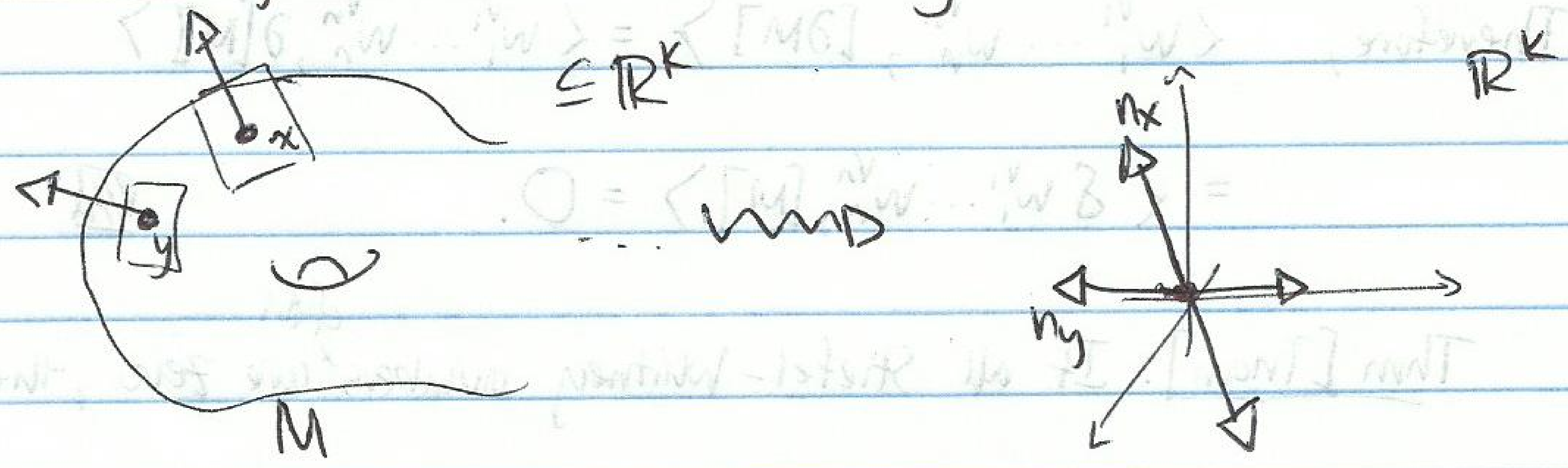
B. Universal bundles

Given a hypersurface / manifold embedded in Euclidean space $M^k \subseteq \mathbb{R}^{k+1}$, the Gauss map sends each pt. of M to its unit normal vector, giving $M \xrightarrow{n} S^k$. This map depends on orientations (The sign of the normal, out vs in). Equivalently, this gives a map to $\mathbb{RP}^k$, sending each pt to the line determined by the normal direction, then translated to the origin under the embedding



We aim to generalize this construction.

Def: An n-frame in $\mathbb{R}^{n+k}$ is an n-tuple of linearly independent vectors in $\mathbb{R}^{n+k}$. The collection of all n-frames is an open subset of $\mathbb{R}^{n+k} \times \dots \times \mathbb{R}^{n+k}$, called the Stiefel manifold & denoted $V_n(\mathbb{R}^{n+k})$. //

Alternatively, one can define $V_n^O(\mathbb{R}^{n+k})$ to be orthonormal n-frames.

$V_n^0(\mathbb{R}^{n+k})$ & $V_n(\mathbb{R}^{n+k})$ are homotopy equivalent Gram-Schmidt (in fact, this gives V_n^0 as a deformation retract of V_n).

<u>Def</u>: The <u>Grassmann manifold</u> $G_n(\mathbb{R}^{n+k})$ is the set of all n-dimensional planes through the origin of $\mathbb{R}^{n+k}$. As a space, its topologized via the quotient map $V_n\mathbb{R}^{n+k} \xrightarrow{\text{span}} G_n(\mathbb{R}^{n+k})$. //

In fact, the commutative diagram

$$\begin{tikzcd}
V_n^0(\mathbb{R}^{n+k}) \arrow[r, phantom, "\subseteq"] \arrow[dr, "\text{span}"'] & V_n(\mathbb{R}^{n+k}) \arrow[r, "\text{Gram-Schmidt}"] \arrow[d, "\text{span}"] & V_n^0(\mathbb{R}^{n+k}) \arrow[dl, "\text{span}"] \\
& G_n(\mathbb{R}^{n+k}) &
\end{tikzcd}$$

shows using V_n^0 or V_n to define G_n gives the same topology.

Further, notice that $G_1\mathbb{R}^{n+1} \cong \mathbb{RP}^n$.

<u>Lemma</u>: $G_n(\mathbb{R}^{n+k})$ is a compact smooth manifold of dimension nk. The correspondence $X \to X^\perp$, sending an n-plane to its orthogonal k-plane, gives a homeomorphism $G_n\mathbb{R}^{n+k} \cong G_k\mathbb{R}^{n+k}$.

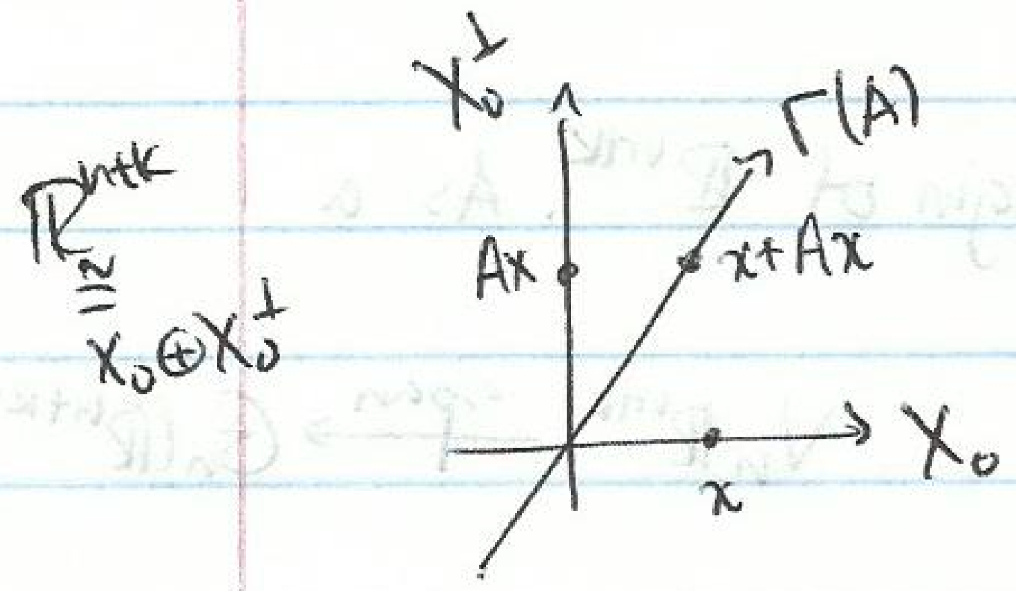
<u>Pf</u>: Its straightforward to show Hausdorff (MS p57). We'll prove every pt. has a Euclidean nbhd. Let $X_0 \in G_n\mathbb{R}^{n+k}$. Then $\mathbb{R}^{n+k} \cong X_0 \oplus X_0^\perp$. To simplify the proof, we prove the following.

$\pi : X_0 \oplus X_0^\perp \to X_0$

<u>Claim</u>: Every elt of $\{ Y \in G_n(\mathbb{R}^{n+k}) \mid \pi(Y) = X_0 \} = \{ Y \in G_n\mathbb{R}^{n+k} \mid Y \cap X_0^\perp = \{0\} \}$

can be written uniquely as $\Gamma(A)$, for some linear transform $A : X_0 \to X_0^\perp$,

where $\Gamma(A) = \{ x + Ax \mid x \in X_0 \} \subseteq \mathbb{R}^{n+k}$. //



"Every n-plane intersecting $X_0^\perp$ in the origin is the graph of a linear transformation, & vice-versa".

<u>Pf of claim</u>: Let $Y \subseteq \mathbb{R}^{n+k}$ such that $Y \cap X_0^\perp = \{0\}$.

Every $y \in Y \subseteq \mathbb{R}^{n+k} \cong X_0 \oplus X_0^\perp$ can be written uniquely as

$y = \alpha + \beta$, $\alpha \in X_0$, $\beta \in X_0^\perp$. Set $A\alpha = \beta$. A is well-defined

since $y_1 = \alpha + \beta_1$ & $y_2 = \alpha + \beta_2 \Rightarrow \underbrace{\beta_1 - \beta_2}_{\in X_0^\perp} = \underbrace{y_1 - y_2}_{\in Y} \in Y \cap X_0^\perp = \{0\}$.

For the other direction, let $A : X_0 \to X_0^\perp$ be linear. Then

$x + Ax \in \Gamma(A) \cap X_0^\perp \Rightarrow x \in X_0^\perp$, since $Ax \in X_0^\perp$. But

$x \in X_0 \cap X_0^\perp = \{0\}$, so $x = 0$ & $\Gamma(A) \cap X_0^\perp = \{0\}$. // Pf of claim.

Given $X_0 \in G_n\mathbb{R}^{n+k}$, define a nbhd $\mathcal{U}$ by

$$\mathcal{U} = \{ Y \in G_n\mathbb{R}^{n+k} \mid Y \cap X_0^\perp = \{0\} \} \subseteq G_n\mathbb{R}^{n+k}$$

which is an open subset. By the claim, to each $Y \in \mathcal{U}$

we associate $A = A(Y) : X_0 \to X_0^\perp$. Thus we can think of

(linear trans)

$A : \mathcal{U} \longrightarrow \hom(X_0, X_0^\perp) \cong \mathbb{R}^{nk}$. We'll show A is a homeo.

First, if $\{x_1, x_2, \ldots, x_n\}$ is an orthonormal basis for X_0, then each $Y \in \mathcal{U}$ has a unique basis $\{y_1, \ldots, y_n\}$ such that $\pi(y_1) = x_1$, $\pi(y_2) = x_2, \ldots, \pi(y_n) = x_n$. Hence $\{y_i\}$ vary continuously on Y. Furthermore, the identity $y_i = x_i + A(Y) \cdot x_i$, implies $A(Y) \cdot x_i \in X_0^\perp$ varies ctsly on Y & hence A is continuous. Finally, the identity shows $\{y_1, \ldots, y_n\}$ depend ctsly on $A(Y)$ & thus Y depends ctsly on $A(Y)$. Hence A^{-1} is continuous. ▨

(See MS p. 59} for $X \mapsto X^\perp$ homeo).

Def: The <u>canonical n-plane bundle over $G_n(\mathbb{R}^{n+k})$</u>, denoted γ^n_{n+k}, is constructed as follows. Let

$$E(\gamma^n_{n+k}) = \{(X, v) \mid X \in G_n(\mathbb{R}^{n+k}), v \in X\} \subseteq G_n\mathbb{R}^{n+k} \times \mathbb{R}^{n+k},$$

and then $E(\gamma^n_{n+k}) \xrightarrow{\pi} G_n(\mathbb{R}^{n+k})$, $(X, v) \longmapsto X$. //

The fibers naturally have the structure of an n-dim'l $\mathbb{R}$-v.s.

$\pi^{-1}(X) = \{(X, v) \mid v \in X \cong \mathbb{R}^n\}$.

<u>Lemma</u>: γ^n_{n+k} is locally trivial.

<u>Pf:</u> Take $\mathcal{U}$, the nbhd of $X_0 \in G_n(\mathbb{R}^{n+k})$, as above.

Then $h: \pi^{-1}(\mathcal{U}) \longrightarrow \mathcal{U} \times X_0$ defined by $h(Y, y) = (Y, py)$,

where this last p is $p: \mathbb{R}^{n+k} \cong X_0 \oplus X_0^{\perp} \to X_0$. This is clearly

continuous, with inverse $h^{-1}(Y, x) = (Y, x + A(Y)x)$. ☒

For smooth n-manifolds in Euclidean space, we have

generalized Gauss maps: $M \subseteq \mathbb{R}^{n+k} \rightsquigarrow \bar{g}: M \to G_n(\mathbb{R}^{n+k})$

given by $x \mapsto T_xM$. This is covered by a bundle map;

that is, we have

$$\begin{tikzcd} TM \arrow[r, "g"] \arrow[d] & \gamma^n_{n+k} \arrow[d] \\ M \arrow[r, "\bar{g}"] & Gr_n(\mathbb{R}^{n+k}) \end{tikzcd}$$

But this isn't special to manifolds & their tangent bundles.

<u>Lemma:</u> Let ξ be an n-plane bundle over a compact base B. Then there exists a bundle map $f: \xi \longrightarrow \gamma^n_{n+k}$, provided k is sufficiently large.

The idea is to construct an auxilary map $\hat{f}: E(\xi) \to \mathbb{R}^m$ which is a linear monomorphism on each fiber; that is $\hat{f}|_b : E_b(\xi) \longrightarrow \mathbb{R}^m$ is linear & injective. Thus $\operatorname{im} \hat{f}|_b$ is a n-dim'l subspace of $\mathbb{R}^m$, & a point in $G_n(\mathbb{R}^m)$.

Then the bundle map will be given by

$$\begin{array}{ccc} (b,e) & \longmapsto & (\operatorname{im}\hat{f}|_b,\ \hat{f}(e)) = (\hat{f}(\text{fiber through } e), \hat{f}(e)) \\ E(\xi) & \longrightarrow & E(\gamma^n_m) \\ \downarrow & & \downarrow \\ B(\xi) & \longrightarrow & G_n(\mathbb{R}^m) \\ b & \longmapsto & \operatorname{im}\hat{f}|_b \end{array}$$

(Unfortunately, we use a partition of unity type argument.)

Pf: Take $\{U_i\}_{i=1}^r$ covering $B := B(\xi)$ so $\xi|_{U_i}$ is trivial. Compact, Hausdorff $\Rightarrow \exists$ open sets $V_1, \dots, V_r$ s.t. $\overline{V}_i \subsetneq U_i$ and $\{V_i\}$ cover B. Further $\exists$ open $W_1, \dots, W_r$ s.t. $\overline{W}_i \subsetneq V_i$, & they cover. Let $\lambda_i : B \to \mathbb{R}$, a continuous function, s.t. $\lambda_i = \begin{cases} 1 & \text{on } \overline{W}_i \\ 0 & \text{outside } V_i. \end{cases}$

Then $\xi|_{U_i}$ trivial $\Rightarrow \exists\ h_i : \pi^{-1}(U_i) \to U_i \times \mathbb{R}^n \xrightarrow{p_2} \mathbb{R}^n$, which maps each fiber of $\xi|_{U_i} = \pi^{-1}(U_i)$ linearly onto $\mathbb{R}^n$. Finally, set $h_i' : E(\xi) \to \mathbb{R}^n$ by $h_i'(e) = \begin{cases} 0 & \pi(e) \notin V_i \\ \lambda_i(\pi(e))\, h_i(e) & \pi(e) \in U_i. \end{cases}$

Then $\hat{f} : E(\xi) \to \mathbb{R}^n \oplus \mathbb{R}^n \oplus \cdots \oplus \mathbb{R}^n \cong \mathbb{R}^{rn}$, by $\hat{f}(e) = (h_1'(e), \dots, h_r'(e))$ is cts & injective on fibers. ▨

(The point is we want to "add" the h_i together, but they don't share a common domain, so using λ_i we can fix the problem.)

This is sufficient for bundles over compact spaces, but ~~by mapping to $Gr_n(\mathbb{R}^{n+\infty})$, we can do better.~~