MTG 7396
Homework 1
Due Monday, October 2

Solve the following problems. Be sure to show all work and prove all statements.

Part I. Do any three problems from this section.

1. Let M and N be two smooth, non-empty manifolds of the same dimension, such that N is compact and M is connected.
   (a) If $g : N \to M$ is a submersion, then show g is surjective.
   (b) If $f : N \to M$ is an embedding, then show f is a diffeomorphism.

2. (a) Let $e_a^k \subset \mathbb{R}^k$ be the open ball
$$e_a^k = \{x \in \mathbb{R}^k \mid |x|^2 < a\} .$$
   Show e_a^k is homeomorphic to $\mathbb{R}^k$.
   (b) If M is a k-dimensional manifold, show that every point of M has a neighborhood homeomorphic to all of $\mathbb{R}^k$. Therefore, charts can always be chosen with all of Euclidean space as their co-domains.
   (c) Suppose N is a non-empty, n-dimensional manifold, with $k \leq n$. Show there exists an embedding of $\mathbb{R}^k$ into N.

3. Prove that a finite dimensional real vector space is a smooth manifold.

4. Show the inclusion $\mathbb{R}^{n+1} \subset \mathbb{R}^{n+2}$ induces an embedding $\mathbb{R}\mathrm{P}^n \subset \mathbb{R}\mathrm{P}^{n+1}$. Show
$$\mathbb{R}\mathrm{P}^{n+1} \setminus \mathbb{R}\mathrm{P}^n \cong \mathbb{R}^{n+1} .$$

Part II. Do any two problems from this section.

5. Define complex projective space $\mathbb{C}\mathrm{P}^n$ as follows. On the complex vector space $\mathbb{C}^{n+1}$, set $x \sim y$ iff there exists $\lambda \in \mathbb{C}$, $\lambda \neq 0$, such that $\lambda x = y$. The resulting quotient space is $\mathbb{C}\mathrm{P}^n := \left(\mathbb{C}^{n+1} \setminus \{0\}\right) / \sim$. Just as for real projective spaces, denote the class of $x = (x_0, x_1, \ldots, x_n) \in \mathbb{C}^{n+1}$ under $\sim$ by $[x] = [x_0, x_1, \ldots, x_n]$.
   (a) Show that $\mathbb{C}\mathrm{P}^n$ is a $2n$-dimensional smooth manifold.
   (b) Show that the mapping $f : \mathbb{C}\mathrm{P}^n \times \mathbb{C}\mathrm{P}^m \to \mathbb{C}\mathrm{P}^{n+m+nm}$, given by
$$(x, y) \mapsto [x_0 y_0, x_0 y_1, \ldots, x_i y_k, \ldots, x_n y_m] ,$$
   is an embedding.
   (c) Write down the analogous map for real projective spaces and show the same is true.

6. Let $M_2(\mathbb{R})$ be the set of all 2 by 2 matrices with real entries.
   (a) Show that $M_2(\mathbb{R})$ is a manifold of dimension 4.

(b) Let $SL_2(\mathbb{R}) \subset M_2(\mathbb{R})$ denote those matrices with determinant $+1$. Show $SL_2(\mathbb{R})$ is a submanifold of dimension 3.

(c) Let $R \subset M_2(\mathbb{R})$ denote those matrices of rank 1. Show R is a submanifold of dimension 3.

7. Let ω be an irrational number and let $T = \mathbb{R}^2/\mathbb{Z}^2$ denote the torus. Define $\alpha : \mathbb{R} \to T$ by $\alpha(t) = \pi(t, \omega t)$, where $\pi : \mathbb{R}^2 \to T^2$ is the standard projection.

   (a) Show that α is an injective immersion.

   (b) Show that the image of α is everywhere dense in T. (Hint: $\mathbb{Z} + \mathbb{Z}\omega \subset \mathbb{R}$ is dense in $\mathbb{R}$.)

   (c) Is α a smooth embedding?

Part III. Do any one problem.

7. For a smooth manifold M, $C^\infty(M)$ is an algebra under pointwise multiplication and addition of functions, as we discussed in class.

   (a) For $p \in M$, set $\mathfrak{M}_p = \{f \in C^\infty(M) \mid f(p) = 0\}$. Show that $\mathfrak{M}_p$ is a maximal ideal in $C^\infty(M)$.

   (b) Set $\overline{\mathfrak{M}}_p = \{\overline{\phi} \in \mathcal{F}_p \mid \overline{\phi}(p) = 0\} \subset \mathcal{F}_p$, where $\mathcal{F}_p$ consists of germs of real-valued functions on M at p. Show that $\overline{\mathfrak{M}}_p$ is the only maximal ideal of $\mathcal{F}_p$.

   (c) Let $\overline{\mathfrak{M}}_p^k$ denote the k^{th} power of the ideal, consisting of all finite linear combinations of k-fold products of elements of $\overline{\mathfrak{M}}_p$. Prove that

$$T_pM \cong \left(\overline{\mathfrak{M}}_p/\overline{\mathfrak{M}}_p^2\right)^* ,$$

   where the $*$ denotes vector space dual.

   (d) (Specialize the above to $\mathbb{R}^n$.) If $\mathcal{E}_n$ denotes the function germs at the origin on $\mathbb{R}^n$, then let $\mathfrak{E}_n$ denote the maximal ideal. Show that $\mathfrak{E}_n^k$ consists of those germs $\overline{\phi}$ for which all partial derivatives of order less than k vanish at the origin.

8. Suppose that $f_1, f_2, \ldots, f_l$ are smooth, real-valued functions on a manifold M of dimension $k \geq l$, and define $f = (f_1, f_2, \ldots, f_l) : M \to \mathbb{R}^l$. The functions $f_1, f_2, \ldots, f_l$ are said to be *independent at x* if $T_x(f) : T_x(M) \to \mathbb{R}^l$ is surjective.

   (a) Set $X = f^{-1}(0)$. Show that if $f_1, f_2, \ldots, f_l$ are independent on every point of X, then X is a submanifold of M. In this case, X is said to be *cut out* by the independent functions $\{f_i\}$. What is the dimension of X?

   (b) Suppose that $g : M \to N$ is a smooth map, and $y \in N$ is a regular value. Prove that the submanifold $f^{-1}(y)$ can be cut out by independent functions.

   (c) Prove that every submanifold can locally be cut out by independent functions.

   (d) Can you construct a submanifold that is not globally cut out by independent functions?

Part IV. Do the following problem.

9. A *Riemannian metric* on a bundle $\xi = (\pi : E \to B)$ is a section of $(E \otimes E)^* \to B$ such that, for every $b \in B$, the bilinear form determined by this section $E_b \otimes E_b \to \mathbb{R}$ is symmetric and positive definite. A *Riemannian metric on a smooth manifold M* is a Riemannian metric on TM, the tangent bundle of M.

(a) Suppose that M is a smooth manifold, immersed in Euclidean space. Show that M can be equipped with a Riemannian metric.

(b) A bundle η is a *sub-bundle* of ξ if $F_b(\eta) \subset F_b(\xi)$ for every $b \in B$. If ξ is equipped with a Riemannian metric and η is a sub-bundle of ξ, show there exists a sub-bundle of ξ given by

$$\eta^\perp = \bigcup_{b \in B} F_b^\perp .$$

(c) Let M be a submanifold of N, and denote the embedding $e : M \subset N$. The tangent bundle TM is a sub-bundle of the restriction $TN|_M$. The *normal bundle of* e is the orthogonal complement: $\nu_e := TM^\perp \subset TN|_M$. Show $TM \oplus \nu_e \cong TN|_M$.

(d) Let $S^n \subset \mathbb{R}^{n+1}$ be the standard embedding. Show the normal bundle of this embedding of S^n is trivial.

Bonus problems.

10. Prove $T(\mathbb{R}P^n) \cong \hom(\gamma_n^1, (\gamma_n^1)^\perp)$.

11. Prove $T(\mathbb{R}P^n) \oplus \varepsilon^1 \cong (\gamma_n^1)^{n+1}$. (Hint: If ξ possesses a Riemannian metric, then $\xi \cong \xi^*$).