Solve the following problems. Be sure to show all work and prove all statements.

**Part I.** Do any three problems from this section.

1. Let \( M \) and \( N \) be two smooth, non-empty manifolds of the same dimension, such that \( N \) is compact and \( M \) is connected.
   
   (a) If \( g : N \to M \) is a submersion, then show \( g \) is surjective.
   
   (b) If \( f : N \to M \) is an embedding, then show \( f \) is a diffeomorphism.

2. (a) Let \( e_k^a \subset \mathbb{R}^k \) be the open ball
   \[
   e_k^a = \{ x \in \mathbb{R}^k \mid |x|^2 < a \}.
   \]
   Show \( e_k^a \) is homeomorphic to \( \mathbb{R}^k \).
   
   (b) If \( M \) is a \( k \)-dimensional manifold, show that every point of \( M \) has a neighborhood homeomorphic to all of \( \mathbb{R}^k \). Therefore, charts can always be chosen with all of Euclidean space as their co-domains.
   
   (c) Suppose \( N \) is a non-empty, \( n \)-dimensional manifold, with \( k \leq n \). Show there exists an embedding of \( \mathbb{R}^k \) into \( N \).

3. Prove that a finite dimensional real vector space is a smooth manifold.

4. Show the inclusion \( \mathbb{R}^{n+1} \subset \mathbb{R}^{n+2} \) induces an embedding \( \mathbb{RP}^n \subset \mathbb{RP}^{n+1} \). Show
   \[
   \mathbb{RP}^{n+1} \setminus \mathbb{RP}^n \cong \mathbb{R}^{n+1}.
   \]

**Part II.** Do any two problems from this section.

5. Define complex projective space \( \mathbb{CP}^n \) as follows. On the complex vector space \( \mathbb{C}^{n+1} \), set \( x \sim y \) iff there exists \( \lambda \in \mathbb{C}, \lambda \neq 0 \), such that \( \lambda x = y \). The resulting quotient space is \( \mathbb{CP}^n := (\mathbb{C}^{n+1} \setminus \{0\}) / \sim \). Just as for real projective spaces, denote the class of \( x = (x_0, x_1, \ldots, x_n) \in \mathbb{C}^{n+1} \) under \( \sim \) by \( [x] = [x_0, x_1, \ldots, x_n] \).
   
   (a) Show that \( \mathbb{CP}^n \) is a \( 2n \)-dimensional smooth manifold.
   
   (b) Show that the mapping \( f : \mathbb{CP}^n \times \mathbb{CP}^m \to \mathbb{CP}^{n+m+nm} \), given by
   \[
   (x, y) \mapsto [x_0y_0, x_0y_1, \ldots, x_iy_k, \ldots, x_ny_m],
   \]
   is an embedding.
   
   (c) Write down the analogous map for real projective spaces and show the same is true.

6. Let \( M_2(\mathbb{R}) \) be the set of all 2 by 2 matrices with real entries.
   
   (a) Show that \( M_2(\mathbb{R}) \) is a manifold of dimension 4.
(b) Let $SL_2(\mathbb{R}) \subset M_2(\mathbb{R})$ denote those matrices with determinant $+1$. Show $SL_2(\mathbb{R})$ is a submanifold of dimension 3.

(c) Let $R \subset M_2(\mathbb{R})$ denote those matrices of rank 1. Show $R$ is a submanifold of dimension 3.

7. Let $\omega$ be an irrational number and let $T = \mathbb{R}^2/\mathbb{Z}^2$ denote the torus. Define $\alpha : \mathbb{R} \to T$ by $\alpha(t) = \pi(t, \omega t)$, where $\pi : \mathbb{R}^2 \to T^2$ is the standard projection.

(a) Show that $\alpha$ is an injective immersion.

(b) Show that the image of $\alpha$ is everywhere dense in $T$. (Hint: $\mathbb{Z} + \mathbb{Z}\omega \subset \mathbb{R}$ is dense in $\mathbb{R}$.)

(c) Is $\alpha$ a smooth embedding?

Part III. Do any one problem.

7. For a smooth manifold $M$, $C^\infty(M)$ is an algebra under pointwise multiplication and addition of functions, as we discussed in class.

(a) For $p \in M$, set $\mathfrak{M}_p = \{ f \in C^\infty(M) \mid f(p) = 0 \}$. Show that $\mathfrak{M}_p$ is a maximal ideal in $C^\infty(M)$.

(b) Set $\overline{\mathfrak{M}}_p = \{ \overline{\varphi} \in \mathcal{F}_p \mid \overline{\varphi}(p) = 0 \} \subset \mathcal{F}_p$, where $\mathcal{F}_p$ consists of germs of real-valued functions on $M$ at $p$. Show that $\overline{\mathfrak{M}}_p$ is the only maximal ideal of $\mathcal{F}_p$.

(c) Let $\overline{\mathfrak{M}}^k_p$ denote the $k^{th}$ power of the ideal, consisting of all finite linear combinations of $k$-fold products of elements of $\overline{\mathfrak{M}}_p$. Prove that

$$T_p M \cong \left( \overline{\mathfrak{M}}_p / \overline{\mathfrak{M}}^2_p \right)^*,$$

where the * denotes vector space dual.

(d) (Specialize the above to $\mathbb{R}^n$.) If $\mathcal{E}_n$ denotes the function germs at the origin on $\mathbb{R}^n$, then let $\mathfrak{E}_n$ denote the maximal ideal. Show that $\mathfrak{E}_n^k$ consists of those germs $\overline{\varphi}$ for which all partial derivatives of order less than $k$ vanish at the origin.

8. Suppose that $f_1, f_2, \ldots, f_l$ are smooth, real-valued functions on a manifold $M$ of dimension $k \geq l$, and define $f = (f_1, f_2, \ldots, f_l) : M \to \mathbb{R}^l$. The functions $f_1, f_2, \ldots, f_l$ are said to be independent at $x$ if $T_x(f) : T_x(M) \to \mathbb{R}^l$ is surjective.

(a) Set $X = f^{-1}(0)$. Show that if $f_1, f_2, \ldots, f_l$ are independent on every point of $X$, then $X$ is a submanifold of $M$. In this case, $X$ is said to be cut out by the independent functions $\{ f_i \}$. What is the dimension of $X$?

(b) Suppose that $g : M \to N$ is a smooth map, and $y \in N$ is a regular value. Prove that the submanifold $f^{-1}(y)$ can be cut out by independent functions.

(c) Prove that every submanifold can locally be cut out by independent functions.

(d) Can you construct a submanifold that is not globally cut out by independent functions?

Part IV. Do the following problem.

9. Let $\xi = (E, \pi, B)$ be a vector bundle. A Riemannian metric on $\xi$ is a section of $(E \otimes E)^* \to B$ such that, for every $b \in B$, the bilinear form determined by this section $E_b \times E_b \to \mathbb{R}$ is symmetric and positive definite.
(a) Suppose that $E$ embeds in Euclidean space. Show that $E$ can be equipped with a Riemannian metric.

(b) A bundle $\eta$ is a sub-bundle of $\xi$ if $F_b(\eta) \subset F_b(\xi)$ for every $b \in B$. If $\xi$ is equipped with a Riemannian metric and $\eta$ is a sub-bundle of $\xi$, show there exists a sub-bundle of $\xi$ given by

$$\eta^\perp = \bigcup_{b \in B} F_b^\perp.$$  

(c) Let $M$ be a submanifold of $N$. The tangent bundle $TM$ is a sub-bundle of the restriction $TN|_M$. The normal bundle of $M$ is the orthogonal complement $TM^\perp \subset TN|_M$. Show $TM \oplus \nu \cong TN|_M$.

(d) Let $S^n \subset \mathbb{R}^{n+1}$ be the standard embedding. Show the normal bundle of this embedding of $S^n$ is trivial.

**Bonus problems.**

10. Prove $T(\mathbb{R}P^n) \cong \text{hom}(\gamma_n^1, (\gamma_n^1)^\perp)$.

11. Prove $T(\mathbb{R}P^n) \oplus \epsilon^1 \cong (\gamma_n^1)^{n+1}$. (Hint: If $\xi$ possesses a Riemannian metric, then $\xi \cong \xi^*$).