using the inner product on $\mathbb{R}^{n+1}$

$$E\left((\gamma^1_n)^\perp\right) = \{([x],v) \mid v \text{ is perpendicular to } \{x\}\} \subseteq \mathbb{RP}^n \times \mathbb{R}^{n+1}.$$

Then $w((\gamma^1_n)^\perp) = 1 + a + a^2 + \dots + a^n$.

Pf: Since $\gamma^1_n \oplus (\gamma^1_n)^\perp \cong \varepsilon^{n+1}$, we have $w(\gamma^1_n)^\perp = \bar{w}(\gamma^1_n)$

$= (1+a)^{-1} = 1+a+a^2+\dots+a^n = 1+w_1+w_1^2+\dots+w_1^n$. ▨

This is our first example where all the SW classes are not zero:

$w_i((\gamma^1_n)^\perp) \neq 0 \in H^i(\mathbb{RP}^n;\mathbb{Z}/2)\ \forall i$. From the homework,

we have $T\mathbb{RP}^n \oplus \varepsilon^1 \cong T\mathbb{RP}^n \oplus \hom(\gamma^1_n,\gamma^1_n) \cong \underbrace{\gamma^1_n \oplus \dots \oplus \gamma^1_n}_{n+1}$.

Therefore $w(T\mathbb{RP}^n) = w(\mathbb{RP}^n) = w(T\mathbb{RP}^n \oplus \varepsilon^1) = (1+a)^{n+1}$.

Hence, to compute $w_i(\mathbb{RP}^n)$, we need to compute $\binom{n+1}{i} \bmod 2$.

For example, $w(\mathbb{RP}^2) = 1+a+a^2 = 1+w_1+w_1^2$,

$w(\mathbb{RP}^3) = 1$

$w(\mathbb{RP}^4) = 1+w_1+w_1^4$, ... Hence, we get:

<u>Lemma (Stiefel)</u>: $w_i(\mathbb{RP}^n) = 0\ \forall i>0$ iff $n+1$ is a power of 2.

Furthermore, the only projective spaces which <u>can</u> be parallelizable are $\mathbb{RP}^1, \mathbb{RP}^3, \mathbb{RP}^7, \mathbb{RP}^{15}, \dots$ ($\mathbb{RP}^{15}$ isn't it turns out)

<u>Pf</u>: We have $(x+y)^2 \equiv x^2+y^2 \bmod 2$, so $(1+a)^{2^r} \equiv 1+a^{2^r}$.

$a^{n+1} \in H^{n+1}(\mathbb{RP}^n) \cong 0$.

Hence, if $n+1 = 2^r$, then $w(\mathbb{RP}^n) = (1+a)^{n+1} = 1 + a^{n+1} = 1$.

On the other hand, if $n+1 = 2^r \cdot m$ w/ $m > 1$ & m-odd, then

$$w(\mathbb{RP}^n) = (1+a)^{n+1} = (1+a)^{2^r \cdot m} = 1 + m \cdot a^{2^r} + \frac{m(m-1)}{2} a^{2 \cdot 2^r} + \dots \neq 1,$$

since $2^r < n+1$. $\boxed{}$

Thm [Stiefel]: Suppose there exists a bilinear product $p: \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ with no zero divisors. Then $\mathbb{RP}^{n-1}$ is parallelizable, & hence n must be a power of 2.

(Note: this operation need not be associative, or have an identity).

Pf: Let $b_1, b_2, \dots, b_n$ be the std. basis for $\mathbb{R}^n$. Then the map $y \mapsto p(y, b_1)$ defines an isomorphism $\mathbb{R}^n \to \mathbb{R}^n$.

Define a linear transformation $v_i : \mathbb{R}^n \to \mathbb{R}^n$ by $v_i(p(y, b_1)) = p(y, b_i)$.

Then $v_1(x), \dots, v_n(x)$ are linearly independent and $v_1(x) = x$.

The maps $v_2, \dots, v_n$ give rise to $(n-1)$-linearly independent sections of $T\mathbb{RP}^{n-1} \cong \hom(\gamma^1_{n-1}, (\gamma^1_{n-1})^\perp)$. (by Hw)

Explicitly, given a line L through the origin, so $L = [\pm x]$ $x \in S^{n-1}$, define $\bar{v}_i : L \to L^\perp$; given $\lambda x \in L$, $\bar{v}_i(\lambda x) := v_i(\lambda x)$ under the orthogonal projection $\mathbb{R}^n \longrightarrow L^\perp$. Clearly $\bar{v}_1 = 0$, but $\bar{v}_2, \dots, \bar{v}_n$ are linearly independent. Thus $T\mathbb{RP}^{n-1}$ is trivial. $\boxtimes$

Therefore, real division algebras $\mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ only exist when $\mathbb{RP}^{n-1}$ is parallelizable, which means n must be a power of 2. We know, $\mathbb{R}$, $\mathbb{R}^2$, $\mathbb{R}^4$, & $\mathbb{R}^8$ are real-division algebras, but no more exist due to work of Bott-Milnor, Kervaire, & Adams (b/c $T\mathbb{RP}^{15}$, $T\mathbb{RP}^{31}$, ... aren't parallelizable.)

Finally, we end w/ Stiefel-Whitney numbers. Recall that for every smooth closed n-manifold M, there is a class $[M] \in H_n(M; \mathbb{Z}/2)$ called the <u>fundamental class of M</u>. Therefore, given any $v \in H^n(M; \mathbb{Z}/2)$, we can evaluate or pair $v([M]) = \langle v, [M] \rangle \in \mathbb{Z}/2$. Let $r_1, r_2, \dots, r_n$ be non-negative integers such that $r_1 + 2r_2 + 3r_3 + \dots + nr_n = n$. Corresponding to any vector bundle ξ, we form $w_1(\xi)^{r_1} w_2(\xi)^{r_2} \cdots w_n(\xi)^{r_n} \in H^n(B(\xi); \mathbb{Z}/2)$. This is of most interest to us when $\xi = TM$.

<u>Def</u>: The (mod 2) number $\cancel{\langle w_1(\xi)^{r_1} \cdots w_n(\xi)^{r_n}, [M] \rangle}$ $\langle w_1(TM)^{r_1} \cdots w_n(TM)^{r_n}, [M] \rangle =$

$\left(w_1(TM)^{r_1}\cdots w_n(TM)^{r_n}\right)[M]$ is called the <u>Stiefel-Whitney number of M associated to $(r_1, r_2, \dots, r_n)$</u>.

We're interested in all possible Stiefel-Whitney numbers of a manifold.

<u>Ex</u>: $\mathbb{R}P^n$, n-even. $w_n(\mathbb{R}P^n) = \binom{n+1}{n} a^n = (n+1)a^n \neq 0$.

Thus $w_n[\mathbb{R}P^n] \neq 0$. $w_1(\mathbb{R}P^n) = \binom{n+1}{1} a = (n+1)\cdot a \neq 0 \Rightarrow w_1^n[\mathbb{R}P^n] \neq 0$.

For $n = 2^j$, $w(\mathbb{R}P^n) = 1 + a + a^n$, hence all other SW numbers are zero. //

<u>Ex</u>: $\mathbb{R}P^n$, n-odd. write $n = 2k-1$ $w(\mathbb{R}P^n) = (1+a)^{2k} = (1+a^2)^k$. Therefore, $w_j(\mathbb{R}P^n) = 0$ for j-odd. Notice that every monomial $(r_1, r_2, \dots, r_n)$ of total dimension $r_1 + 2r_2 + \dots + nr_n = 2k-1$ must contain a non-zero r_j w/ j-odd.

Therefore, $w_j = 0$ & so all Stiefel-Whitney numbers vanish for odd projective spaces. //

<u>Thm [Pontrjagin]</u>: If M is a smooth, compact manifold, then all Stiefel-Whitney numbers of ∂M are zero.

<u>Pf</u>: Let $\dim M = n$. Then $H_n(M, \partial M; \mathbb{Z}/2) \cong \mathbb{Z}/2$, gen'd by $[M]$ the fundamental class of M & $[\partial M]$ for the gen of $H_{n-1}(\partial M; \mathbb{Z}/2)$.

Equip M with a Riemannian metric. Then there is a unique outward normal vector field along ∂M, which trivializes the normal bundle $\nu_{\partial M}$. Thus $TM|_{\partial M} \cong T(\partial M) \oplus \nu_{\partial M} \cong T(\partial M) \oplus \varepsilon^1$.

Thus $w_i(\partial M) = \overset{=(w_i TM)|_{\partial M}}{w_i(TM|_{\partial M})}$. So in the long exact sequence,

$$\cdots \to H^{n-1}(M;\mathbb{Z}/2) \to H^{n-1}(\partial M;\mathbb{Z}/2) \xrightarrow{\delta} H^n(M,\partial M;\mathbb{Z}/2) \to \cdots$$
$$w_i(TM) \longmapsto w_i(\partial M)$$

we have $\delta(w_1^{r_1} w_2^{r_2} \cdots w_n^{r_n}) = 0$. On the other hand, in homology:

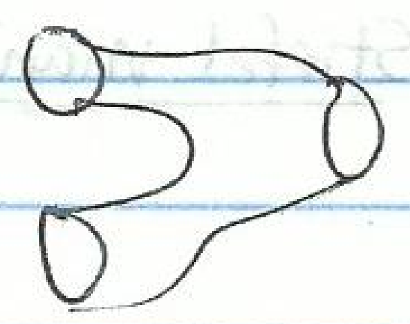
$$\cdots \to H_n(M,\partial M;\mathbb{Z}/2) \xrightarrow{\partial} H_{n-1}(\partial M;\mathbb{Z}/2) \to \cdots, \text{ i.e. } \partial[M] = [\partial M].$$
$$[M] \longmapsto [\partial M]$$

Therefore, $\langle w_1^{r_1} \cdots w_n^{r_n}, [\partial M] \rangle = \langle w_1^{r_1} \cdots w_n^{r_n}, \partial[M] \rangle$

$$= \langle \delta\, w_1^{r_1} \cdots w_n^{r_n}, [M] \rangle = 0. \qquad \blacksquare$$

Thm [Thom]: If all Stiefel-Whitney numbers of M are zero, then $M = \partial K$ for some smooth, cpt manifold K.

Def: Two smooth closed n-manifolds M_1, M_2 are cobordant iff $M_1 \amalg M_2 = \partial K$, K - smooth, cpt $(n+1)$-manifold.



$S^1 \amalg S^1 \sim S^1$

$S^1 \sim S^1$

Cor: Stiefel-Whitney numbers are cobordism invariants!