MTG 7396
Homework 2
Due: Monday, November 20

Solve the following problems. Be sure to show all work and prove all statements.

1. (a) (Milnor-Stasheff problem 4a) Prove
$$w_k(\xi \times \eta) = \sum_{i=0}^{k} w_i(\xi) \times w_{k-i}(\eta)\,.$$
   (b) Prove that $\mathbb{RP}^2 \times \mathbb{RP}^2$ is not cobordant to $\mathbb{RP}^4$.
   (c) Prove that the torus $T = S^1 \times S^1$ is cobordant to S^2.
   (d) Suppose $n = k + j$ for some natural numbers n, k, j. Is $S^k \times S^j$ is cobordant to S^n?

2. (a) Let ξ be an n-plane bundle over a compact base X. Show there exists a vector bundle η over X such that $\xi \oplus \eta$ is a trivial vector bundle.
   (b) Show compactness of X is necessary in (a). (Hint: Consider the canonical line bundle over $\mathbb{RP}^\infty$.)
   (c) Two real vector bundles ξ and η over X are *stably isomorphic* if there exists an n such that $\xi \oplus \varepsilon^n \cong \eta \oplus \varepsilon^n$, where $\varepsilon^n = X \times \mathbb{R}^n$ is the trivial n-plane bundle over X. If ξ and η are stably isomorphic, prove $w_i(\xi) = w_i(\eta)$ for every i.

3. Prove that the orthogonal group $O(n+k)$ acts transitively on $G_n(\mathbb{R}^{n+k})$. Identify the stabilizer of the n-plane $\mathbb{R}^n \oplus 0 \subset \mathbb{R}^n \times \mathbb{R}^k$ under this action; call it S. Show $G_n(\mathbb{R}^{n+k}) \cong O(n+k)/S$.

4. Let $E = S^1 \times I/\sim$, where $(z, 0) \sim (e^{\pi i} \cdot z, 1)$. This is similar to the standard construction of the torus (which is $(z, 0) \sim (z, 1)$), but we rotate one end by π before gluing it to the other end. Let $B = S^1$ and let $p : E \to B$ be given by $p(z, t) = z$.

   (a) Show E is not trivial as a $\mathbb{Z}/2\mathbb{Z}$-bundle.
   (b) Show E is trivial as a G-bundle, where G is the full group of rotations of S^1.

   (Hint: Consider the transition functions of the bundle.) Notice that we have specified both spaces and the projection map, but yet we can't tell if its a trivial bundle until we specify the group of the bundle!

5. Let $\text{vect}_\mathbb{R}$ denote the category of real vector spaces with morphisms given by vector space isomorphisms. A functor $T : \text{vect}_\mathbb{R} \times \text{vect}_\mathbb{R} \to \text{vect}_\mathbb{R}$ assigns a vector space $T(V_1, V_2)$ to each pair of vector spaces V_1, V_2, and to each pair of vector space isomorphisms $f_1 : V_1 \to V_1', f_2 : V_2 \to V_2'$, a vector space isomorphism $T(f_1, f_2) : T(V_1, V_2) \to T(V_1', V_2')$. Functors must satisfy $T(\text{id}_V, \text{id}_W) = \text{id}_{T(V,W)}$ and $T(f_1 \circ f_2, g_1 \circ g_2) = T(f_1, g_1) \circ T(f_2, g_2)$.

   Such a functor is *continuous* if $T(f, g)$ varies continuously on f and g. This notion is well-defined since the set of vector space isomorphisms from one vector space to another has a natural topology.

   (a) Describe this topology.

(b) Let T be a continuous functor as above and let ξ_1 and ξ_2 be 2 vector bundles over a common base B. Construct a new bundle over B by taking the fiber over $b \in B$ to be

$$F_b = T(F_b(\xi_1), F_b(\xi_2))\,.$$

Set the total space E to be the disjoint union of the vector space F_b. and define $\pi(F_b) = b$. Prove there is a canonical topology on E so that E is the total space of a vector bundle with projection π and fibers F_b. (This is in Milnor-Stasheff, Theorem 3.6. Be sure to fill in all the details if using their proof.)

6. (a) Let P_n denote the set of all real $n \times n$ positive definite matrices. Equivalently, these are upper triangular. Prove that P_n is contractible.

   (b) Extra credit: Prove the QR decomposition: every invertible matrix is the product of a positive-definite matrix and an orthogonal matrix. So $GL_n(\mathbb{R}) \cong R_n \times O(n)$. In what category is this isomorphism?

   (c) Prove that a vector bundle admits a Riemannian metric if and only if its structure group can be reduced to $O(n)$.

   (d) Using (a) and (b), show that every vector bundle can be equipped with a Riemannian metric. (This method is the algebraic-topologist's proof.)