty(M) \) is an algebra under pointwise multiplication and addition of functions, as we discussed in class.

(a) For \( p \in M \), set \( \mathfrak{M}_p = \{ f \in C^\infty(M) \mid f(p) = 0 \} \). Show that \( \mathfrak{M}_p \) is a maximal ideal in \( C^\infty(M) \).

(b) Set \( \mathfrak{M}_p = \{ \bar{\varnothing} \in \mathcal{F}_p \mid \bar{\varnothing}(p) = 0 \} \subset \mathcal{F}_p \), where \( \mathcal{F}_p \) consists of germs of real-valued functions on \( M \) at \( p \). Show that \( \mathfrak{M}_p \) is the only maximal ideal of \( \mathcal{F}_p \).

(c) Let \( \mathfrak{M}_p^k \) denote the \( k \)-th power of the ideal, consisting of all finite linear combinations of \( k \)-fold products of elements of \( \mathfrak{M}_p \). Prove that

\[
T_pM \cong \left( \mathfrak{M}_p/\mathfrak{M}_p^2 \right)^* ,
\]

where the \(*\) denotes vector space dual.

(d) (Specialize the above to \( \mathbb{R}^n \).) If \( \mathcal{E}_n \) denotes the function germs at the origin on \( \mathbb{R}^n \), then let \( \mathfrak{E}_n \) denote the maximal ideal. Show that \( \mathfrak{E}_n^k \) consists of those germs \( \bar{\varnothing} \) for which all partial derivatives of order less than \( k \) vanish at the origin.

8. Suppose that \( f_1, f_2, \ldots, f_l \) are smooth, real-valued functions on a manifold \( M \) of dimension \( k \geq l \), and define \( f = (f_1, f_2, \ldots, f_l) : M \to \mathbb{R}^l \). The functions \( f_1, f_2, \ldots, f_l \) are said to be independent at \( x \) if \( T_x(f) : T_x(M) \to \mathbb{R}^l \) is surjective.

(a) Set \( X = f^{-1}(0) \). Show that if \( f_1, f_2, \ldots, f_l \) are independent on every point of \( X \), then \( X \) is a submanifold of \( M \). In this case, \( X \) is said to be cut out by the independent functions \{\( f_i \}\}. What is the dimension of \( X \)?

(b) Suppose that \( g : M \to N \) is a smooth map, and \( y \in N \) is a regular value. Prove that the submanifold \( f^{-1}(y) \) can be cut out by independent functions.

(c) Prove that every submanifold can locally be cut out by independent functions.

(d) Can you construct a submanifold that is not globally cut out by independent functions?

Part IV. Do the following problem.

9. A Riemannian metric on a bundle \( \xi = (\pi : E \to B) \) is a section of \((E \otimes E)^* \to B\) such that, for every \( b \in B \), the bilinear form determined by this section \( E_b \otimes E_b \to \mathbb{R} \) is symmetric and positive definite. A Riemannian metric on a smooth manifold \( M \) is a Riemannian metric on \( TM \), the tangent bundle of \( M \).
(a) Suppose that $M$ is a smooth manifold, immersed in Euclidean space. Show that $M$ can be equipped with a Riemannian metric.

(b) A bundle $\eta$ is a sub-bundle of $\xi$ if $F_b(\eta) \subset F_b(\xi)$ for every $b \in B$. If $\xi$ is equipped with a Riemannian metric and $\eta$ is a sub-bundle of $\xi$, show there exists a sub-bundle of $\xi$ given by
\[ \eta^\perp = \bigcup_{b \in B} F_b^\perp. \]

(c) Let $M$ be a submanifold of $N$, and denote the embedding $e : M \subset N$. The tangent bundle $TM$ is a sub-bundle of the restriction $TN|_M$. The normal bundle of $e$ is the orthogonal complement: $\nu_e := TM^\perp \subset TN|_M$. Show $TM \oplus \nu_e \cong TN|_M$.

(d) Let $S^n \subset \mathbb{R}^{n+1}$ be the standard embedding. Show the normal bundle of this embedding of $S^n$ is trivial.

Bonus problems.

10. Prove $T(\mathbb{R}P^n) \cong \text{hom}(\gamma_n^1, (\gamma_n^1)^\perp)$.

11. Prove $T(\mathbb{R}P^n) \oplus \varepsilon^1 \cong (\gamma_n^1)^{n+1}$. (Hint: If $\xi$ possesses a Riemannian metric, then $\xi \cong \xi^\ast$).