with a left G-action, but the action need not be as nice as in principal G-bundles.

Def: If W is a right G-space and X is a left G-space, the balanced product $W \times_G X$ is the quotient space $W \times X/\sim$ where $(wg, x) \sim (w, gx)$. //

Alternatively, it will be useful to turn a left G-action on X into a right action, via $x \cdot g := g^{-1}x$. Then $(x', x) \cdot g := (x'g, g^{-1}x)$ equips $X \times X$ with the diagonal action of G; thus $X \times_G X = X \times X / G$.

There are 2 special cases for us:

a) take $X = pt$, then $W \times X/\sim \;\cong W/G$

b) if $X = G$, w/ G-action given by the natural left action of G on itself, then $W \times_G G$ is a right G-space (again by right G-action on G), and $W \times G \xrightarrow{\text{action}} W \Rightarrow W \times_G G \xrightarrow{\cong} W$ is a G-equiv. homeo. //

The balanced product doesn't have a G-action in general. However, this will be our main tool for transforming certain types of bundles to others.

<u>Def:</u> Let $\xi = P \xrightarrow{\pi} X$ be a principal G-bundle and let F be a left G-space. Then $P \times F$ inherits a right G-structure by $(p,f)\cdot g = (p\cdot g, g^{-1}\cdot f)$. ~~Then~~ Finally, we build

$$\begin{array}{c} P \times_G F \\ \pi_F \downarrow \\ X \end{array}$$

where $\pi_F([p,f]\cdot g) = \pi(p)$, from the original bundle. Then $P \times_G F \xrightarrow{\pi_F} X$ is called the <u>fiber bundle over X w/ associated principal bundle ξ</u>. G still acts on F and is called the <u>structure group</u> of the bundle. //

Notice that we have $\begin{array}{c} P \times_G F \\ \pi_F \downarrow \\ X \end{array}$ built out of ξ and <u>any</u> left G-space F.

Furthermore, $\begin{array}{c} G \times_G F \cong F \\ \downarrow \\ P \times_G F \\ \downarrow \\ X \end{array}$ we've changed the fiber to F, but still have retained the "structure of ξ".

<u>Ex:</u> Consider the principal $\mathbb{Z}/2$-bundle

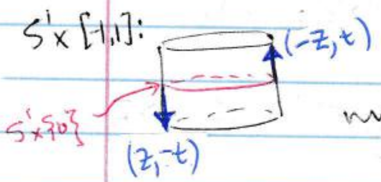
$$\begin{array}{c} \mathbb{Z}/2 \\ \downarrow \\ S^1 \\ \downarrow \\ S^1 \cong \mathbb{R}P^1 \end{array}$$

and let $F = [-1,1]$ be a $G = \mathbb{Z}/2$-space by $(\pm 1)\cdot t = \pm t$, $t \in F$. Then

$$\begin{array}{c} \mathbb{Z}/2 \times_{\mathbb{Z}/2} [-1,1] \cong [-1,1] \\ \downarrow \\ S^1 \times_{\mathbb{Z}/2} [-1,1] \\ \downarrow \\ S^1 \end{array}$$

(*) the vector bundle is the Möbius bundle.

$S^1 \times [-1,1]$:



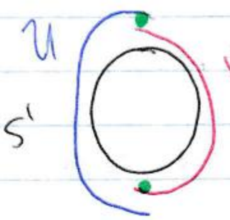
$\rightsquigarrow$

$\downarrow$

$S^1 = \mathbb{R}P^1$

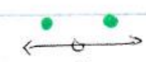
(*) There are two principal $\mathbb{Z}/2$-bundles over S^1, corresponding to whether the $\mathbb{Z}/2$-action is trivial or non-trivial. The non-trivial one, which we just described, turns into a bundle w/ fiber $[-1,1]$ via a process known as "clutching" or "clamping". The idea/terminology is that we glue the bundle together out of local data by clutching together the two pieces:

$\mathbb{R}P^1 \cong S^1$



transition functions: $U \cap V \longrightarrow GL_1\mathbb{R}$

$S^0 \longrightarrow \mathbb{R}^\times$



If the map is homotopic to a constant, then the bundle will be trivial. If not, then the Möbius bundle.

<u>Prop</u>: For $\begin{array}{c} P \times_G F \\ \pi_F \downarrow \\ X \end{array}$ as above, $\pi_F^{-1}(x) \underset{\text{homeo}}{\cong} F$.

<u>Proof</u>: Let $\pi: P \to X$ be the associated principal G-bundle, and let $\pi(p_0) = x$ f.s. $p_0 \in P$. Define $h: F \to P \times_G F$ by $h(f) = (p_0, f)G$. Since $\pi_F((p_0, f)G) = \pi(p_0) = b$, the image of h can be restricted to $\pi_F^{-1}(b)$; hence $h: F \to \pi_F^{-1}(b)$.

Define an inverse to h by first constructing $j: \pi^{-1}(b) \times F \to F$ by sending $(p, f) \mapsto g \cdot f$, where g is the unique group element of G

such that $p_0 \cdot g = p$. Then $j(pg, g^{-1}f) = \tilde{g} \cdot g^{-1}f$, where $\tilde{g}$ satisfies $p_0 \cdot \tilde{g} = pg$. Thus, $j(pg, g^{-1}f) = \tilde{g} \cdot g^{-1}f = g \cdot g \cdot g^{-1}f$ $= gf = j(p,f)$. Thus, j factors through $P \times_G F$, and hence by restriction we get $j : \pi_F^{-1}(x) \to F$, which is inverse to h. ☒

One can precisely define a "fiber bundle" or "G-bundle" by defining them to be the result of applying "$- \times_G F$" to a principal G-bundle.

C. Translating vector bundles to principal bundles and back

Let $\xi = E \xrightarrow{p} X$ be an n-plane vector bundle. There are two equivalent constructions for the principal $GL_n\mathbb{R}$ bundle associated to ξ.

Given ξ, let $P = P_\xi$ = n-frame bundle of $\xi = V_n\xi$, where $V_n\xi = \{(x, (v_1, v_2, \ldots, v_n)) \mid x \in X, (v_1, \ldots, v_n) \in V_n(p^{-1}(x))\}$ consists of points of X, together w/ n-frames for the fiber over x.

P is a topological space as before: $P \subseteq \overbrace{E \oplus E \oplus \cdots \oplus E}^{n}$ (whitney sum).

$GL_n\mathbb{R}$ acts on $E^{\oplus n}$ on the right via $(v_1 \cdots v_n) \cdot A$, right matrix multiplication. This action is free (check/contemplate $[v_1 \cdots v_n]A = [v_1 \cdots v_n]$), and makes P into a principal $GL_n\mathbb{R}$ - bundle over X.

<u>Thm</u>: For any space X, there is a bijection:

$$\left\{\begin{matrix}\text{principal} \\ GL_n^{\mathbb{R}}\text{-bundles}/X\end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix}\text{real } n\text{-plane} \\ \text{bundles } /X\end{matrix}\right\}$$

Given $\xi \in$ Principal $GL_n\mathbb{R}$-bundle, $\xi = P \xrightarrow{\pi} X \longmapsto$

$$\begin{matrix} & \mathbb{R}^n \\ & \swarrow \\ P \times_{GL_n\mathbb{R}} \mathbb{R}^n \\ \downarrow \\ X \end{matrix}$$

$$\begin{matrix} & GL_n\mathbb{R} \\ & \downarrow \\ \{n\text{-frames}\} = & V_n E \\ & \downarrow \\ & X \end{matrix} \longleftarrow\!\!\shortmid \text{ Given } E \xrightarrow{p} X \text{ an } n\text{-plane bundle}/X$$

The braces refer to isomorphism classes of such objects. //

In order to simplify the proof we'll take the alternate viewpoint on frames. For a vectors space W, $V_q W = q$-frames of W = linear isomorphisms $\mathbb{R}^q \to W$. For a bundle E, we consider the bundle $Iso(\varepsilon^q, E) \subseteq Hom(\varepsilon^q, E)$, where the fiber over $x \in X$ is the space of all isomorphisms $\mathbb{R}^q \to E_x = p^{-1}(x)$. There's a map of bundles given by $Iso(\varepsilon^q, \xi) \to V_n \xi$, sending $(x, f: \mathbb{R}^q \to E_x) \longmapsto (x, f(e_1), \ldots, f(e_q))$ where $\{e_i\}$ is the standard basis. This is $GL_q\mathbb{R}$ equivariant & hence an isomorphism.

<u>Pf:</u> Let ξ be an n-plane bundle. Define

$$f_1 : \mathrm{Iso}(\varepsilon^n, \xi) \times_{GL_n\mathbb{R}} \mathbb{R}^n \longrightarrow E(\xi) \text{ by}$$

$$f_1([x,f,v]) = (x, f(v)), \text{ where } f: \mathbb{R}^n \xrightarrow{\cong} E_x,\ v \in \mathbb{R}^n.$$

f_1 is well-defined on the balanced product, since $(fg)(v) = f(gv)$.

f_1 is continuous since on a trivializing nbhd of ξ, f_1 takes the form

$$U \times GL_n\mathbb{R} \times_{GL_n\mathbb{R}} \mathbb{R}^n \xrightarrow{\cong} U \times \mathbb{R}^n.$$ Furthermore, f_1 linear iso. on fibers, so f_1 gives an isomorphism of vector bundles.

On the other hand, let $P \xrightarrow{\pi} X$ be a principal $GL_n\mathbb{R}$-bundle & set $\xi = P \times_{GL_n\mathbb{R}} \mathbb{R}^n$. Define $\tau : P \to V_n\xi$ by $\tau(p) = ([p,e_1], \dots, [p,e_n])$.

The map τ is continuous & by identifying $V_n\xi$ with $\mathrm{Iso}(\varepsilon^n, \xi)$, we have τ is equivariant: $\tau : P \to V_n\xi \cong \mathrm{Iso}(\varepsilon^n, \xi)$

$$p \longmapsto (b, f_p), \text{ where } b = \pi(p)$$

and $f_p : \mathbb{R}^n \to E_b$ sends $e_i \longmapsto [p, e_i]$. Therefore, for $g \in GL_n\mathbb{R}$,

$f_{pg}(e_i) = [pg, e_i] = [p, ge_i]$. So $\tau(pg) = (b, f_{pg}) = (b, f_p \cdot g) = \tau(p)\cdot g$.

Summarizing, f_1 shows for a v.b. ξ, $\underbrace{V_n\xi \times_{GL_n\mathbb{R}} \mathbb{R}^n \cong \xi}_{\text{as v.b.}}$

and τ shows for a princ. $GL_n\mathbb{R}$-bundle P, $V_n(P \times_{GL_n\mathbb{R}} \mathbb{R}^n) \cong P$ as princ. $GL_n\mathbb{R}$-bundles.

Hence, we're done. ▨