Def: Using the std. inclusions $\mathbb{R}^1 \subseteq \mathbb{R}^2 \subseteq \cdots$, we define

$\mathbb{R}^\infty := \operatorname{colim}_n \mathbb{R}^n = \bigcup_n \mathbb{R}^n$, where $x \in \mathbb{R}^\infty$ satisfies $x = (x_1, x_2, \ldots)$

for which all but finitely many x_i are zero.

Def: The infinite Grassmannian $G_n = G_n(\mathbb{R}^\infty)$ is the set

of n-dim'l linear subspaces of $\mathbb{R}^\infty$. Its topologized via

$$Gr_n(\mathbb{R}^n) \subset G_n(\mathbb{R}^{n+1}) \subset G_n(\mathbb{R}^{n+2}) \subset \cdots \subset \operatorname{colim}_k G_n(\mathbb{R}^{n+k}) = G_n(\mathbb{R}^\infty).$$

Thus, $\mathcal{U} \subseteq G_n(\mathbb{R}^\infty)$ is open iff $\mathcal{U} \cap G_n(\mathbb{R}^{n+k})$ is open $\forall k$. //

As a special case, $\mathbb{R}P^\infty = G_1(\mathbb{R}^\infty) = \operatorname{colim}_n (\mathbb{R}P^1 \subset \mathbb{R}P^2 \subseteq \cdots \subseteq \mathbb{R}P^n \subseteq \cdots)$.

Def: The canonical line bundle γ^n over G_n is just

what you think: $E(\gamma^n) = \{(X, v) \mid X \in G_n, v \in X\} \subseteq G_n \times \mathbb{R}^\infty$. //

That γ^n is locally trivial is essentially the same as γ^n_{n+k}.

Thm: Any $\mathbb{R}^n$-bundle ξ over a paracompact base

admits a bundle map $\xi \to \gamma^n$.

Thm: Any two bundle maps $\xi \underset{g}{\overset{f}{\rightrightarrows}} \gamma^n$ are bundle-homotopic.

These two assert that γ^n is the universal n-plane bundle.

Def: Two bundle maps $f, g: \xi \to \gamma$ are bundle-homotopic if there is a one parameter family of bundle maps $h_t: \xi \to \gamma$, $t \in [0,1]$ w/ $h_0 = f$, $h_1 = g$, and h is continuous. Equivalently, the map $h: E(\xi) \times [0,1] \to E(\gamma)$ is cts. //

The "covering homotopy theorem" says that "homotopic bundles" are isomorphic, i.e., bundle-homotopic maps determine an isomorphism. This leads us to the following classification theorem.

<u>Thm [Bundle Classification]</u>: Let X be a paracompact space. Then n-plane bundles over X are in bijection w/ homotopy classes of maps from X into $Gr_n(\mathbb{R}^\infty)$. In symbols,

$$\left\{ \begin{matrix} \mathbb{R}^n\text{-vector} \\ \text{bundles} \\ \text{over } X \end{matrix} \right\} \longleftrightarrow \left[X, Gr_n(\mathbb{R}^\infty) \right]$$

Given $\xi = \begin{matrix} E \\ \downarrow \\ X \end{matrix} \longmapsto \hat{f}: B(\xi) = X \longrightarrow G_n(\mathbb{R}^m) \subseteq G_n(\mathbb{R}^\infty)$, as constructed in the previous Lemma.

Given f:
$$\begin{tikzcd} f^*\gamma^n \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{P.b.}"] & \gamma^n \arrow[d] \\ X \arrow[r, "f"] & G_n(\mathbb{R}^\infty) \end{tikzcd}$$

So either take the canonical map on base spaces, or pullback the universal n-plane bundle.

This classification sits in a much more general framework that we'll explore in the rest of class.

<u>Recall the following</u>: Before moving on, recall the transition functions that give an alternate description of bundles:

Given ξ, a real v.b. over B, and a bundle atlas $\{(\varphi_\alpha, \mathcal{U}_\alpha)\}$ for ξ, we have

$$\varphi_\beta \circ \varphi_\alpha^{-1} : \mathcal{U}_\alpha \cap \mathcal{U}_\beta \times \mathbb{R}^n \longrightarrow \mathcal{U}_\alpha \cap \mathcal{U}_\beta \times \mathbb{R}^n$$
$$(p, v) \longmapsto (p, \varphi_{\beta\alpha}(p) \cdot v)$$

The $\{\varphi_{\beta\alpha} : \mathcal{U}_\beta \cap \mathcal{U}_\alpha \to GL_n\mathbb{R}\}$ are called the <u>transition functions</u> of ξ. These tell us how to put the trivial pieces together to get something non-trivial.

This ~~is~~ perspective is important b/c it stresses $GL_n\mathbb{R}$, the structure group of the bundle, which is crucial. The fiber is not so important as we shall see. The transition functions satisfy

1. $\varphi_{\alpha\alpha}(p) = Id_{\mathcal{U}_\alpha} \;\; \forall \; p \in \mathcal{U}_\alpha$

2. $\varphi_{\alpha\beta}(p)\,\varphi_{\beta\gamma}(p)\,\varphi_{\gamma\alpha}(p) = Id_{\mathcal{U}_\alpha \cap \mathcal{U}_\beta \cap \mathcal{U}_\gamma}$, known as the <u>co-cycle condition</u>.

From these, it follows (a) $\varphi_{\alpha\beta}^{-1}(p) = \varphi_{\beta\alpha}(p)$

and (b) $\varphi_{\alpha\beta}(p)\,\varphi_{\beta\gamma}(p) = \varphi_{\alpha\gamma}(p)$.

IV. Principal bundles

A. Principal G-bundles

Def: A topological group G is a topological space w/ a group structure so that

$$G \times G \longrightarrow G$$
$$(s,t) \longmapsto st^{-1}$$

is continuous. //

Def: For a topological group G, a right G-space X is a top. space, together w/ an action $X \times G \to X$. // (Equivalently, left G-spaces $G \times X \to X$)

Ex: $\mathbb{R}^n$ is a left $GL_n\mathbb{R}$ space, or left $O(n)$-space, by matrix multiplication: $A \in GL_n\mathbb{R}$, $v \in \mathbb{R}^n$ $A \cdot v$.

Ex: $V_q^0(\mathbb{R}^n)$ = orthonormal q-frames in $\mathbb{R}^n$. $V_q^0(\mathbb{R}^n)$ is a right $O(r)$-space $\forall\, r \leq q$. The action is given by changing the first r-vectors in the frame by the $r \times r$ orthogonal matrix.

Equivalently $(v_1, \dots, v_q) \in V_q^0$, $A \in O(r)$, the action is $\begin{bmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_q \\ | & | & & | \end{bmatrix} \cdot \left[\begin{array}{c|c} A & 0 \\ \hline 0 & \underbrace{I}_{q-r} \end{array}\right]$.

Def: A G-action is free if $\exists\, x \in X$ s.t. $g \cdot x = x$, then $g = e$. //

That is, if g has at least one-fixed point, then $g = e$. Equivalently, if $\exists\, x \in X$ s.t. $g \cdot x = h \cdot x \Rightarrow g = h$. ("fixed-point free")

Def: A G-action is transitive if $\forall\, x, y \in X$, $\exists\, g \in G$ s.t. $g \cdot x = y$.

Def: Let G be a topological group. A <u>principal</u> <u>G-bundle</u> consists of a map $\pi: P \to X$, w/ a (continuous) right action $P \times G \to P$ such that ① G preserves the fibers $\pi^{-1}(x)$, i.e., $g \cdot y \in \pi^{-1}(x)$ $\forall y \in \pi^{-1}(x)$, $\forall g \in G$, ② G acts freely and transitively on the fibers, and ③ π is locally trivial:

$\forall x \in X$, $\exists$ nbhd $\mathcal{U}$ of x such that

$$\begin{tikzcd} \pi^{-1}(\mathcal{U}) \arrow[rr, "\cong", "\text{homeo}"'] \arrow[dr, "\pi"'] & & \mathcal{U} \times G \arrow[dl, "\text{proj}"] \\ & \mathcal{U} & \end{tikzcd}$$

commutes. //

<u>Notice</u>: Since G preserves the fibers and acts transitively on them, the orbits of the G-action coincides with these fibers. Furthermore, $P/G \cong X$.

<u>Notice</u>: Since the action of G is free, the fibers are homeomorphic to G itself, although only as a space, since the actual homeomorphism isn't specified by this information; the fiber's alg. structure isn't specified. Equivalently, there is no preferred identity element in the fiber; these spaces are known as <u>G-torsors</u>.

<u>Note</u>: All of this works in the smooth category, replacing P & X by smooth manifolds, π a submersion, & G a Lie group.

<u>Ex</u>: Let $\begin{array}{c}\mathbb{R}\\ \downarrow \\ S^1\end{array}$ be the map sending $x \longmapsto e^{2\pi i x}$. Then $\mathbb{Z} = \pi_1(S^1)$ acts on $\mathbb{R}$ by translation. This action is fiberwise (meaning it preserves fibers) + is free + transitive. Hence this is a principal $\mathbb{Z} = \pi_1(S^1)$-bundle. In fact, $\mathbb{R}$ is the universal cover of S^1 + this example extends to covering spaces more generally.

<u>Ex</u>: $V_q(\mathbb{R}^n)$ = q-frames in $\mathbb{R}^n$. To each q-frame, associate a pt in $G_q(\mathbb{R}^n)$ by taking the span. The group $GL_q(\mathbb{R})$ acts on V_q by multiplication on the right, + the orbit space / quotient is clearly $G_q(\mathbb{R}^n)$.

<u>Ex</u>: $S^1 \xrightarrow{2} S^1$, $z \longmapsto z^2$. This is a principal $\mathbb{Z}/2$-bundle.

<u>Def</u>: A <u>morphism</u> of principal G-bundles over X is an G-equivariant map σ such that

$$\begin{tikzcd} P \arrow[r, "\sigma"] \arrow[d] & Q \arrow[d] \\ X \arrow[r, equal] & X \end{tikzcd}$$

commutes.

A map f is <u>G-equivariant</u> (or just a G-map) if $f(x \cdot g) = f(x) \cdot g$.

<u>Prop</u>: Any morphism of principal G-bundles is an isomorphism.

Pf:
$$\begin{tikzcd} P \arrow[rr, "\sigma"] \arrow[dr, "\pi"'] & & Q \arrow[dl, "\pi'"] \\ & X & \end{tikzcd}$$
Suppose $\sigma(p_1) = \sigma(p_2)$. Then $\pi(p_1) = \pi'\sigma(p_2) = \pi'(p_2)$. Hence p_1 + p_2 are in the same fiber of π + $\exists\, g \in G$ s.t. $p_1 \cdot g = p_2$. But $\sigma(p_1)\cdot g = \sigma(p_2) = \sigma(p_1) \Rightarrow g = e$, $\Rightarrow p_1 = p_2$. ✓

For every $q \in Q$, let $p \in P$ such that $\pi(p) = \pi'(q)$. Then $\pi'(q) = \pi(p) = \pi'(\sigma(p))$, so $\sigma(p)$ + q are in the same fiber of π'. So $\sigma(p)\cdot g = q$ f.s. $g \in G$, and $\sigma(p \cdot g) = q = \sigma(p)\cdot g$.

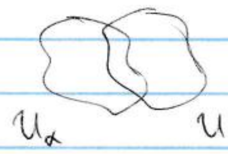
So σ is injective + surjective. Its not hard to see/construct σ^{-1}. ▨

Prop: A principal G-bundle $\pi : P \to X$ is trivial iff it admits a section.

Pf: If $P = G \times X$ is trivial, then take $s(x) = (x, e)$. Conversely, let $s : X \to P$ be a section. Then

$$\begin{tikzcd} X \times G \arrow[rr] \arrow[dr] & & P \arrow[dl] \\ & X & \end{tikzcd}$$
given by $(x,g) \longmapsto s(x)\cdot g$ is a morphism of principal bundles + hence an iso. ▨

Finally, we end with a local description. Let $P \xrightarrow{\pi} X$ be a principal G-bundle, $\{\{U_\alpha\}, \{\varphi_\alpha\}\}$ bundle atlas for π, and consider the intersections: on each intersection, we have



2 trivializations φ_α, & φ_β. Thus, we have

$$\varphi_\beta \circ \varphi_\alpha^{-1} : \mathcal{U}_\alpha \cap \mathcal{U}_\beta \times G \longrightarrow \mathcal{U}_\alpha \cap \mathcal{U}_\beta \times G \quad , \quad (x, g) \longmapsto (x, \psi_{\beta\alpha}(x) \cdot g).$$

The $\{\psi_{\beta\alpha}\}$ are known as transition functions. These functions satisfy

1. $\psi_{\beta\alpha} = \psi_{\alpha\beta}^{-1}$ and 2. $\psi_{\beta\alpha}\psi_{\alpha\gamma} = \psi_{\beta\gamma}$, with each

$\psi_{\alpha\beta} : \mathcal{U}_\alpha \cap \mathcal{U}_\beta \to \text{homeo}(G)$.

For principal G-bundles, $\psi_{\alpha\beta}(x)$ is given by left translation of G.

Aside: these two conditions show that transition functions define a 1-co-cycle for Čech cohomology. That is, if X is a manifold, let $\tilde{G}$ = smooth functions from M to G. Such functions have the structure of a sheaf of groups. Conversely, given a 1-co-cycle $\{\psi_{ab}\}$, glue together $\{\mathcal{U}_i \times G\}$ by $(x, g) \longmapsto (x, \psi_{ab}(x) \cdot g)$ & define P in this way. Hence, Principal G-bundles $/ X \longleftrightarrow H^1(X; \tilde{G})$

B. G-bundles & fiber bundles

The most general construction is that of a G-bundle, a bundle (locally trivial fibration) whose fibers are equipped