

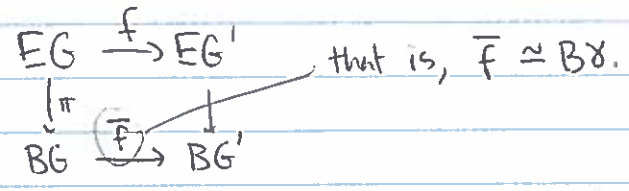
⊕ A local cross-section of $H \in G$ is a nbhd of $e \in H \in G/H$ call it U , & a map $f: U \rightarrow G$ s.t. $U \xrightarrow{f} G \rightarrow G/H$ is the identity on U .

Ex: ~~sub~~ subgrps of discrete, of Lie grps, etc.

⊖ $H \in G$ has a local cross-section $\Rightarrow H$ -admissible

Some useful facts about the classifying space functor B :

- Up to homotopy, B is a product preserving functor from top. spaces to top. spaces.
- Let $\gamma: G \rightarrow G'$ is a group hom., $EG \xrightarrow{\pi} BG$, $EG' \xrightarrow{\pi'} BG'$, universal bundles. If $f: EG \rightarrow EG'$ is any map s.t. $f(yg) = f(y)\gamma(g) \forall y \in EG, \forall g \in G$, then f descends to $B\gamma$:



These subgroups are called admissible.

(Ex: $\gamma: G \rightarrow G$ given by conjugation, $\gamma_g(h) = ghg^{-1} \Rightarrow B\gamma \simeq BId$)

• Suppose that $i: H \in G$ such that $G \rightarrow G/H$ is a principal H -bundle. Then

$$BH = EG/H \xrightarrow{B_i} BG = EG/G \text{ is a bundle w/ fiber } G/H.$$

• Further assume H is normal in G . Then the SES

$$1 \rightarrow H \rightarrow G \rightarrow G/H \rightarrow 1 \text{ gives rise to a fiber sequence } BH \xrightarrow{B_i} BG \xrightarrow{B_j} B(G/H).$$

Suppose $P \rightarrow X$ is a principal G -bundle and $H \in G$ is

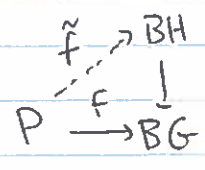
a subgroup. We say P is induced from an H -bundle if \exists a

principal H -bundle Q and an iso. $Q \times_H G \cong P$. In this case,

we say the structure group can be reduced to H .

Thm: If H is an admissible subgroup, then TFAE:

- P is induced from an H -bundle,
- $P \times_G G/H$ admits a section
- The classifying map of P lifts to BH .



Pf: Suppose $P \cong Q \times_H G$ f.c. prin. H -bundle Q . Then

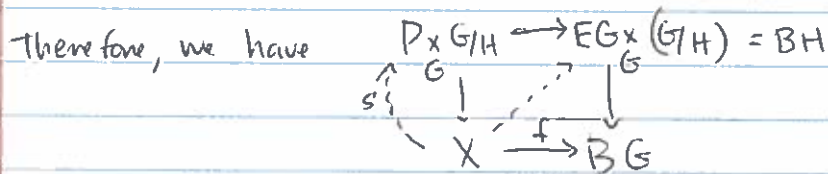
$$P \times_G (G/H) = P \times_G G \times_H X \cong Q \times_H G \times_G G \times_H X \cong Q \times_H G \times_H X \cong Q \times_H (G/H).$$

Define an H -equiv. map $x \rightarrow G/H$ by mapping to the

identity coset, which is an H -fixed point. Then $Q \times_H (x) \rightarrow Q \times_H (G/H)$

can be identified with $Q \times_H (x) \cong X \rightarrow P \times_G G/H$, a section. $a \Rightarrow b \checkmark$

If H is admissible, then $BH = EG/H = EG \times_G (G/H)$.



and $f^*(EG \times_G (G/H)) \cong P \times_G G/H$. $b \Rightarrow c \checkmark$

If $X \xrightarrow{f} BH$ exists, take $Q = f^*EH$. $c \Rightarrow a$. \square

⊛ ⊛ Reduction of structure group will be a major theme in the remainder of our course. This is such a useful concept because bundles are specified by maps into classifying spaces.

Ex: A G -bundle is trivial iff the classifying map is nullhomotopic, that is if $\{e\} \subseteq G$ is the trivial subgroup, $X \xrightarrow{\quad} BG$

Def: Let G be a topological group. A characteristic class c for G -bundles associates to each G -bundle \mathcal{E} over X , a cohomology class $c(\mathcal{E}) \in H^*X$, that is natural wrt G -bundle maps:

$$\begin{array}{ccc} E(\mathcal{E}') & \xrightarrow{f} & E(\mathcal{E}) \\ \downarrow & & \downarrow \\ Y & \xrightarrow{f} & X \end{array}$$
 is a map of G -bundles, then $f^*c(\mathcal{E}) = c(\mathcal{E}')$.

Lemma: Characteristic classes for G -bundles are in 1:1 correspondence

with elements of H^*BG .

Proof: We may restrict to principal G -bundles. Let $\begin{array}{c} EG \\ \downarrow \\ BG \end{array}$ be

the universal G -bundle, c a char. class, $\circ f: X \rightarrow BG$ classify \mathcal{E} .

Then $c(f) = f^*c(\pi)$, so c is completely determined by $c(\pi) \in H^*BG$.

Conversely, $y \in H^*BG$ gives a char. class c by $c(\pi) = y \circ \text{natural}$. \square

Ex: Stiefel-Whitney classes. $G = GL_n \mathbb{R} \cong O(n)$. A calculation

shows $H^*(BO(n); \mathbb{Z}/2) \cong \mathbb{Z}/2[w_1, w_2, \dots, w_n]$.

Ex: Chern classes. $G = U(n)$. $H^*(BU(n); \mathbb{Z}) \cong \mathbb{Z}[c_1, c_2, \dots, c_n]$.

Remark: You should think of this using the Yoneda Lemma, Natural

transformations from $[-, Y] \rightarrow F(-)$ are in 1:1 correspondence with $F(Y)$.

Before moving on, lets peek towards K-theory. We saw that n-plane bundles on X are the same as maps $X \rightarrow G_n \mathbb{R}^\infty$, or equivalently, $X \rightarrow BGL_n \mathbb{R}$. One might consider all bundles on X letting n-vary. Take $G_{\infty} \mathbb{R}^\infty = \text{colim}_n G_n \mathbb{R}^\infty = \bigcup_n G_n \mathbb{R}^\infty$, or similarly, $BGL_{\infty} \mathbb{R}$. Then $X \rightarrow G_{\infty} \mathbb{R}^\infty$ specifies an n-plane on X for some n. So $[X, G_{\infty} \mathbb{R}^\infty] = [X, BGL_{\infty} \mathbb{R}]$ gives all n-plane bundles on X, $\forall n$. For reasons we'll see later, lets work with $O(n)$ instead of $GL_n \mathbb{R}$, since $O(n)$ is compact.

Thus $[X, BGL_{\infty} \mathbb{R}] = [X, BO(\infty)] = [X, BO]$, defining $BO = BO(\infty)$.

We "define" $\tilde{K}O^0(X) = [X, BO]$, reduced real 0^{th} K-theory.

You should think about what information a map $X \rightarrow BO$ actually encodes...

V. Orientations, Characteristic Classes, - \mathbb{C} -bundles

A. Orientations

We will now introduce orientations, which will allow us to work with \mathbb{Z} -coefficients (instead of $\mathbb{Z}/2$).

Def: An orientation of a real vector space V of $\dim n > 0$

is an equiv. class of ordered basis, where $(v_1, \dots, v_n) \sim (v'_1, \dots, v'_n)$

iff the matrix A s.t. $v'_i = \sum_j A_{ij} v_j$ has $\det A > 0$.

Every vector space has 2 orientations, a \mathbb{R}^n has a canonical orientation, given by its canonical ordered basis.

In algebraic topology, we orient simplices by ordering their vertices. Let Σ^n be an n -simplex, linearly embedded in V , w/ ordered vertices $a_0 < a_1 < \dots < a_n$. Take the first vector in V to be from a_0 to a_1 , take the second to be from a_1 to a_2 , and so on. This gives an orientation of V . Notice that a choice of orientation of V corresponds to a choice of generator for $H_n(V, V_0; \mathbb{Z})$,

where $V_0 = V \setminus \{0\}$. If Δ^n denotes the std. n -simplex w/ its

canonical ordering, then choose an orientation preserving linear embedding

$\sigma: \Delta^n \rightarrow V$, sending the barycenter of Δ^n to 0 - thus

$\partial \Delta^n$ to V_0 . Then $\sigma \in Z_n(V, V_0; \mathbb{Z})$ gives rise to a preferred generator μ_V of $H_n(V, V_0; \mathbb{Z})$. You can similarly give a preferred generator μ_V of $H^n(V, V_0; \mathbb{Z})$ by forcing $\mu_V(\mu_V) = +1$.

Def. An orientation for an n -plane bundle ξ ($n > 0$) is a function which assigns an orientation to each fiber of ξ in a consistent way. That is, $\forall b_0 \in B(\xi)$, \exists bundle chart (U, φ) with $b_0 \in U$, $\varphi: \pi^{-1}(U) \xrightarrow{\cong} U \times \mathbb{R}^n$ such that $\forall b \in U$ the homomorphism $\pi^{-1}(b) \xrightarrow{\varphi|_{\pi^{-1}(b)}} \{b\} \times \mathbb{R}^n$ is orientation preserving.

This is equivalent to saying \exists n sections $s_1, \dots, s_n: U \rightarrow \pi^{-1}(U)$ s.t. $s_1(b), \dots, s_n(b)$ forms an oriented basis w/ the same orientation as $\pi^{-1}(b)$. \nearrow

The previous discussion implies that to each fiber F of ξ we have a preferred generator $\mu_F \in H^n(F, F_0; \mathbb{Z})$. Local consistency implies $\forall b \in B, \exists V \ni b$ a cohomology class $u \in H^n(\pi^{-1}(V), \pi^{-1}(V)_0; \mathbb{Z})$ s.t. \forall fiber F over V we

④ Equivalently, for an n-plane bundle $E \rightarrow X$, $\wedge^n E \rightarrow X$ is a rank 1 bundle, so put a relation on $\wedge^n E$ zero section ξ , $x \sim y \Leftrightarrow x = \lambda y$ f.s. $\lambda > 0$.
Set $\tilde{X}(E) = \wedge^n E - \xi / \sim$, then $\tilde{X}(E) \rightarrow X$ is a two sheeted cover. $\tilde{X}(E)$ is trivial iff E is orientable = a section of it is an orientation.

have $u|_{(F, F_0)} \in H^n(F, F_0; \mathbb{Z})$ coincides with u_F .

Ex: Clearly any trivial bundle is orientable, since (F, F_0)

$\cong (\mathbb{R}^n, \mathbb{R}^n - \{0\}) \cong (D^n, S^{n-1})$, for every point in the base.

Ex: The Möbius bundle is not orientable. Every section

of $\tilde{\gamma}$ has a zero.

Thm: Let ξ be an oriented n-plane bundle w/ $E = E(\xi)$.

Then $H^i(E, E_0; \mathbb{Z}) \cong 0$ $i < n$, and $H^n(E, E_0; \mathbb{Z})$ contains

a unique class u such that

$$u|_{(F, F_0)} \in H^n(F, F_0; \mathbb{Z})$$

is equal to u_F \forall fiber F . Furthermore, the correspondence

$$\begin{array}{ccc} H^k(E; \mathbb{Z}) & \xrightarrow{\cong} & H^{n+k}(E, E_0; \mathbb{Z}) \\ \downarrow \gamma & & \downarrow \gamma \cup u \end{array}$$

is an isomorphism $\forall k \in \mathbb{Z}$.

This says $H^*(E, E_0; \mathbb{Z})$ is a free $H^*(E; \mathbb{Z})$ -module on one

generator u of degree n . This leads to the Thm isomorphism

$$\begin{array}{ccc} \tilde{\varphi}: H^k(B; \mathbb{Z}) & \rightarrow & H^{n+k}(E, E_0; \mathbb{Z}) & E \\ & & & \downarrow \pi \\ \alpha \downarrow & \rightarrow & (\pi^* \alpha) \cup u & B \end{array}$$

Let ξ be an oriented n -plane bundle. Then $E \in (E, E_0)$ gives $H^*(E, E_0) \rightarrow H^*E$, denoted by $y \mapsto y|_E$. Taking the preferred generator $u \in H^n(E, E_0; \mathbb{Z})$ gives $u|_E \in H^n(E; \mathbb{Z}) \cong H^n(B; \mathbb{Z})$.

Def: The Euler class of an oriented n -plane bundle ξ is $e(\xi) \in H^n(B; \mathbb{Z})$ corresponding to $u|_E \in H^n(E; \mathbb{Z}) \cong H^n(B; \mathbb{Z})$. //

Properties: a) Naturality: $B \xrightarrow{f} B'$ covered by an orientation preserving map $\xi \rightarrow \xi'$, then $e(\xi) = f^* e(\xi')$.

b) If $-\xi$ is ξ with opposite orientation, then $e(-\xi) = -e(\xi)$.

c) if ξ is a trivial n -plane bundle $e(\xi) = 0$.

d) if n is odd, then $2e(\xi) = 0$.

(since u lies in an odd dimension, $2(u \cup u) = 0$. Now use \mathbb{Z})

e) The map $H^n(B; \mathbb{Z}) \rightarrow H^n(B; \mathbb{Z}/2)$ sends $e(\xi) \mapsto W_n(\xi)$.

f) $e(\xi \oplus \xi') = e(\xi) \cup e(\xi')$.

g) if ξ possesses a nowhere zero cross-section, then $e(\xi) = 0$. //

The Euler class is unstable in the sense that $e(\xi \oplus \varepsilon) \neq e(\xi)$, since $e(\xi \oplus \varepsilon) = 0$. This is in contrast to SW classes! But,