II. Vector bundles

A. The tangent bundle & constructions

<u>Def:</u> Let B be a topological space. A <u>real vector bundle ξ over B</u> consists of:

(a) a topological space $E = E(\xi)$, called the <u>total space</u>,

(b) a continuous map $\pi : E \to B$, called the <u>projection map</u>,

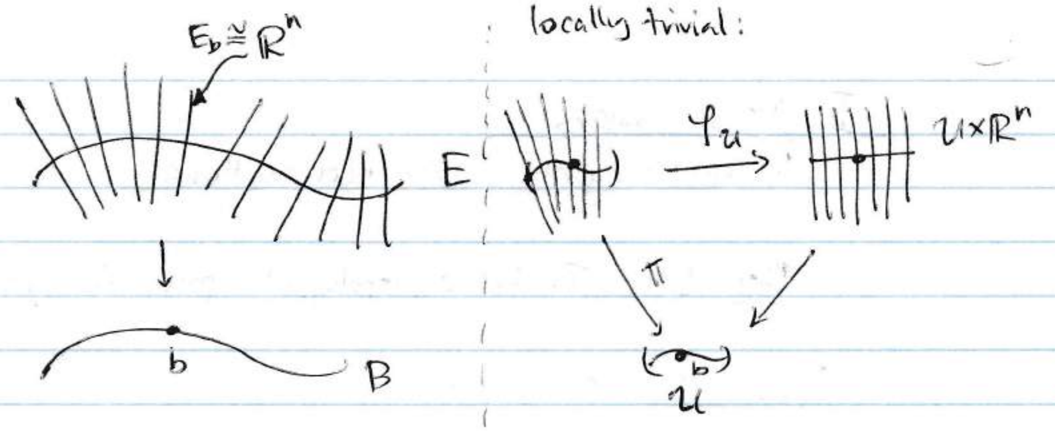
and (c) $\forall\, b \in B$, a $\mathbb{R}$-vector space structure on $\pi^{-1}(b) = E_b$,

which is <u>locally trivial</u>: $\forall\, b \in B$, $\exists$ nbhd $\mathcal{U} \ni b$, an integer $n \geq 0$, and a homeomorphism $\varphi_{\mathcal{U}} : \pi^{-1}(\mathcal{U}) \xrightarrow{\cong} \mathcal{U} \times \mathbb{R}^n$, so that $\varphi_{\mathcal{U}}\big|_{E_b} : \pi^{-1}(\{b\}) \longrightarrow \{b\} \times \mathbb{R}^n \cong \mathbb{R}^n$ is an isomorphism.

The vector space $\pi^{-1}(b)$ is called the <u>fiber over b</u>. It is sometimes denoted F_b, $F_b(\xi)$, $E_b(\xi)$ (& its never empty). The dimension n is the <u>rank</u> of the bundle, and is locally constant w.r.t. b. If it is constant, then we have an <u>n-plane bundle</u>. The local triviality is best thought of as: $\forall\, b \in B, \exists\, \mathcal{U} \ni b$, $n \in \mathbb{N}$, & $\varphi_{\mathcal{U}}$ s.t.

$$\begin{tikzcd} \pi^{-1}(\mathcal{U}) \arrow[r, "\varphi_{\mathcal{U}}", "\cong"'] \arrow[d, "\pi"] & \mathcal{U} \times \mathbb{R}^n \arrow[dl, "\text{projection}"] \\ \mathcal{U} & \end{tikzcd}$$

Pictorially:



<u>Def</u>: A pair $(\mathcal{U}, \varphi_{\mathcal{U}})$ is called a <u>bundle chart</u>. If (X, φ_X) is a bundle chart, then ξ is called a <u>trivial bundle</u>.

One can also define a <u>smooth vector bundle</u> by requiring E & B to be smooth manifolds, π to be a smooth submersion, and each bundle chart a diffeomorphism. We'll see an equivalent definition later in terms of bundle atlases.

<u>Def</u>: Suppose ξ and ξ' are vector bundles over B. A continuous map $f: E(\xi) \to E(\xi')$, written $f: \xi \to \xi'$ for short, is called a <u>bundle homomorphism</u> if

$$\begin{tikzcd} E(\xi) \arrow[rr, "f"] \arrow[dr, "\pi"'] & & E(\xi') \arrow[dl, "\pi'"] \\ & B & \end{tikzcd}$$

commutes and $f_b: E_b \to E'_b$ is linear. If each $f_b: E_b \to E'_b$ is an isomorphism, we say f is an <u>isomorphism</u>.

<u>Ex</u>: For any space B, consider the bundle $\begin{array}{c} B\times\mathbb{R}^n \xrightarrow{\pi} B \\ (b,v)\longmapsto b \end{array}$.

Define a vector space structure on the fibers by the obvious formula.

This is the <u>trivial bundle over B</u> & sometimes denoted $\mathcal{E}^n_B$. Any other $\mathbb{R}^n$-bundle over B is trivial iff its isomorphic to $\mathcal{E}^n_B$.

<u>Ex</u>: Let M be a smooth manifold. The <u>tangent bundle of M</u>, denoted TM (or T_M), has total space (also denoted TM) $TM=\{(m,v)\mid m\in M, v\in T_mM\}$. The projection map $\pi: TM\to M$ sends $(m,v)\mapsto m$.

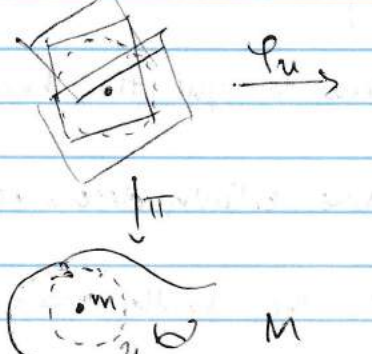
The vector space structure is the obvious: $\lambda_1\cdot(m,v_1)+\lambda_2\cdot(m,v_2) = (m, \lambda_1 v_1+\lambda_2 v_2)$, for $\lambda_i\in\mathbb{R}$, $v_i\in T_mM$. The local triviality comes from the atlas on M: $\forall\, m\in M$, take a chart $(h,\mathcal{U})$

$$\begin{array}{c} TM \\ \downarrow \\ \mathbb{R}^n \xleftarrow{h} \mathcal{U}\subseteq M \end{array}$$

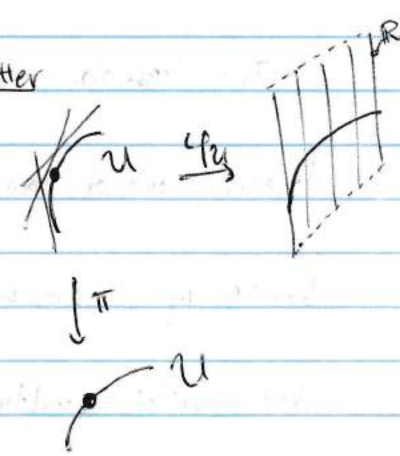
write $h=(h_1,\dots,h_n)$

and $\varphi_{\mathcal{U}}: \pi^{-1}(\mathcal{U})\to \mathcal{U}\times\mathbb{R}^n$ to be

$$T_mM\ni X\longmapsto (m, X(h_1), X(h_2),\dots,X(h_n)).$$



or <u>better</u>



<u>Def:</u> A manifold is ~~is~~ <u>parallelizable</u> if TM is trivial.

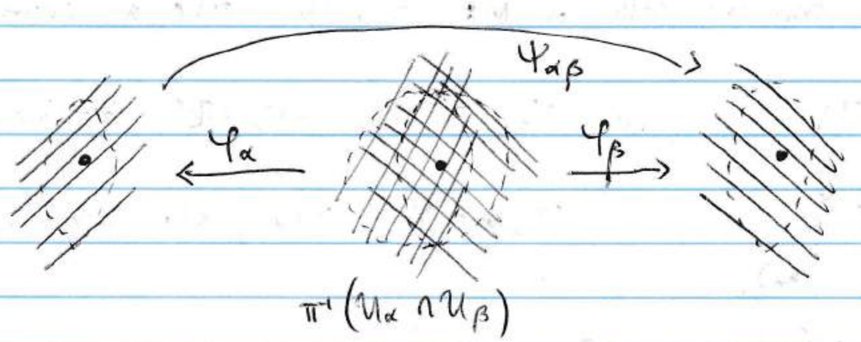
<u>Ex:</u> S^1 is parallelizable. S^2 is not parallelizable.

<u>Def:</u> Let ξ be real v.b. over B. A set $\{(\varphi_\alpha, U_\alpha)\}$ of bundle charts is a <u>bundle atlas for ξ</u> if $\bigcup_\alpha U_\alpha = B$.

The continuous mappings on overlaps give rise to

$$\varphi_\beta \circ \varphi_\alpha^{-1} : U_\alpha \cap U_\beta \times \mathbb{R}^n \longrightarrow U_\alpha \cap U_\beta \times \mathbb{R}^n$$
$$(p, v) \longmapsto (p, \varphi_{\beta\alpha}(p) \cdot v).$$

By the definition of vector bundle, this composition restricted to $b \in U_\alpha \cap U_\beta$ is a linear isomorphism of $\mathbb{R}^n$. The maps $\varphi_{\beta\alpha} : U_\beta \cap U_\alpha \longrightarrow GL_n\mathbb{R}$ are called <u>transition functions</u> of ξ.



$\pi^{-1}(U_\alpha \cap U_\beta)$

The transition functions tell one how to glue the bundle together out of local data. In fact, these entirely determine the bundle up to isomorphism. This underlies the bundle classification that we've building towards.

Ex: Let $E(\gamma_n^1)$ be the subset of $\mathbb{R}P^n \times \mathbb{R}^{n+1}$ consisting of all pairs $([x], v)$ such that $v = \lambda x$ f.s. $\lambda \in \mathbb{R}$. Define $\pi: E(\gamma_n^1) \to \mathbb{R}P^n$ by $\pi([x], v) = [x]$. The fibers $\pi^{-1}([x])$ are identified with $\{([x], \lambda x) \mid \lambda \in \mathbb{R}\}$, + hence is the line in $\mathbb{R}^{n+1}$ through $\pm x \in S^n$. This vector bundle is the <u>canonical line bundle over $\mathbb{R}P^n$</u>, denoted γ_n^1. Locally trivial: take $\mathcal{U} \subseteq S^n$ open so that it doesn't contain antipodal pts, + let $\mathcal{U}_1$ denote its projection to $\mathbb{R}P^n$. Define $\varphi_{\mathcal{U}_1}: \pi^{-1}(\mathcal{U}_1) \to \mathcal{U}_1 \times \mathbb{R}$ by sending $([x], \lambda x) \mapsto ([x], \lambda)$. This well-defined (and a homeomorphism) by our choice of $\mathcal{U}, \mathcal{U}_1$.

* Def: A <u>section</u> of a vector bundle ξ over B is a continuous function $s: B \to E(\xi)$ such that $\pi \circ s = id_B$.

This definition implies $s(b) \in \pi^{-1}(b)$ $\forall b$. A section is <u>no-where zero</u> or <u>non-vanishing</u> if $s(b) \neq 0 \in \pi^{-1}(b)$ $\forall b$.

<u>Prop</u>: γ_n^1 has no non-vanishing sections.

<u>Pf.</u> Let $s: \mathbb{R}P^n \to E(\gamma_n^1)$ be any section and

Consider $S^n \xrightarrow{prj} \mathbb{RP}^n \xrightarrow{s} E(\gamma_n^1)$, sending $x \longmapsto ([x], \lambda(x)\cdot x)$

for some continuous real-valued function $\lambda(x)$. Furthermore,

$-x \longmapsto ([x], -\lambda(-x)\cdot x)$, implying $-\lambda(-x) = \lambda(x)$. ~~So~~ Since

$\lambda : S^n \to \mathbb{R}$ is cts and S^n-connected, the intermediate

value thm $\Rightarrow$ $\lambda(x_0)=0$ f.s. $x_0 \in S^n$ $\Rightarrow$ $s([x_0]) = ([x_0], 0)$. $\boxtimes$

Clearly, a trivial vector bundle admits non-vanishing

sections: $X \times \mathbb{R}^n \underset{s}{\overset{\pi}{\rightleftarrows}} X$; take $s(x) = (x, (1,1,\dots,1))$.

<u>Cor</u>: The canonical line bundle is not trivial $\forall n$.

Let's take a closer look at the canonical line bundle

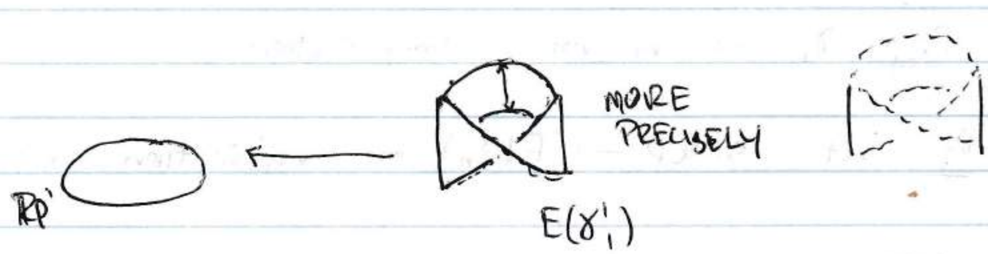
over $\mathbb{RP}^1 \cong S^1$. Any pt of $E(\gamma_1^1)$ looks like $([x], v)$, or

$([\cos\theta, \sin\theta], \lambda(\cos\theta, \sin\theta))$ for $\theta \in [0,\pi]$, $\lambda \in \mathbb{R}$. This fails

to be unique at $\theta = 0, \pi$, because $([\cos 0, \sin 0], \lambda(\cos 0, \sin 0))$

$= ([\cos\pi, \sin\pi], -\lambda(\cos\pi, \sin\pi))$. Therefore, $E(\gamma_1^1) = [0,\pi]\times\mathbb{R} / (0,\lambda) \sim (\pi, -\lambda)$.

The total space of γ_1^1 is an (open) Möbius band:



<u>Lemma</u>: Suppose ξ & η are vector bundles over B & $f: \xi \to \eta$ is a bundle map. If $f_b: E_b(\xi) \to E_b(\eta)$ is an isomorphism $\forall b \in B$, so f is a bundle isomorphism, then $f: E(\xi) \to E(\eta)$ is a homeomorphism.

We'll skip this proof, although you should work out the details.

<u>Def</u>: A collection of sections $\{s_1, \dots, s_k\}$ of a vector bundle ξ are <u>nowhere dependent</u> if $\forall b \in B(\xi)$, $s_1(b), s_2(b), \dots, s_k(b)$ are linearly independent.

<u>Thm</u>: An $\mathbb{R}^n$-bundle ξ is trivial iff ξ admits n nowhere dependent sections.

<u>Pf</u>: Given n nowhere dependent sections $\{s_1, \dots, s_n\}$ of ξ, define $f: \mathcal{E}^n_B \to E$ by $f(b, v_1, \dots, v_n) = v_1 \cdot s_1(b) + \dots + v_n s_n(b)$. This map is a fibewise isomorphism, since $\{s_1(b), \dots, s_n(b)\}$ form a basis for $E_b(\xi)$. Given a trivial bundle over B, define $s_i(b) = (b, 0, \dots, 0, \overset{i^{\text{th}}\text{ spot}}{1}, 0, \dots, 0)$ $\forall b \in B$. These are clearly nowhere dep. ☒

In order for us to explore the geometry of the tangent

$\xi = \begin{array}{c} E \\ \downarrow \pi \\ B \end{array}$

bundle, we need some auxiliary definitions.

Def: Let ξ be a vector bundle and X an arbitrary topological space. Given any map $f: X \to B$, construct the pullback bundle, or induced bundle, as follows: take the pullback

$$\begin{array}{ccc} f^*\xi & \xrightarrow{\hat{f}} & E \\ \pi' \downarrow & & \downarrow \pi \\ X & \xrightarrow{f} & B \end{array}$$

so that $f^*\xi = \{(x,e) \mid f(x) = \pi(e)\} \subseteq X \times E$.

The map $\pi'(x,e) = x$, the vector space structure on $\pi'^{-1}(x)$ is the usual, and so $\hat{f}: E_b(f^*\xi) \xrightarrow{\cong} E_{f(b)}(\xi)$. Show this is locally trivial.

Q: What is the pullback of a trivial bundle?

Def: Suppose ξ & η are two vector bundles over B & let $\Delta: B \to B \times B$ be the diagonal map. The bundle $\Delta^*(\xi \times \eta)$ over B is the Whitney sum of ξ & η, denoted $\xi \oplus \eta$.

Each fiber of $\xi \oplus \eta$ is isomorphic to $\xi_b \oplus \eta_b$.

Def: The cartesian product of ξ_1 over B_1 and ξ_2 over B_2 is defined to be $\xi_1 \times \xi_2 : E_1 \times E_2 \xrightarrow{\pi_1 \times \pi_2} B_1 \times B_2$. Clearly, the fibers, and vector space structure are the obvious things.