Solve the following problems. Be sure to show all work and prove all statements.

1. (a) (Milnor-Stasheff problem 4a) Prove

\[ w_k(\xi \times \eta) = \sum_{i=0}^{k} w_i(\xi) \times w_{k-i}(\eta). \]

(b) Prove that \( \mathbb{R}P^2 \times \mathbb{R}P^2 \) is not cobordant to \( \mathbb{R}P^4 \).

(c) Prove that the torus \( T = S^1 \times S^1 \) is cobordant to \( S^2 \).

(d) Suppose \( n = k + j \) for some natural numbers \( n, k, j \). Is \( S^k \times S^j \) is cobordant to \( S^n \)?

2. (a) Let \( \xi \) be an \( n \)-plane bundle over a compact base \( X \). Show there exists a vector bundle \( \eta \) over \( X \) such that \( \xi \oplus \eta \) is a trivial vector bundle.

(b) Show compactness of \( X \) is necessary in (a). (Hint: Consider the canonical line bundle over \( \mathbb{R}P^\infty \).

(c) Two real vector bundles \( \xi \) and \( \eta \) over \( X \) are \textit{stably isomorphic} if there exists an \( n \) such that \( \xi \oplus \varepsilon^n \cong \eta \oplus \varepsilon^n \), where \( \varepsilon^n = X \times \mathbb{R}^n \) is the trivial \( n \)-plane bundle over \( X \). If \( \xi \) and \( \eta \) are stably isomorphic, prove \( w_i(\xi) = w_i(\eta) \) for every \( i \).

3. Prove that the orthogonal group \( O(n+k) \) acts transitively on \( G_n(\mathbb{R}^{n+k}) \). Identify the stabilizer of the \( n \)-plane \( \mathbb{R}^n \oplus 0 \subset \mathbb{R}^n \times \mathbb{R}^k \) under this action; call it \( S \). Show \( G_n(\mathbb{R}^{n+k}) \cong O(n+k)/S \).

4. Let \( E = S^1 \times I / \sim \), where \((z, 0) \sim (e^{\pi i} \cdot z, 1)\). This is similar to the standard construction of the torus (which is \((z, 0) \sim (z, 1)\)), but we rotate one end by \( \pi \) before gluing it to the other end. Let \( B = S^1 \) and let \( p : E \to B \) be given by \( p(z, t) = z \).

(a) Show \( E \) is not trivial as a \( \mathbb{Z}/2\mathbb{Z} \)-bundle.

(b) Show \( E \) is trivial as a \( G \)-bundle, where \( G \) is the full group of rotations of \( S^1 \).

(Hint: Consider the transition functions of the bundle.) Notice that we have specified both spaces and the projection map, but yet we can’t tell if its a trivial bundle until we specify the group of the bundle!

5. Let \( \text{vect}_\mathbb{R} \) denote the category of real vector spaces with morphisms given by vector space isomorphisms. A functor \( T : \text{vect}_\mathbb{R} \times \text{vect}_\mathbb{R} \to \text{vect}_\mathbb{R} \) assigns a vector space \( T(V_1, V_2) \) to each pair of vector spaces \( V_1, V_2 \), and to each pair of vector space isomorphisms \( f_1 : V_1 \to V_1', f_2 : V_2 \to V_2' \), a vector space isomorphism \( T(f_1, f_2) : T(V_1, V_2) \to T(V_1', V_2') \). Functors must satisfy \( T(id_V, id_W) = id_{T(V,W)} \) and \( T(f_1 \circ f_2, g_1 \circ g_2) = T(f_1, g_1) \circ T(f_2, g_2) \).

Such a functor is \textit{continuous} if \( T(f, g) \) varies continuously on \( f \) and \( g \). This notion is well-defined since the set of vector space isomorphisms from one vector space to another has a natural topology.

(a) Describe this topology.
(b) Let $T$ be a continuous functor as above and let $\xi_1$ and $\xi_2$ be 2 vector bundles over a common base $B$. Construct a new bundle over $B$ by taking the fiber over $b \in B$ to be

$$F_b = T(F_b(\xi_1), F_b(\xi_2)).$$

Set the total space $E$ to be the disjoint union of the vector space $F_b$. and define $\pi(F_b) = b$. Prove there is a canonical topology on $E$ so that $E$ is the total space of a vector bundle with projection $\pi$ and fibers $F_b$. (This is in Milnor-Stasheff, Theorem 3.6. Be sure to fill in all the details if using their proof.)

6. (a) Let $P_n$ denote the set of all real $n \times n$ positive definite matrices. Equivalently, these are upper triangular. Prove that $P_n$ is contractible.

(b) Extra credit: Prove the QR decomposition: every invertible matrix is the product of a positive-definite matrix and an orthogonal matrix. So $GL_n(\mathbb{R}) \cong \mathbb{R}^n \times O(n)$. In what category is this isomorphism?

(c) Prove that a vector bundle admits a Riemannian metric if and only if its structure group can be reduced to $O(n)$.

(d) Using (a) and (b), show that every vector bundle can be equipped with a Riemannian metric. (This method is the algebraic-topologist’s proof.)