B. Universal bundles

Given a hypersurface/manifold embedded in Euclidean space $M^k \subseteq \mathbb{R}^{k+1}$, the Gauss map sends each pt. of $M$ to its unit normal vector, giving $M \to S^k$. This map depends on orientations (the sign of the normal, out vs in). Equivalently, this gives a map to $\mathbb{R}^k$, sending each pt to the line determined by the normal direction, then translated to the origin under the embedding.

We aim to generalize this construction.

Def: An $n$-frame in $\mathbb{R}^{mk}$ is an $n$-tuple of linearly independent vectors in $\mathbb{R}^{mk}$. The collection of all $n$-frames is an open subset of $\mathbb{R}^{mk} \times \cdots \times \mathbb{R}^{mk}$, called the Stiefel manifold $V_n(\mathbb{R}^{mk})$. Alternatively, one can define $V_n^o(\mathbb{R}^{mk})$ to be orthonormal $n$-frames.
$V_n^0(\mathbb{R}^{m+k}) \cup V_n(\mathbb{R}^{m+k})$ are homotopy equivalent Gram-Schmidt

(in fact, this gives $V_n^0$ as a deformation retract of $V_n$).

**Def:** The Grassmann manifold $G_n(\mathbb{R}^{m+k})$ is the set of all $n$-dimensional planes through the origin of $\mathbb{R}^{m+k}$. As a space, its topologized via the quotient map $V_n(\mathbb{R}^{m+k}) \to G_n(\mathbb{R}^{m+k})$.

In fact, the commutative diagram

$$
\begin{array}{ccc}
V_n^0(\mathbb{R}^{m+k}) & \xrightarrow{\text{Gram-Schmidt}} & V_n^0(\mathbb{R}^{m+k}) \\
\downarrow \text{span} & & \downarrow \text{span} \\
G_n(\mathbb{R}^{m+k}) & \xrightarrow{\text{span}} & G_n(\mathbb{R}^{m+k})
\end{array}
$$

shows using $V_n^0$ or $V_n$ to define $G_n$ gives the same topology.

Further, notice that $G_n(\mathbb{R}^{m}) \cong \mathbb{R}P^n$.

**Lemma:** $G_n(\mathbb{R}^{m+k})$ is a compact smooth manifold of dimension $nk$. The correspondence $X \to X^\perp$, sending an $n$-plane to its orthogonal $k$-plane, gives a homeomorphism $G_n(\mathbb{R}^{m+k}) \cong G_k(\mathbb{R}^{m+k})$.

**Pf:** It's straightforward to show Hausdorff (MS p.57). We'll prove every pt. has a Euclidean nbhd. Let $x_0 \in G_n(\mathbb{R}^{m+k})$. Then $\mathbb{R}^{m+k} \cong x_0 + x_0^\perp$. To simplify the proof, we prove the following.
\[ \Pi: X_0 \otimes X_0^\perp \to X_0 \]

Claim: Every \( \xi \in G_{\Pi}(\mathbb{P}^{n+k}) \) \( \Pi(\xi) = X_0 \sum \xi = \{ \xi \in G_{\Pi}(\mathbb{P}^{n+k}) \mid \xi \cap X_0^\perp = \{0\} \} \)

can be written uniquely as \( \Gamma(A) \), linear transformation \( A: X_0 \to X_0^\perp \),

where \( \Gamma(A) = \{ \gamma x + Ax \mid x \in X_0 \sum \xi \subseteq \mathbb{P}^{n+k} \} \).

"Every \( n \)-plane intersecting \( X_0^\perp \) in the origin is the graph of a linear transformation, and vice-versa."

**Proof of Claim:** Let \( Y \subseteq \mathbb{P}^{n+k} \) such that \( Y \cap X_0^\perp = \{0\} \).

Every \( y \in Y \subseteq \mathbb{P}^{n+k} \) can be written uniquely as \( y = \alpha + \beta, \alpha \in X_0, \beta \in X_0^\perp \). Set \( A\alpha = \beta \). \( A \) is well-defined.

Since \( y_1 = \alpha + \beta_1, y_2 = \alpha + \beta_2 \Rightarrow \beta_1 - \beta_2 = y_1 - y_2 \in Y \cap X_0^\perp = \{0\} \).

For the other direction, let \( A: X_0 \to X_0^\perp \) be linear. Then
\[ x + Ax \in \Gamma(A) \cap X_0^\perp \Rightarrow x \in X_0^\perp \text{, since } Ax \in X_0^\perp. \]

Given \( X_0 \in G_{\Pi}(\mathbb{P}^{n+k}) \), define a nbhd \( U \) by
\[ U = \{ Y \in G_{\Pi}(\mathbb{P}^{n+k}) \mid Y \cap X_0^\perp = \{0\} \} = G_{\Pi}(\mathbb{P}^{n+k}) \]

which is an open subset. By the claim, to each \( Y \in U \) we associate \( A = A(Y): X_0 \to X_0^\perp \). Thus we can think of...
(linear trans)

\[ A : U \rightarrow \text{hom}(X_0, X^1) \cong \mathbb{R}^{nk} \]. Well show A is a homeo.

First, if \( \xi \in X_0 \), \( \xi_1, \ldots, \xi_n \) is an orthonormal basis for \( X_0 \), then each \( y \in U \) has a unique basis \( \tilde{y}_1, \ldots, \tilde{y}_n \) such that

\[ T_i(y) = x_i, \quad T_i(y_2) = x_2, \ldots, \quad T_i(y_n) = x_n \]. Hence \( \tilde{y}_i \) vary continuously on \( Y \). Furthermore, the identity \( \tilde{y}_i = x_i + A(Y) \cdot x_i \), implies \( A(Y) \cdot x_i \in X^1 \) varies otsly on \( Y \) and hence A is continuous.

Finally, the identity shows \( \tilde{y}_i, \ldots, \tilde{y}_n \) depend otsly on \( A(Y) \). Thus \( Y \) depends otsly on \( A(Y) \). Hence \( A^{-1} \) is continuous. \[ \square \]

(See MS p. 593 for \( X \rightarrow X^1 \) homeo).

Def: The canonical n-plane bundle over \( G_n(\mathbb{R}^{nk}) \), denoted \( \gamma_{nk}^n \), is constructed as follows. Let

\[ E(\gamma_{nk}^n) = \{ (X, v) \mid X \in G_n(\mathbb{R}^{nk}), v \in X \times \mathbb{R}^{nk} \times \mathbb{R}^{nk} \}, \]

and then \( E(\gamma_{nk}^n) \overset{T_i}{\rightarrow} G_n(\mathbb{R}^{nk}), (X, v) \mapsto X \).\]

The fibers naturally have the structure of an n-clim'\( \mathbb{R} \)-v.s.

\[ T_i^{-1}(X) = \{ (X, v) \mid v \in \mathbb{R}^{nk} \times \mathbb{R}^{nk} \}. \]

Lemma: \( \gamma_{nk}^n \) is locally trivial.
Pf: Take $U$, the nbhd of $X_0 \in \text{Gr}_n(\mathbb{R}^{m+k})$, as above.

Then $h: T^{-1}(U) \to U \times X_0$ defined by $h(Y, y) = (Y, py)$,
where this last $p$ is $p: \mathbb{R}^{m+k} \cong X_0 \oplus X_0^\perp \to X_0$. This is clearly continuous, with inverse
$h^{-1}(Y, x) = (Y, x + A(Y)x)$.

For smooth $n$-manifolds in Euclidean space, we have

generalized Gauss maps: $M \subseteq \mathbb{R}^{m+k}$ and $\tilde{g}: M \to \text{Gr}_n(\mathbb{R}^{m+k})$
given by $x \mapsto T_xM$. This is covered by a bundle map;

that is, we have

$$
\begin{array}{ccc}
TM & \xrightarrow{\tilde{g}} & \mathbb{R}^{m+k} \\
\downarrow & & \downarrow \\
M & \xrightarrow{\tilde{g}} & \text{Gr}_n(\mathbb{R}^{m+k})
\end{array}
$$

But this isn't special to manifolds or their tangent bundles.

**Lemma:** Let $\mathcal{E}$ be an $n$-plane bundle over a compact base $B$. Then there exists a bundle map $f: \mathcal{E} \to \mathbb{R}^{m+k}$,
provided $k$ is sufficiently large.

The idea is to construct an auxiliary map $\hat{f}: E(\mathcal{E}) \to \mathbb{R}^m$

which is a linear monomorphism on each fiber; that is

$\hat{f}|_b: E_b(\mathcal{E}) \to \mathbb{R}^m$ is linear and injective. Thus $\text{im}\hat{f}|_b$ is a
$n$-dimensional subspace of $\mathbb{R}^m$, a a point in $\text{Gr}_n(\mathbb{R}^m)$. 

Then the bundle map will be given by
\[
(\mathfrak{b}, e) \rightarrow (\text{im} \mathcal{F}_{\mathfrak{b}}, f(e)) = (\mathcal{F} \text{(fiber through } e), \hat{f}(e))
\]
\[
E(\mathfrak{g}) \rightarrow E(Y^m_n)
\]
\[
\downarrow \quad \downarrow
\]
\[
B(\mathfrak{g}) \rightarrow G_n(\mathbb{R}^m)
\]
\[
b \mapsto \text{im} \mathcal{F}_{\mathfrak{b}}
\]

\textbf{Pf.} Take \( \mathcal{U}_i \) covering \( B := B(\mathfrak{g}) \) so \( \mathcal{U}_i \) is

trivial. Compact, Hausdorff \( \Rightarrow \) \( \mathcal{U}_i \) open sets \( V_1, \ldots, V_r \) s.t. \( \overline{V}_i \subset U_i \) and \( \mathcal{U}_i \) cover \( B \). Further \( \mathcal{U}_i \) open \( W_1, \ldots, W_r \) s.t. \( \overline{W}_i \subset \overline{V}_i \), \( \cap \) they

cover. Let \( \lambda_i : B \rightarrow \mathbb{R} \), a continuous function, s.t. \( \lambda_i |_{\overline{W}_i} = 1 \) on \( \overline{W}_i \).

Then \( \mathcal{U}_i \), trivial = \( \exists \) \( \lambda_i : \mathcal{U}_i \rightarrow U_i \times \mathbb{R}^m \rightarrow \mathbb{R}^m \), which

maps each fiber of \( \mathcal{U}_i \) linearly onto \( \mathbb{R}^m \). Finally,

set \( \lambda_i : E(\mathcal{U}_i) \rightarrow \mathbb{R}^m \) by \( \lambda_i(e) = \begin{cases} 
0 & e \in U_i \\
\lambda_i(e) & e \in \mathcal{U}_i \end{cases} \)

Then \( \hat{f} : E(\mathcal{U}_i) \rightarrow \mathbb{R}^m \otimes \mathbb{R}^m \rightarrow \mathbb{R}^n \), by \( \hat{f}(e) = (\hat{h}_1(e), \ldots, \hat{h}_r(e)) \)

is cts. \& injective on fibers. \( \mathcal{U}_i \)

(The point is we want to "add" the \( \hat{h}_i \) together, but they don't

have a common domain, so using \( \lambda_i \) we can fix the problem.)

This is sufficient for bundles over compact spaces, but

by mapping to \( Gr_n(\mathbb{R}^m) \), we can do better.