Differential Equations Exam One NAME:

1. Solve (explicitly) the separable Differential Equation
$\frac{dy}{dx} = \frac{y^2+1}{y(x+1)}$ with $y(0) = 2$.

Separate: $\frac{ydy}{y^2+1} = \frac{dx}{x+1}$.

Integrate: $\frac{1}{2}ln(y^2 + 1) = ln(x + 1) + c$

Simplify: $y^2 + 1 = k(x + 1)^2$.

Initial Value: $5 = k(1)$

Final Answer: $y = \sqrt{5(x + 1)^2 - 1}$.

2. Solve (implicitly) the Exact Differential Equation

$2x + y^3 sec^2 x)dx + (1 + 3y^2 tan\ x)dy = 0$.

Integrate: $[x^2 + y^3\ tan\ x] \bigcup [y + y^3 tan\ x] = c$.

Final Answer: $x^2 + y^3\ tan\ x + y = c$.

3. Use Euler's method with $h = .2$, for 3 steps, to find an approximate value $y(1.6)$ for the solution to the initial value problem $dy/dx = y - x$ with $y(1) = 3$.

$x_0 = 1$ and $y_0 = 3$ and $y_{n+1} = y_n + hf(x_n, y_n) = y_n + (.2)(y_n - x_n)$.

$x_1 = 1.2$ and $y_1 = 3 + (.2)(3 - 1) = 3.4$.

$x_2 = 1.4$ and $y_2 = 3.4 + (.2)(3.4 - 1.2) = 3.84$.

$x_3 = 1.6$ and $y_3 = 3.84 + (.2)(3.84 - 1.4) = 4.33$.

Final Answer: $y(1.6) \sim 4.33$.

4. Solve the equation $y' = y^2(y - 2)(y - 5)$ graphically.

(a) Find the critical values c such that $y = c$ is a solution.

(b) Find the sign of y' and the limiting value L on each interval.

(c) Sketch solutions with $y(0) = -1$, $y(0) = 1$ and $y(0) = 6$.

(a) $y = 0$, $y = 2$, and $y = 5$.

(b) $y' > 0$ on $(-\infty, 0)$, $(0, 2)$ and $(5, \infty)$; $y' < 0$ on $(2, 5)$.

The limiting value is $+\infty$ on $(5, \infty)$.

The limiting value is 2 on $(0, 5)$.

The limiting value is 0 on $(-\infty, 0]$.

5. A packet dropped from a height of 400 meters with initial velocity 0 and air resistance $.7mv$, so that the velocity $v(t)$ is subject to the differential equation $v' = -.7v - 9.8$.

Find the approximate time of impact and the limiting velocity.

$L = \frac{-b}{a} = -\frac{9.8}{7} = -14.$

This should be the approximate limiting velocity.

$v = -14 + 14e^{-.7t}$.

Integrating, $y = -14t - 20e^{-.7t} + 420$

Assuming that $20e^{-.7t}$ is negligible,

$y = 0$ when t is approximately $420/14 = 30$ seconds.

6. Solve (implicitly) the homogeneous differential equation $y' = \frac{2y^2}{xy - x^2}$. Simplify your answer.

Hint: You may want to use partial fractions.

Let $v = y/x$ so that $y = vx$ and $y' = v + xv'$

Substituting, we get the separable equation

$v + xv' = \frac{2v^2}{v-1}$.

Separating, we get

$dx/x = \frac{v-1}{v^2+v}$

Using partial fractions,

$\frac{v-1}{v(v+1} = \frac{A}{v} + \frac{B}{v+1}$

$v - 1 = A(v+1) + Bv$, so substituting $v = -1$ and $v = 0$:

$A = -1$ and $B = 2$.

Integrating

$ln\ x = 2ln(v+1) - ln\ v + c$

Exponentiating,

$x = C\frac{(v+1)^2}{v}$

Substituting $v = y/x$ and multiplying by x^2/x^2:

$x = C\frac{(x+y)^2}{xy}$

Simplifying: $x^2y = C(x+y)^2$.

7. A brine solution of salt with concentration .2 kg/L flows at a constant rate of 4 L/min into a tank that initially held 20 L of brine solution containing 5 kg of salt. The well-stirred solution flows out of the tank at the rate of 3 L/min. Find the mass $x(t)$ of salt in the tank at time t and also the concentration $y(t)$ and the limiting value of the concentration.

The volume in the tank at time t is $t+20$, .8 kg per minute of salt is flowing into the tank and $\frac{3x}{t+20}$ kg is flowing out. So the differential equation is $dx/dt+\frac{3x}{t+20} = .8$

This is a linear equation with $P = \frac{3}{t+20}$ and $Q = .8$.

$\int \frac{3}{t+20} = 3\ ln(t+20)$, so the integrating factor $\mu = e^{3\ ln(t+20)} = (t+20)^3$. Then

$x = (t+20)^{-3} \int .8(t+20)^3 = (t+20)^{-3}[.2(t+20)^4+C] = .2(t+20)+C(t+20)^{-3}$.

$x(0) = 5 = .2(20) + C(20)^{-3} = 4 + C/8000$, so $C = 8000$.

Thus $x = .2(t + 20) + 8000/(t + 20)^3$

The concentration $y(t) = x/(t + 20) = .2 + 8000(t + 20)^{-4}$

which has limiting value .2.