⊛ A <u>local cross-section</u> of $H \leq G$ is a nbhd of $eH \in G/H$ call it $\mathcal{U}$, + a map $f: \mathcal{U} \to G$ s.t. $\mathcal{U} \xrightarrow{f} G \to G/H$ is the identity on $\mathcal{U}$.

<u>Ex</u>: ~~[illegible]~~ subgps of discrete, of Lie gps, etc.

⊛ $H \leq G$ has a local cross-section $\Rightarrow$ H-admissible

Some useful facts about the classifying space functor B:

- Up to homotopy, B is a product preserving functor from top. spaces to top. spaces.
- Let $\gamma: G \to G'$ is a group hom., $EG \xrightarrow{\pi} BG$, $EG' \xrightarrow{\pi'} BG'$, universal bundles. If $f: EG \to EG'$ is any map s.t. $f(yg) = f(y)\gamma(g)$ $\forall y \in EG$, $\forall g \in G$, then f descends to $B\gamma$:

$$\begin{tikzcd} EG \arrow[r, "f"] \arrow[d, "\pi"] & EG' \arrow[d] \\ BG \arrow[r, "\bar{f}"] & BG' \end{tikzcd}$$

that is, $\bar{f} \simeq B\gamma$.

(Ex: $\gamma_g: G \to G$ given by conjugation, $\gamma_g(h) = ghg^{-1}$ $\Rightarrow$ $B\gamma_g \simeq B\mathrm{Id}$)

These subgroups are called <u>admissible</u>.

- Suppose that $i: H \leq G$ such that $G \to G/H$ is a principal H-bundle. Then

$$BH = EG/H \xrightarrow{Bi} BG = EG/G \text{ is}$$

a bundle w/ fiber G/H.

- Further assume H is normal in G. Then the SES

$$1 \to H \xrightarrow{i} G \xrightarrow{j} G/H \to 1$$ gives rise to a fiber sequence $BH \xrightarrow{Bi} BG \xrightarrow{Bj} B(G/H)$.

Suppose $P \to X$ is a principal G-bundle and $H \leq G$ is a subgroup. We say P is <u>induced from an H-bundle</u> if $\exists$ a principal H-bundle Q and an iso. $Q \times_H G \cong P$. In this case, we say the structure group can be <u>reduced</u> to H.

<u>Thm</u>: If H is an admissible subgroup, then TFAE:

a) P is induced from an H-bundle,

b) $P \times_G G/H$ admits a section

c) The classifying map of P lifts to BH

$$\begin{tikzcd} & BH \arrow[d] \\ P \arrow[r, "f"] \arrow[ur, dashed, "\tilde{f}"] & BG \end{tikzcd}$$

Pf: Suppose $P \cong Q \times_H G$ f.s. prin. H-bundle Q. Then

$$P \times_G (G/H) = P \times_G G \times_H * \cong Q \times_H G \times_G G \times_H * \cong Q \times_H G \times_H * \cong Q \times_H (G/H).$$

Define an H-equiv. map $* \longrightarrow G/H$ by mapping to the identity coset, which is an H-fixed point. Then $Q \times_H (*) \longrightarrow Q \times_H (G/H)$ can be identified with $Q \times_H (*) \cong X \longrightarrow P \times_G G/H$, a section. $a \Rightarrow b$ ✓

If H is admissible, then $BH = EG/H = EG \times_G (G/H)$.

Therefore, we have

$$\begin{tikzcd}
P \times_G G/H \arrow[r] \arrow[d] & EG \times_G (G/H) = BH \arrow[d] \\
X \arrow[u, bend left, dashed, "s"] \arrow[r, "f"] \arrow[ur, dashed] & BG
\end{tikzcd}$$

and $f^*(EG \times_G (G/H)) \cong P \times_G G/H$. $b \Rightarrow c$ ✓

If $X \xrightarrow{f} BH$ exists, take $Q = f^* EH$. $c \Rightarrow a$. ☐

(*) (*) Reduction of structure group will be a major theme in the remainder of our course. This is such a useful concept because bundles are specified by maps into classifying spaces.

Ex: A G-bundle is trivial iff the classifying map is nullhtpc, that is if $\{e\} \leq G$ is the trivial subgp,

$$\begin{tikzcd}
 & B\{e\} \arrow[d] \\
X \arrow[r] \arrow[ur, dashed] & BG
\end{tikzcd}$$

<u>Def</u>: Let G be a topological group. A <u>characteristic class</u> c for G-bundles associates to each G-bundle ξ over X, a cohomology class $c(\xi) \in H^*X$, that is natural wrt G-bundle maps:

$$\begin{array}{ccc} E(\xi') & \xrightarrow{\tilde{f}} & E(\xi) \\ \downarrow & & \downarrow \\ Y & \xrightarrow{f} & X \end{array}$$

is a map of G-bundles, then $f^* c(\xi) = c(\xi')$.

<u>Lemma</u>: Characteristic classes for G-bundles are in 1:1 correspondence with elements of H^*BG.

<u>Proof</u>: We may restrict to principal G-bundles. Let $\pi \begin{array}{c} EG \\ \downarrow \\ BG \end{array}$ be the universal G-bundle, c a char. class, & $f: X \to BG$ classify ξ.

Then $c(f) = f^* c(\pi)$, so c is completely determined by $c(\pi) \in H^*BG$.

Conversely, $y \in H^*BG$ gives a char. class c by $c(\pi) = y$ & naturality. ▨

<u>Ex</u>: Stiefel-Whitney classes. $G = GL_n\mathbb{R} \cong O(n)$. A calculation shows $H^*(BO(n); \mathbb{Z}/2) \cong \mathbb{Z}/2[w_1, w_2, \dots, w_n]$.

<u>Ex</u>: Chern classes. $G = U(n)$. $H^*(BU(n); \mathbb{Z}) \cong \mathbb{Z}[c_1, c_2, \dots, c_n]$.

<u>Remark</u>: You should think of this using the Yoneda Lemma, Natural transformations from $[-, Y] \to F(-)$ are in 1:1 correspondence with $F(Y)$.

Before moving on, lets peek towards K-theory. We saw that n-plane bundles on X are the same as maps $X \to G_n\mathbb{R}^\infty$, or equivalently, $X \to BGL_n\mathbb{R}$. One might consider all bundles on X letting n vary. Take $G_\infty\mathbb{R}^\infty = \operatorname{colim}_n G_n\mathbb{R}^\infty = \bigcup_n G_n\mathbb{R}^\infty$, or similarly, $BGL_\infty\mathbb{R}$. Then $X \to G_\infty\mathbb{R}^\infty$ specifies an n-plane on X for some n. So $[X, G_\infty\mathbb{R}^\infty] = [X, BGL_\infty\mathbb{R}]$ gives all n-plane bundles on X, $\forall n$. For reasons we'll see later, lets work with $O(n)$ instead of $GL_n\mathbb{R}$, since $O(n)$ is compact. Thus $[X, BGL_\infty\mathbb{R}] = [X, BO(\infty)] = [X, BO]$, defining $BO = BO(\infty)$. We "define" $\widetilde{KO}^0(X) = [X, BO]$, reduced real 0^{th} K-theory. You should think about what information a map $X \to BO$ actually encodes...

V. Orientations, Characteristic Classes, + $\mathbb{C}$-bundles

A. Orientations

We will now introduce orientations, which will allow us to work with $\mathbb{Z}$-coefficients (instead of $\mathbb{Z}/2$).

Def: An orientation of a real vector space V of dim $n>0$ is an equiv. class of ordered basis, where $(v_1, \dots, v_n) \sim (v_1', \dots, v_n')$ iff the matrix A s.t. $v_i' = \sum_{j} A_{ij} v_j$ has $\det A > 0$.

Every vector space has 2 orientations, + $\mathbb{R}^n$ has a canonical orientation, given by its canonical ordered basis.

In algebraic topology, we orient simplices by ordering their vertices. Let Σ^n be an n-simplex, linearly embedded in V, w/ ordered vertices $a_0 < a_1 < \cdots < a_n$. Take the first vector in V to be from a_0 to a_1, take the second to be from a_1 to a_2, and so on. This gives an orientation of V. Notice that a choice of orientation of V corresponds to a choice of generator for $H_n(V, V_0; \mathbb{Z})$, where $V_0 = V \setminus \{0\}$. ~~Even~~ If Δ^n denotes the std. n-simplex w/ its canonical ordering, then choose an orientation preserving linear embedding $\sigma : \Delta^n \to V$, sending the barycenter of Δ^n to 0 + thus

$\partial\Delta^n$ to V_0. Then $\sigma \in Z_n(V, V_0; \mathbb{Z})$ gives rise to a preferred generator μ_V of $H_n(V, V_0; \mathbb{Z})$. You can similarly give a preferred generator u_V of $H^n(V, V_0; \mathbb{Z})$ by forcing $u_V(\mu_V) = +1$.

Def: An orientation for an n-plane bundle ξ $(n>0)$ is a function which assigns an orientation to each fiber of ξ in a consistent way. That is, $\forall\ b_0 \in B(\xi)$, $\exists$ bundle chart $(\mathcal{U}, \varphi)$ with $b_0 \in \mathcal{U}$, $\varphi: \pi^{-1}(\mathcal{U}) \xrightarrow{\cong} \mathcal{U} \times \mathbb{R}^n$ such that $\forall\ b \in \mathcal{U}$ the homomorphism $\pi^{-1}(b) \xrightarrow{\varphi|_{\{b\}}} \{b\} \times \mathbb{R}^n$ is orientation preserving. This is equivalent to saying $\exists$ n sections $s_1, \dots, s_n: \mathcal{U} \to \pi^{-1}(\mathcal{U})$ s.t. $s_1(b), \dots, s_n(b)$ forms an oriented basis w/ the same orientation as $\pi^{-1}(b)$.

The previous discussion implies that to each fiber F of ξ we have a preferred generator $u_F \in H^n(F, F_0; \mathbb{Z})$. Local consistency implies $\forall\ b \in B$, $\exists\ V \ni b$ a cohomology class $u \in H^n(\pi^{-1}(V), \pi^{-1}(V)_0; \mathbb{Z})$ s.t. $\forall$ fiber F over V we

(*) Equivalently, for an n-plane bundle $E \to X$, $\Lambda^n E \to X$ is a rank 1 bundle, so put a relation on $\Lambda^n E \setminus \{\text{zero section}\}$, $x \sim y \Leftrightarrow x = \lambda y$ f.s. $\lambda > 0$, set $\tilde{X}(E) = \Lambda^n E \setminus \{0\}/\sim$, then $\tilde{X}(E) \to X$ is a two sheeted cover. $\tilde{X}(E)$ is trivial iff E is orientable $\Leftrightarrow$ a section of it is an orientation.

have $u|_{(F,F_0)} \in H^n(F,F_0;\mathbb{Z})$ coincides with u_F.

Ex: Clearly any trivial bundle is orientable, since $(F,F_0) \cong (\mathbb{R}^n, \mathbb{R}^n \setminus \{0\}) \simeq (D^n, S^{n-1})$, for every point in the base.

Ex: The Möbius bundle is not orientable. Every section of γ^1 has a zero.

Thm: Let ξ be an oriented n-plane bundle w/ $E = E(\xi)$. Then $H^i(E,E_0;\mathbb{Z}) \cong 0$ $i < n$, and $H^n(E,E_0;\mathbb{Z})$ contains a unique class u such that

$$u|_{(F,F_0)} \in H^n(F,F_0;\mathbb{Z})$$

is equal to u_F $\forall$ fiber F. Furthermore, the correspondence

$$\begin{aligned} H^k(E;\mathbb{Z}) &\longrightarrow H^{n+k}(E,E_0;\mathbb{Z}) \\ y &\longmapsto y \cup u \end{aligned}$$

is an isomorphism $\forall k \in \mathbb{Z}$. //

This says $H^*(E,E_0;\mathbb{Z})$ is a free $H^*(E;\mathbb{Z})$-module on one generator u of degree n. This leads to the Thom isomorphism

$$\begin{aligned} \Phi : H^k(B;\mathbb{Z}) &\longrightarrow H^{n+k}(E,E_0;\mathbb{Z}) \\ x &\longmapsto (\pi^* x) \cup u \end{aligned} \qquad \begin{array}{c} E \\ \downarrow \pi \\ B \end{array}$$

Let ξ be an oriented n-plane bundle. Then $E \subseteq (E, E_0)$ gives $H^*(E, E_0) \to H^*E$, denoted by $y \mapsto y|E$. Taking the preferred generator $u \in H^n(E, E_0; \mathbb{Z})$ gives $u|_E \in H^n(E; \mathbb{Z}) \cong H^n(B; \mathbb{Z})$

<u>Def:</u> The <u>Euler class</u> of an oriented n-plane bundle ξ is $e(\xi) \in H^n(B; \mathbb{Z})$ corresponding to $u|_E \in H^n(E; \mathbb{Z}) \cong H^n(B; \mathbb{Z})$. //

<u>Properties:</u> a) <u>Naturality</u>: $B \xrightarrow{f} B'$ covered by an orientation preserving map $\xi \to \xi'$, then $e(\xi) = f^* e(\xi')$.

b) If $-\xi$ is ξ with <u>opposite orientation</u>, then $e(-\xi) = -e(\xi)$.

c) if ξ is a trivial n-plane bundle $e(\xi) = 0$.

d) if n is odd, then $2e(\xi) = 0$.

(since u lies in an odd dimension, $2(u \cup u) = 0$. Now use Φ)

e) The map $H^n(B; \mathbb{Z}) \to H^n(B; \mathbb{Z}/2)$ sends $e(\xi) \mapsto w_n(\xi)$.

f) $e(\xi \oplus \xi') = e(\xi) \cup e(\xi')$.

g) if ξ possesses a nowhere zero cross-section, then $e(\xi) = 0$. //

The Euler class is unstable in the sense that $e(\xi \oplus \varepsilon) \neq e(\xi)$, since $e(\xi \oplus \varepsilon) = 0$. This is in contrast to SW classes! But,