Foundations of Mathematics

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Chapter 1

Introduction

The first problem in the study of the foundations of mathematics is to determine the nature of mathematics. That is, what are mathematicians supposed to do?

For the sake of discussion, let us consider four kinds of mathematical activity.

The first and most natural activity is computation. That is, the creation and application of algorithms to use in the solution of mathematical or scientific problems. For example, we learn algorithms for addition and multiplication of integers in elementary school. In college we may learn the Euclidean Algorithm, which is used to compute the least common denominator of two positive integers. It is non-trivial to prove that the Euclidean Algorithm actually works. In this course we will consider the concept of algorithms and the question of whether certain problems may or may not be solvable by an algorithm.

The second kind of mathematical activity consists of discovering properties of natural mathematical structures such as the integers, the real line and Euclidean geometry, in somewhat the same way that a physicist discovers properties of the universe of matter and energy. That is, by experiment and by thought. Thus we have commutative law of addition, the density of the ordering of the real line and the incidence axioms for points and lines. I am thinking in particular here of properties which cannot be derived from previous principles but which will be taken as axioms or definitions of our structures. This can also include the discovery of complicated theorems which it is hoped will follow from previously accepted principles. In this course we will discuss the Incompleteness Theorem of Godel, which implies that there will always be new properties of the natural numbers \{0, 1, 2, \ldots\} with addition and multiplication remaining to be discovered.

The third kind of mathematical activity consists of deriving (or proving) theorems from a given set of axioms, usually those abstracted from the study of the first kind. These theorems will now apply to a whole family of models, those which satisfy the axioms. For example, one may discover that all reasonably small positive integers may be obtained as a sum of 5 or fewer squares and conjecture that this property is true of all positive integers. Now it remains to prove the conjecture, using known properties of the integers. Most of the time a proof will apply to more than one mathematical structure. For example, we have in finite group theory Lagrange's Theorem that the order of a subgroup divides the number of elements of the group. This applies to any structure which satisfies the axioms of group theory. In this course we will consider the concept of mathematical proof. The Completeness Theorem of Godel tells us that any true theorem can be proved (eventually).

The fourth kind of mathematical activity consists of constructing new models. As an example, we have the construction of the various finite groups, culminating in the monster simple group recently found by Griess. Most of the models are based on the natural structures of the integers and the reals, but the powerful ideas of set theory have led to many models which could not have been found otherwise. This fourth kind of mathematics includes demonstrating the independence of the axioms found in the first kind and also includes the "give a counterexample" part of the standard mathematical question: "Prove or give a counterexample." In this course we will consider the concept of models and use models to show that various conjectures are independent. The most
famous result of this kind is the independence of the parallel Postulate of Euclidean Geometry. A more recent example is Cohen’s model in which the Continuum Hypothesis is false.

The study of the foundations of mathematics is sometimes called meta-mathematics. The primary tool in this study is mathematical logic. In particular, mathematical logic provides the formal language of mathematics, in which theorems are stated. Therefore we begin with the propositional and predicate calculus and the notions of truth and models.

Set Theory and Mathematical Logic compose the foundation of pure mathematics. Using the axioms of set theory, we can construct our universe of discourse, beginning with the natural numbers, moving on with sets and functions over the natural numbers, integers, rationals and real numbers, and eventually developing the transfinite ordinal and cardinal numbers. Mathematical logic provides the language of higher mathematics which allow one to frame the definitions, lemmas, theorems and conjectures which form the everyday work of mathematicians. The axioms and rules of deduction set up the system in which we can prove our conjectures, thus turning them into theorems.

A separate volume on set theory begins with a chapter introducing the axioms of set theory, including a brief review of the notions of sets, functions, relations, intersections, unions, complements and their connection with elementary logic. The second chapter introduces the notion of cardinality, including finite versus infinite, and countable versus uncountable sets. We define the Von Neumann natural numbers \( \omega = \{0, 1, 2, \ldots \} \) in the context of set theory. The methods of recursive and inductive definability over the natural numbers are used to define operations including addition and multiplication on the natural numbers. These methods are also used to define the transitive closure of a set \( A \) as the closure of \( A \) under the union operator and to define the hereditarily finite sets as the closure of 0 under the power set operator. The notion of a model of set theory is introduced. Conditions are given under which a given set \( A \) can satisfy certain of the axioms, such as the union axiom, the power set axiom, and so on. It is shown that the hereditarily finite sets satisfy all axioms except for the Axiom of Infinity.

Several topics covered here are not typically found in a standard textbook.

An effort is made to connect foundations with the usual mathematics major topics of algebra, analysis, geometry and topology. Thus we have chapters on Boolean algebras, on non-standard analysis, and on the foundations of geometry. There is an introduction to descriptive set theory, including cardinality of sets of real numbers. The topics of inductive and recursive definability plays an important role in all areas of logic, including set theory, computability theory, and proof theory. As part of the material on the axioms of set theory, we consider models of various subsets of the axioms, as an introduction to consistency and independence. Our development of computability theory begins with the study of finite state automata and is enhanced by an introduction to algorithmic randomness, the preeminent topic in computability in recent times. This additional material gives the instructor options for creating a course which provides the basic elements of set theory and logic, as well as making a solid connection with many other areas of mathematics.
Chapter 2

Foundations of Geometry

2.1 Introduction

Plane geometry is an area of mathematics that has been studied since ancient times. The roots of the word "geometry" are the Greek words *ge* meaning "earth" and *metria* meaning "measuring". This name reflects the computational approach to geometric problems that had been used before the time of Euclid, (ca. 300 B.C.), who introduced the axiomatic approach in his book, *Elements*. He was attempting to capture the reality of the space of our universe through this abstraction. Thus the theory of geometry was an attempt to capture the essence of a particular model.

Euclid did not limit himself to plane geometry in the *Elements*, but also included chapters on algebra, ratio, proportion and number theory. His book set a new standard for the way mathematics was done. It was so systematic and encompassing that many earlier mathematical works were discarded and thereby lost for historians of mathematics.

We start with a discussion of the foundations of plane geometry because it gives an accessible example of many of the questions of interest.

2.2 Axioms of plane geometry

Definition 2.2.1. The theory of Plane Geometry, PG, has two one-place predicates, *Pt* and *Ln*, to distinguish the two kinds of objects in plane geometry, and a binary incidence relation, *In*, to indicate that a point is on or incident with a line.

By an abuse of notation, write $P \in Pt$ for $Pt(P)$ and $\ell \in Ln$ for $Ln(\ell)$.

There are five axioms in the theory PG:

$(A_0)$ (Everything is either a point or line, but not both; only points are on lines.)

$(\forall x)((x \in Pt \lor x \in Ln) \land \neg(x \in Pt \land x \in Ln)) \land (\forall x, y)(x In y \rightarrow (x \in Pt \land y \in Ln))$.

$(A_1)$ (Any two points belong to a line.)

$(\forall P, Q \in Pt)(\exists \ell \in Ln)(P In \ell \land Q In \ell)$.

$(A_2)$ (Every line has at least two points.)

$(\forall \ell \in Ln)(\exists P, Q \in Pt)(P In \ell \land Q In \ell \land P \neq Q)$.

$(A_3)$ (Two lines intersect in at most one point.)

$(\forall \ell, g \in Ln)(\forall P, Q \in Pt)((\ell \neq g \land P In \ell \land P In g) \rightarrow P = Q)$.
(A₄) (There are four points no three on the same line.) \((\exists P_0, P_1, P_2, P_3 \in Pt)(P_0 \neq P_1 \neq P_2 \neq P_3 \& P_0 \neq P_5 \& P_1 \neq P_3 \& P_3 \neq P_1 \& (\forall \ell \in Ln))(\neg(P_0 \text{In} \& P_1 \text{In} \& P_2 \text{In} \& P_3 \text{In} \ell) \& \neg(P_0 \text{In} \& P_1 \text{In} \& P_2 \text{In} \ell) \& \neg(P_0 \text{In} \& P_2 \text{In} \& P_3 \text{In} \ell) \& \neg(P_0 \text{In} \& P_1 \text{In} \& P_3 \text{In} \ell) \& \neg(P_3 \text{In} \& P_1 \text{In} \& P_2 \text{In} \ell))\).

The axiom labeled 0 simply says that our objects have the types we intend, and is of a different character than the other axioms. In addition to these axioms, Euclid had one that asserted the existence of circles of arbitrary center and arbitrary radius, and one that asserted that all right angles are equal. He also had another axiom for points and lines, called the parallel postulate, which he attempted to show was a consequence of the other axioms.

**Definition 2.2.2.** Two lines are parallel if there is no point incident with both of them.

**Definition 2.2.3.** For \(n \geq 0\), the \(n\text{-parallel postulate}, P_n\), is the following statement:

\((P_n)\) For any line \(\ell\) and any point \(Q\) not on the line \(\ell\), there are \(n\) lines parallel to \(\ell\) through the point \(Q\).

\(P_1\) is the familiar parallel postulate.

For nearly two thousand years, people tried to prove what Euclid had conjectured. Namely, they tried to prove that \(P_1\) was a consequence of the other axioms. In the 1800's, models of the other axioms were produced which were not models of \(P_1\).

## 2.3 Non-Euclidean models

Nikolai Lobachevski (1793-1856), a Russian mathematician, and Janos Bolyai (1802-1860), a Hungarian mathematician, both produced models of the other axioms together with the parallel postulate \(P_{\infty}\), that there are infinitely many lines parallel to a given line through a given point. This geometry is known as *Lobachevskian Geometry*. It is enough to assume \(P_{\geq 2}\) together with the circle and angle axioms to get \(P_{\infty}\).

**Example 2.3.1.** (A model for Lobachevskian Geometry): Fix a circle, \(C\), in a Euclidean plane. The points of the geometry are the interior points of \(C\). The lines of the geometry are the intersection of lines of the Euclidean plane with the interior of the circle. Given any line \(\ell\) of the geometry and any point \(Q\) of the geometry which is not on \(\ell\), every Euclidean line through \(Q\) which intersects \(\ell\) on or outside of \(C\) gives rise to a line of the geometry which is parallel to \(\ell\).

**Example 2.3.2.** (A model for Riemannian Geometry): Fix a sphere, \(S\), in Euclidean 3-space. The points of the geometry may be thought of as either the points of the upper half of the sphere, or as equivalence classes consisting of the pairs of points on opposite ends of diameters of the sphere (antipodal points). If one chooses to look at the points as coming from the upper half of the sphere, one must take care to get exactly one from each of the equivalence classes. The lines of the geometry are the intersection of the great circles with the points. Since any two great circles meet in two antipodal points, every pair of lines intersects. Thus this model satisfies \(P_0\).
Bernhard Riemann (1826-1866), a German mathematician, was a student of Karl Gauss (1777-1855), who is regarded as the greatest mathematician of the nineteenth century. Gauss made contributions in the areas of astronomy, geodesy and electricity as well as mathematics. While Gauss considered the possibility of non-Euclidean geometry, he never published anything about the subject.

2.4 Finite geometries

Next we turn to finite geometries, ones with only finitely many points and lines. To get the theory of the finite projective plane of order \( q \), denoted \( \text{PG}(q) \), in addition to the five axioms given above, we add two more:

\( (A_5) \) Every line contains exactly \( q+1 \) points.

\( (A_6) \) Every point lies on exactly \( q+1 \) lines.

The first geometry we look at is the finite projective plane of order 2, \( \text{PG}(2) \), also known as the Fano Plane.

**Theorem 2.4.1.** The theory \( \text{PG}(2) \) consisting of \( \text{PG} \) together with the two axioms \((5)_2 \) and \((6)_2 \) determines a finite geometry of seven points and seven lines, called the Fano plane.

**Proof.** See Exercise 5 to prove from the axioms and Exercise 4 that the following diagram gives a model of \( \text{PG}(2) \) and that any model must have the designated number of points and lines. \( \square \)

Next we construct a different model of this finite geometry using a vector space. The vector space underlying the construction is the vector space of dimension three over the field of two elements, \( Z_2 = \{0, 1\} \). The points of the geometry are one dimensional subspaces. Since a one-dimensional subspace of \( Z_2 \) has exactly two triples in it, one of which is the triple \((0, 0, 0)\), we identify the points with the triples of 0's and 1's that are not all zero. The lines of the geometry are the two
dimensional subspaces. The incidence relation is determined by a point is on a line if the one dimensional subspace is a subspace of the two dimensional subspace. Since a two dimensional subspace is picked out as the orthogonal complement of a one-dimensional subspace, each two dimensional subspace is identified with the non-zero triple, and to test if point \((i, j, k)\) is on line \([\ell, m, n]\), one tests the condition

\[
i\ell + jm + kn \equiv 0 \pmod{2}.
\]

There are exactly \(2^3 = 8\) ordered triples of 0’s and 1’s, of which one is the all zero vector. Thus the ordered triples pick out the correct number of points and lines. The following array gives the incidence relation, and allows us to check that there are three points on every line and three lines through every point.

<table>
<thead>
<tr>
<th>In ((i, j, k))</th>
<th>([1,0,0])</th>
<th>([0,1,0])</th>
<th>([0,0,1])</th>
<th>([1,1,0])</th>
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<td>((1,0,0))</td>
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<td>0</td>
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</table>

The vector space construction works over other finite fields as well. The next bigger example is the projective geometry of order 3, PG(3). The points are the one dimensional subspaces of the vector space of dimension 3 over the field of three elements, \(\mathbb{Z}_3 = \{0, 1, 2\}\). This vector space has \(3^3 - 1 = 27 - 1 = 26\) non-zero vectors. Each one dimensional subspace has two non-zero elements, so there are \(26/2 = 13\) points in the geometry. As above, the lines are the orthogonal or perpendicular complements of the subspaces that form the lines, so there are also 13 of them. The test for incidence is similar to the one above, except that one must work \((\text{mod } 3)\) rather than \((\text{mod } 2)\).

This construction works for each finite field. In each case the order of the projective geometry is the size of the field.

The next few lemmas list a few facts about projective planes.

**Lemma 2.4.2.** In any model of PG\((q)\), any two lines intersect in a point.

**Lemma 2.4.3.** In any model of PG\((q)\) there are exactly \(q^2 + q + 1\) points.

**Lemma 2.4.4.** In any model of PG\((q)\) there are exactly \(q^2 + q + 1\) lines.

For models built using the vector space construction over a field of \(q\) elements, it is easy to compute the number of points and lines as \(\frac{q^3 - 1}{q - 1} = q^2 + q + 1\). However there are non-isomorphic projective planes of the same order. For a long time four non-isomorphic planes of order nine were known, each obtained by a variation on the above vector space construction. Recently it has been shown with the help of a computer that there are exactly four non-isomorphic planes of order nine.

Since the order of a finite field is always a prime power, the models discussed so far all have prime power order. Much work has gone into the search for models of non-prime power order. A well-publicized result showed that there were no projective planes of order 10. This proof required many hours of computer time.

### 2.5 Exercises

1. Translate Axioms \((5)_q\) and \((6)_q\) into the formal language.
2. Label the three points of intersection of the lines in the illustration of Riemannian geometry which form a triangle above the “equator”.

3. Define an isomorphism between the two models of $PG(2)$.

4. Prove from the axioms that any two lines in $PG(q)$ must intersect in a point. (Hint: Show that if $g$ and $h$ do not intersect and $P$ is incident with $g$, then $P$ is on at least one more line than $h$ has points.)

5. Construct a model for $PG(2)$ starting with four non-collinear points $A$, $B$, $C$ and $D$ and denoting the additional point on the line $AB$ by $E$, the additional point on the line $AC$ by $F$, and the additional point on the line $BC$ by $G$. Use the axioms and exercise 1.4 to justify the construction.

6. Prove from the axioms that in any model of $PG(q)$ there are exactly $q^2 + q + 1$ points.

7. Prove from the axioms that in any model of $PG(q)$ there are exactly $q^2 + q + 1$ lines.

8. List the 13 one-dimensional subspaces of $Z_3^3$ by giving one generator of each. (Hint: Proceeding lexicographically, four of them begin with “0” and the other nine begin with “1”.) These are the points of $PG(3)$. Identify the 13 two dimensional subspaces of $Z_3^3$ as orthogonal complements of these one-dimensional spaces. These are the lines of $PG(3)$. For each line, list the four points on the line.

9. Show that the axioms 1,2,3,4 for Plane Geometry are independent by constructing models which satisfy exactly 3 of the axioms. (There are 4 possible cases here.)
CHAPTER 2. FOUNDATIONS OF GEOMETRY
Chapter 3

Propositional Logic

3.1 The basic definitions

Propositional logic concerns relationships between sentences built up from primitive proposition symbols with logical connectives.

The symbols of the language of predicate calculus are

1. Logical connectives: ¬, & , ∨, →, ↔
2. Punctuation symbols: ( , )

A propositional variable is intended to represent a proposition which can either be true or false. Restricted versions, L, of the language of propositional logic can be constructed by specifying a subset of the propositional variables. In this case, let PVar(L) denote the propositional variables of L.

Definition 3.1.1. The collection of sentences, denoted Sent(L), of a propositional language L is defined by recursion.

1. The basis of the set of sentences is the set PVar(L) of propositional variables of L.
2. The set of sentences is closed under the following production rules:
   (a) If A is a sentence, then so is (¬A).
   (b) If A and B are sentences, then so is (A & B).
   (c) If A and B are sentences, then so is (A ∨ B).
   (d) If A and B are sentences, then so is (A → B).
   (e) If A and B are sentences, then so is (A ↔ B).

Notice that as long as L has at least one propositional variable, then Sent(L) is infinite. When there is no ambiguity, we will drop parentheses.

In order to use propositional logic, we would like to give meaning to the propositional variables. Rather than assigning specific propositions to the propositional variables and then determining their truth or falsity, we consider truth interpretations.
Definition 3.1.2. A truth interpretation for a propositional language \( \mathcal{L} \) is a function

\[ I : \text{PVar}(\mathcal{L}) \rightarrow \{0, 1\}. \]

If \( I(A_i) = 0 \), then the propositional variable \( A_i \) is considered to represent a false proposition under this interpretation. On the other hand, if \( I(A_i) = 1 \), then the propositional variable \( A_i \) is considered to represent a true proposition under this interpretation.

There is a unique way to extend the truth interpretation to all sentences of \( \mathcal{L} \) so that the interpretation of the logical connectives reflects how these connectives are normally understood by mathematicians.

Definition 3.1.3. Define an extension of a truth interpretation \( I : \text{PVar}(\mathcal{L}) \rightarrow \{0, 1\} \) for a propositional language to the collection of all sentences of the language by recursion:

1. On the basis of the set of sentences, \( \text{PVar}(\mathcal{L}) \), the truth interpretation has already been defined.

2. The definition is extended to satisfy the following closure rules:
   
   (a) If \( I(A) \) is defined, then \( I(\neg A) = 1 - I(A) \).
   
   (b) If \( I(A) \) and \( I(B) \) are defined, then \( I(A \& B) = I(A) \cdot I(B) \).
   
   (c) If \( I(A) \) and \( I(B) \) are defined, then \( I(A \lor B) = \max \{I(A), I(B)\} \).
   
   (d) If \( I(A) \) and \( I(B) \) are defined, then

   \[
   I(A \rightarrow B) = \begin{cases} 
   0 & \text{if } I(A) = 1 \text{ and } I(B) = 0, \\
   1 & \text{otherwise}. 
   \end{cases} \tag{3.1}
   
   (e) If \( I(A) \) and \( I(B) \) are defined, then \( I(A \leftrightarrow B) = 1 \) if and only if \( I(A) = I(B) \).

Intuitively, tautologies are statements which are always true, and contradictions are ones which are never true. These concepts can be defined precisely in terms of interpretations.

Definition 3.1.4. A sentence \( \varphi \) is a tautology for a propositional language \( \mathcal{L} \) if every truth interpretation \( I \) has value 1 on \( \varphi \), \( I(\varphi) = 1 \). \( \varphi \) is a contradiction if every truth interpretation \( I \) has value 0 on \( \varphi \), \( I(\varphi) = 0 \). Two sentences \( \varphi \) and \( \psi \) are logically equivalent, in symbols \( \varphi \leftrightarrow \psi \), if every truth interpretation \( I \) takes the same value on both of them, \( I(\varphi) = I(\psi) \). A sentence \( \varphi \) is satisfiable if there is some truth interpretation \( I \) with \( I(\varphi) = 1 \).

The notion of logical equivalence is an equivalence relation; that is, it is a reflexive, symmetric and transitive relation. The equivalence classes given by logical equivalence are infinite for non-trivial languages (i.e., those languages containing at least one propositional variable). However, if the language has only finitely many propositional variables, then there are only finitely many equivalence classes.

Notice that if \( \mathcal{L} \) has \( n \) propositional variables, then there are exactly \( d = 2^n \) truth interpretations, which we may list as \( \mathcal{I} = \{I_0, I_1, \ldots, I_{2^n-1}\} \). Since each \( I_i \) maps the truth values 0 or 1 to each of the \( n \) propositional variables, we can think of each truth interpretation as a function from the set \( \{0, \ldots, n-1\} \) to the set \( \{0, 1\} \). The collection of such functions can be written as \( \{0, 1\}^n \), which can also be interpreted as the collection of binary strings of length \( n \).

Each sentence \( \varphi \) gives rise to a function \( TF_{\varphi} : \mathcal{I} \rightarrow \{0, 1\} \) defined by \( TF_{\varphi}(I_i) = I_i(\varphi) \). Informally, \( TF_{\varphi} \) lists the column under \( \varphi \) in a truth table. Note that for any two sentences \( \varphi \) and \( \psi \), if \( TF_{\varphi} = TF_{\psi} \), then \( \varphi \) and \( \psi \) are logically equivalent. Thus there are exactly \( 2^d = 2^{2^n} \) many equivalence classes.

Lemma 3.1.5. The following pairs of sentences are logically equivalent as indicated by the metalogical symbol \( \leftrightarrow \).
3.2. DISJUNCTIVE NORMAL FORM THEOREM

1. \( \neg \neg A \iff A \).
2. \( \neg A \lor \neg B \iff \neg (A \land B) \).
3. \( \neg A \land \neg B \iff \neg (A \lor B) \).
4. \( A \rightarrow B \iff \neg A \lor B \).
5. \( A \leftrightarrow B \iff (A \rightarrow B) \land (B \rightarrow A) \).

Proof. Each of these statements can be proved using a truth table, so from one example the reader may do the others. Notice that truth tables give an algorithmic approach to questions of logical equivalence.

\[
\begin{array}{cccccc}
I_0 & A & B & \neg A & \neg B & (\neg A) \lor (\neg B) & (A \land B) & (\neg (A \land B)) \\
I_1 & 0 & 0 & 1 & 1 & 1 & 0 & 1 \\
I_2 & 1 & 0 & 0 & 1 & 1 & 0 & 1 \\
I_3 & 0 & 1 & 1 & 0 & 1 & 0 & 1 \\
I_4 & 1 & 1 & 0 & 0 & 1 & 0 & 0
\end{array}
\]

Using the above equivalences, one could assume that \( \neg \) and \( \lor \) are primitive connectives, and define the others in terms of them. The following list gives three pairs of connectives each of which is sufficient to get all our basic list:

\(-, \lor\)
\(-, \land\)
\(-, \rightarrow\)

In logic, the word “theory” has a technical meaning, and refers to any set of statements, whether meaningful or not.

**Definition 3.1.6.** A set \( \Gamma \) of sentences in a language \( \mathcal{L} \) is **satisfiable** if there is some interpretation \( I \) with \( I(\varphi) = 1 \) for all \( \varphi \in \Gamma \). A set of sentences \( \Gamma \) **logically implies** a sentence \( \varphi \), in symbols, \( \Gamma \models \varphi \), if for every interpretation \( I \), if \( I(\psi) = 1 \) for all \( \psi \in \Gamma \), then \( I(\varphi) = 1 \). A (propositional) theory in a language \( \mathcal{L} \) is a set of sentences \( \Gamma \subseteq \text{Sent}(\mathcal{L}) \) which is closed under logical implication.

Notice that a theory as a set of sentences matches with the notion of the theory of plane geometry as a set of axioms. In studying that theory, we developed several models. The interpretations play the role here that models played in that discussion. Here is an example of the notion of logical implication defined above.

**Lemma 3.1.7.** \( \{ (A \land B), (\neg C) \} \models (A \lor B) \).

### 3.2 Disjunctive Normal Form Theorem

In this section we will show that the language of propositional calculus is sufficient to represent every possible truth function.

**Definition 3.2.1.**

1. A **literal** is either a propositional variable \( A_i \), or its negation \( \neg A_i \).
2. A **conjunctive clause** is a conjunction of literals and a **disjunctive clause** is a disjunction of literals. We will assume in each case that each propositional variable occurs at most once.
3. A propositional sentence is in **disjunctive normal form** if it is a disjunction of conjunctive clauses and it is in **conjunctive normal form** if it is a conjunction of disjunctive clauses.
Lemma 3.2.2.

(i) For any conjunctive clause \( C = \phi(A_1, \ldots, A_n) \), there is a unique interpretation \( I_C : \{A_1, \ldots, A_n\} \rightarrow \{0, 1\} \) such that \( I_C(\phi) = 1 \).

(ii) Conversely, for any interpretation \( I : \{A_1, \ldots, A_n\} \rightarrow \{0, 1\} \), there is a unique conjunctive clause \( C_I \) (up to permutation of literals) such that \( I(C_I) = 1 \) and for any interpretation \( J \neq I \), \( J(C_I) = 0 \).

Proof. (i) Let \( B_i = \begin{cases} A_i & \text{if } C \text{ contains } A_i \text{ as a conjunct} \\ \neg A_i & \text{if } C \text{ contains } \neg A_i \text{ as a conjunct} \end{cases} \).

It follows that \( C = B_1 \land \ldots \land B_n \). Now let \( I_C(A_i) = 1 \) if and only if \( A_i = B_i \). Then clearly \( I(B_i) = 1 \) for \( i = 1, 2, \ldots, n \) and therefore \( I_C(C) = 1 \). To show uniqueness, if \( J(C) = 1 \) for some interpretation \( J \), then \( \phi(B_i) = 1 \) for each \( i \) and hence \( J = I_C \).

(ii) Let \( B_i = \begin{cases} A_i & \text{if } I(A_i) = 1 \\ \neg A_i & \text{if } I(A_i) = 0 \end{cases} \).

Let \( C_I = B_1 \land \ldots \land B_n \). As above \( I(C_I) = 1 \) and \( J(C_I) = 1 \) implies that \( J = I \).

It follows as above that \( I \) is the unique interpretation under which \( C_I \) is true. We claim that \( C_I \) is the unique conjunctive clause with this property. Suppose not. Then there is some conjunctive clause \( C' \) such that \( I(C') = 1 \) and \( C' \neq C_I \). This implies that there is some literal \( A_i \) in \( C' \) and \( \neg A_i \) in \( C_I \) (or vice versa). But \( I(C') = 1 \) implies that \( I(A_i) = 1 \) and \( I(C_I) = 1 \) implies that \( I(\neg A_i) = 1 \), which is clearly impossible. Thus \( C_I \) is unique. \( \Box \)

Here is the Disjunctive Normal Form Theorem.

Theorem 3.2.3. For any truth function \( F : \{0, 1\}^n \rightarrow \{0, 1\} \), there is a sentence \( \phi \) in disjunctive normal form such that \( F = TF\phi \).

Proof. Let \( I_1, I_2, \ldots, I_k \) be the interpretations in \( \{0, 1\}^n \) such that \( F(I_i) = 1 \) for \( i = 1, \ldots, k \). For each \( i \), let \( C_i = C_i \) be the conjunctive clauses guaranteed to hold by the previous lemma. Now let \( \phi = C_1 \lor C_2 \lor \ldots \lor C_k \). Then for any interpretation \( I \),

\[
TF\phi(I) = 1 \text{ if and only if } I(\phi) = 1 \text{ (by definition)}
\]

\[
\begin{align*}
&\text{if and only if } I(C_i) = 1 \text{ for some } i = 1, \ldots, k \\
&\text{if and only if } I = I_i \text{ for some } i \text{ (by the previous lemma)} \\
&\text{if and only if } F(I) = 1 \text{ (by the choice of } I_1, \ldots, I_k \text{)}
\end{align*}
\]

Hence \( TF\phi = F \) as desired. \( \Box \)

Example 3.2.4. Suppose that we want a formula \( \phi(A_1, A_2, A_3) \) such that \( I(\phi) = 1 \) only for the three interpretations \((0, 1, 0), (1,1,0) \) and \((1,1,1) \). Then

\[
\phi = (\neg A_1 \land A_2 \land \neg A_3) \lor (A_1 \land A_2 \land \neg A_3) \lor (A_1 \land A_2 \land A_3).
\]

It follows that the connectives \( \neg, \land, \lor \) are sufficient to express all truth functions. By the de Morgan laws \((2,3 \text{ of Lemma 2.5})\) \( \neg, \lor \) are sufficient and \( \neg, \land \) are also sufficient.

### 3.3 Proofs

One of the basic tasks that mathematicians do is proving theorems. This section develops the Propositional Calculus, which is a system of rules of inference for propositional languages. With it
one formalizes the notion of proof. Then one can ask questions about what can be proved, what
cannot be proved, and how the notion of proof is related to the notion of interpretations.

The basic relation in the Propositional Calculus is the relation \( \text{proves} \) between a set, \( \Gamma \) of sentences and a sentence \( B \). A more long-winded paraphrase of the relation “\( \Gamma \) proves \( B \)” is “there is a proof of \( B \) using what ever hypotheses are needed from \( \Gamma \)”. This relation is denoted \( \Gamma \vdash B \), with the following abbreviations for special cases:

<table>
<thead>
<tr>
<th>Formal Version</th>
<th>Abbreviation</th>
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<tbody>
<tr>
<td>( \Gamma \vdash { B } )</td>
<td>( \Gamma \vdash B )</td>
</tr>
<tr>
<td>( { A } \vdash B )</td>
<td>( A \vdash B )</td>
</tr>
<tr>
<td>( \emptyset \vdash B )</td>
<td>( \vdash B )</td>
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</table>

Let \( \bot \) be a new symbol that we will add to our propositional language. The intended interpretation of \( \bot \) is ‘falsehood,’ akin to asserting a contradiction.

**Definition 3.3.1.** A formal proof or derivation of a propositional sentence \( \phi \) from a collection of propositional sentences \( \Gamma \) is a finite sequence of propositional sentences terminating in \( \phi \) where each sentence in the sequence is either in \( \Gamma \) or is obtained from sentences occurring earlier in the sequence by means of one of the following rules.

1. (Given rule) Any \( B \in \Gamma \) may be derived from \( \Gamma \) in one step.
2. (&-Elimination) If \( (A \& B) \) has been derived from \( \Gamma \) then either of \( A \) or \( B \) may be derived from \( \Gamma \) in one further step.
3. (\( \lor \)-Elimination) If \( (A \lor B) \) has been derived from \( \Gamma \), under the further assumption of \( A \) we can derive \( C \) from \( \Gamma \), and under the further assumption of \( B \) we can derive \( C \) from \( \Gamma \), then we can derive \( C \) from \( \Gamma \) in one further step.
4. (\( \rightarrow \)-Elimination) If \( (A \rightarrow B) \) and \( A \) have been derived from \( \Gamma \), then \( B \) can be derived from \( \Gamma \) in one further step.
5. (\( \bot \)-Elimination) If \( \bot \) has been deduced from \( \Gamma \), then we can derive any sentence \( A \) from \( \Gamma \) in one further step.
6. (\( \neg \)-Elimination) If \( \neg \neg A \) has been deduced from \( \Gamma \), then we can derive \( A \) from \( \Gamma \) in one further step.
7. (&-Introduction) If \( A \) and \( B \) have been derived from \( \Gamma \), then \( (A \& B) \) may be derived from \( \Gamma \) in one further step.
8. (\( \lor \)-Introduction) If \( A \) has been derived from \( \Gamma \), then either of \( (A \lor B) \), \( (B \lor A) \) may be derived from \( \Gamma \) in one further step.
9. (\( \rightarrow \)-Introduction) If under the assumption of \( A \) we can derive \( B \) from \( \Gamma \), then we can derive \( A \rightarrow B \) from \( \Gamma \) in one further step.
10. (\( \bot \)-Introduction) If \( (A \& \neg A) \) has been deduced from \( \Gamma \), then we can derive \( \bot \) from \( \Gamma \) in one further step.
11. (\( \neg \)-Introduction) If \( \bot \) has been deduced from \( \Gamma \) and \( A \), then we can derive \( \neg A \) from \( \Gamma \) in one further step.

The relation \( \Gamma \vdash A \) can now be defined to hold if there is a formal proof of \( A \) from \( \Gamma \) that uses the rules given above. The symbol \( \vdash \) is sometimes called a (single) turnstile. Here is a more precise, formal definition.
Definition 3.3.2. The relation $\Gamma \vdash B$ is the smallest subset of pairs $(\Gamma, B)$ from $\mathcal{P}(\text{Sent}) \times \text{Sent}$ which contains every pair $(\Gamma, B)$ such that $B \in \Gamma$ and is closed under the above rules of deduction.

We now provide some examples of proofs.

Proposition 3.3.3. For any sentences $A, B, C$

1. $\vdash A \rightarrow A$

2. $A \rightarrow B \vdash \neg B \rightarrow \neg A$

3. $\{A \rightarrow B, B \rightarrow C\} \vdash A \rightarrow C$

4. $A \vdash A \lor B$ and $A \vdash B \lor A$

5. $\{A \lor B, \neg A\} \vdash B$

6. $A \lor A \vdash A$

7. $A \vdash \neg \neg A$

8. $A \lor B \vdash B \lor A$ and $A \& B \vdash B \& A$

9. $(A \lor B) \lor C \vdash A \lor (B \lor C)$ and $A \lor (B \lor C) \vdash (A \lor B) \lor C$

10. $(A \& B) \& C \vdash A \& (B \& C)$ and $A \& (B \& C) \vdash (A \& B) \& C$

11. $A \& (B \lor C) \vdash (A \& B) \lor (A \& C)$ and $(A \& B) \lor (A \& C) \vdash A \& (B \lor C)$

12. $A \lor (B \& C) \vdash (A \lor B) \& (A \lor C)$ and $(A \lor B) \& (A \lor C) \vdash A \lor (B \& C)$

13. $\neg (A \& B) \vdash \neg A \lor \neg B$ and $\neg A \lor \neg B \vdash \neg (A \& B)$

14. $\neg (A \lor B) \vdash \neg A \& \neg B$ and $\neg A \& \neg B \vdash \neg (A \lor B)$

15. $\neg A \lor B \vdash A \rightarrow B$ and $A \rightarrow B \vdash \neg A \lor B$

16. $\vdash A \lor \neg A$

We give brief sketches of some of these proofs to illustrate the various methods.

Proof.

1. $\vdash A \rightarrow A$

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<thead>
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<tbody>
<tr>
<td>1</td>
<td>$A$</td>
<td>Assumption</td>
</tr>
<tr>
<td>2</td>
<td>$A$</td>
<td>Given</td>
</tr>
<tr>
<td>3</td>
<td>$A \rightarrow A$</td>
<td>$\rightarrow$-Introduction (1-2)</td>
</tr>
</tbody>
</table>
3.3. PROOFS

3. \{A \rightarrow B, B \rightarrow C\} \vdash A \rightarrow C

1. \text{Given}
2. \text{Given}
3. \text{Assumption}
4. \text{→-Elimination 1,3}
5. \text{→-Elimination 2,4}
6. \text{→-Introduction 3-5}

4. A \vdash A \lor B \text{ and } A \vdash B \lor A

1. \text{Given}
2. \text{V-Introduction 1}

1. \text{Given}
2. \text{V-Introduction 1}

5. \{A \lor B, \neg A\} \vdash B

1. \text{Given}
2. \text{Given}
3. \text{Assumption}
4. \text{&-Introduction 2,3}
5. \text{⊥-Introduction 4}
6. \text{⊥-Elimination 5}
7. \text{Assumption}
8. \text{Given}
9. \text{V-Elimination 1-8}

6. A \lor A \vdash A

1. \text{Given}
2. \text{Assumption}
3. \text{Given}
4. \text{Assumption}
5. \text{Given}
6. \text{V-Elimination 1-5}
7. $A \vdash \neg \neg A$

1. $A$  
   Given

2. $\neg A$  
   Assumption

3. $A \& \neg A$  
   $\&$-Introduction 1,2

4. $\bot$  
   $\bot$-Introduction 3

5. $\neg \neg A$  
   $\neg$-Introduction 1-4

8. $A \lor B \vdash B \lor A$ and $A \& B \vdash B \& A$

1. $A \lor B$  
   Given

2. $A$  
   Assumption

3. $B \lor A$  
   $\lor$-Introduction 2

4. $B$  
   Assumption

5. $B \lor A$  
   $\lor$-Introduction 2

6. $B \lor A$  
   $\lor$-Elimination 1-5

1. $A \& B$  
   Given

2. $A$  
   $\&$-Elimination

3. $B$  
   $\&$-Elimination

4. $B \& A$  
   $\&$-Introduction 2-3

10. $(A \& B) \& C \vdash A \& (B \& C)$ and $A \& (B \& C) \vdash (A \& B) \& C$

1. $(A \& B) \& C$  
   Given

2. $A \& B$  
   $\&$-Elimination 1

3. $A$  
   $\&$-Elimination 2

4. $B$  
   $\&$-Elimination 2

5. $C$  
   $\&$-Elimination 1

6. $B \& C$  
   $\&$-Introduction 4,5

7. $A \& (B \& C)$  
   $\&$-Introduction 3,6

13. $\neg (A \& B) \vdash \neg A \lor \neg B$ and $\neg A \lor \neg B \vdash \neg (A \& B)$
3.3. PROOFS

1. \( \neg(A \& B) \)  Given
2. \( A \lor \neg A \) Item 6
3. \( \neg A \) Assumption
4. \( \neg A \lor \neg B \) \lor\text{-Introduction} 4
5. \( A \) Assumption
6. \( B \lor \neg B \) Item 6
7. \( \neg B \) Assumption
8. \( \neg A \lor \neg B \) \lor\text{-Introduction} 7
9. \( B \) Assumption
10. \( A \& B \) \&\text{-Introduction} 5,9
11. \( \bot \) \bot\text{-Introduction} 1,10
12. \( \neg A \lor \neg B \) Item 5
13. \( \neg A \lor \neg B \) \lor\text{-Elimination} 6-12
14. \( \neg A \lor \neg B \) \lor\text{-Elimination} 2-13

1. \( \neg A \lor \neg B \) Given
2. \( A \& B \) Assumption
3. \( A \) \&\text{-Elimination} 2
4. \( \neg \neg A \) Item 8, 3
5. \( \neg B \) Disjunctive Syllogism 1,4
6. \( B \) \&\text{-Elimination} 2
7. \( \bot \) \bot\text{-Introduction} 5,6
8. \( \neg(A \& B) \) Proof by Contradiction 2-7

15. \( \neg A \lor B \vdash A \rightarrow B \) and \( A \rightarrow B \vdash \neg A \lor B \)

1. \( \neg A \lor B \) Given
2. \( A \) Assumption
3. \( \neg \neg A \) Item 8, 2
4. \( B \) Item 5 1,3
5. \( A \rightarrow B \) \rightarrow\text{-Introduction} 2-4
CHAPTER 3. PROPOSITIONAL LOGIC

1. \( A \rightarrow B \) Given

2. \( \neg(\neg A \lor B) \) Assumption

3. \( \neg
\neg A \land \neg B \) Item 14, 1

4. \( \neg
\neg A \) &-Introduction 3

5. \( A \) \( \neg\)-Elimination 4

6. \( B \) \( \rightarrow\)-Elimination 1,5

7. \( \neg B \) &-Introduction 3

8. \( B \land \neg B \) &-Introduction 6,7

9. \( \bot \) \( \bot\)-Introduction 8

10. \( \neg
\neg A \lor B \) \( \bot\)-Elimination 2-9

16. \( \vdash A \lor \neg A \)

1. \( \neg(A \lor \neg A) \) Assumption

2. \( \neg A \land \neg A \) Item 14, 1

3. \( \bot \) \( \bot\)-Introduction 3

4. \( \neg A \lor A \) \( \neg\lor\)-Rule (1-2)

The following general properties about \( \vdash \) will be useful when we prove the soundness and completeness theorems.

**Lemma 3.3.4.** For any sentences \( A \) and \( B \), if \( \Gamma \vdash A \) and \( \Gamma \cup \{A\} \vdash B \), then \( \Gamma \vdash B \).

**Proof.** \( \Gamma \cup \{A\} \vdash B \) implies \( \Gamma \vdash A \rightarrow B \) by \( \rightarrow\)-Introduction. Combining this latter fact with the fact that \( \Gamma \vdash A \) yields \( \Gamma \vdash B \) by \( \rightarrow\)-Elimination.

**Lemma 3.3.5.** If \( \Gamma \vdash B \) and \( \Gamma \subseteq \Delta \), then \( \Delta \vdash B \).

**Proof.** This follows by induction on proof length. For the base case, if \( B \) follows from \( \Gamma \) on the basis of the Given Rule, then it must be the case that \( B \in \Gamma \). Since \( \Gamma \subseteq \Delta \) it follows that \( B \in \Delta \) and hence \( \Delta \vdash B \) by the Given Rule.

If the final step in the proof of \( B \) from \( \Gamma \) is made on the basis of any one of the rules, then we may assume by the induction hypothesis that the other formulas used in these deductions follow from \( \Delta \) (since they follow from \( \Gamma \)). We will look at two cases and leave the rest to the reader.

Suppose that the last step comes by \( \rightarrow\)-Elimination, where we have derived \( A \rightarrow B \) and \( A \) from \( \Gamma \) earlier in the proof. Then we have \( \Gamma \vdash A \rightarrow B \) and \( \Gamma \vdash B \). By the induction hypothesis, \( \Delta \vdash A \) and \( \Delta \vdash A \rightarrow B \). Hence \( \Delta \vdash B \) by \( \rightarrow\)-Elimination.

Suppose that the last step comes from \&-Elimination, where we have derived \( A \land B \) from \( \Gamma \) earlier in the proof. Since \( \Gamma \vdash A \land B \), by inductive hypothesis it follows that \( \Delta \vdash A \land B \). Hence \( \Delta \vdash B \) by \&-elimination.

Next we prove a version of the Compactness Theorem for our deduction system.

**Theorem 3.3.6.** If \( \Gamma \vdash B \), then there is a finite set \( \Gamma_0 \subseteq \Gamma \) such that \( \Gamma_0 \vdash B \).
3.4. The Soundness Theorem

We now determine the precise relationship between \( \vdash \) and \( \models \) for propositional logic. Our first major theorem says that if one can prove something in \( A \) from a theory \( \Gamma \), then \( \Gamma \) logically implies \( A \).

**Theorem 3.4.1** (Soundness Theorem). If \( \Gamma \vdash A \), then \( \Gamma \models A \).

**Proof.** Again we argue by induction on proofs. For the base case, if \( B \) follows from \( \Gamma \) on the basis of the Given Rule, then \( B \in \Gamma \) and we can let \( \Gamma_0 = \{B\} \).

If the final step in the proof of \( B \) from \( \Gamma \) is made on the basis of any one of the rules, then we may assume by the induction hypothesis that the other formulas used in these deductions follow from some finite \( \Gamma_0 \subseteq \Gamma \). We will look at two cases and leave the rest for the reader.

Suppose that the last step of the proof comes by \( \lor \)-Introduction, so that \( B \) is of the form \( C \lor D \). Then, without loss of generality, we can assume that we derived \( C \) from \( \Gamma \) earlier in the proof. Thus \( \Gamma \vdash C \). By the induction hypothesis, there is a finite \( \Gamma_0 \subseteq \Gamma \) such that \( \Gamma_0 \vdash C \). Hence by \( \lor \)-Introduction, \( \Gamma_0 \vdash C \lor D \).

Suppose that the last step of the proof comes by \( \lor \)-Elimination. Then earlier in the proof

(i) we have derived some formula \( C \lor D \) from \( \Gamma \),

(ii) under the assumption of \( C \) we have derived \( B \) from \( \Gamma \), and

(iii) under the assumption of \( D \) we have derived \( B \) from \( \Gamma \).

Thus, \( \Gamma \vdash C \lor D \), \( \Gamma \cup \{C\} \vdash B \), and \( \Gamma \cup \{D\} \vdash B \). Then by assumption, by the induction hypothesis, there exist finite sets \( \Gamma_0, \Gamma_1, \text{ and } \Gamma_2 \) of \( \Gamma \) such that \( \Gamma_0 \vdash C \lor D \), \( \Gamma_1 \cup \{C\} \vdash B \) and \( \Gamma_2 \cup \{D\} \vdash B \). By Lemma 3.3.5,

(i) \( \Gamma_0 \cup \Gamma_1 \lor \Gamma_2 \vdash C \lor D \)

(ii) \( \Gamma_0 \cup \Gamma_1 \lor \Gamma_2 \\{C\} \vdash B \)

(iii) \( \Gamma_0 \cup \Gamma_1 \lor \Gamma_2 \\{D\} \vdash B \)

Thus by \( \lor \)-Elimination, we have \( \Gamma_0 \cup \Gamma_1 \lor \Gamma_2 \vdash B \). Since \( \Gamma_0 \cup \Gamma_1 \lor \Gamma_2 \) is finite and \( \Gamma_0 \cup \Gamma_1 \lor \Gamma_2 \subseteq \Gamma \), the result follows. \( \square \)

### 3.4 The Soundness Theorem

We now determine the precise relationship between \( \vdash \) and \( \models \) for propositional logic. Our first major theorem says that if one can prove something in \( A \) from a theory \( \Gamma \), then \( \Gamma \) logically implies \( A \).

**Theorem 3.4.1** (Soundness Theorem). If \( \Gamma \vdash A \), then \( \Gamma \models A \).

**Proof.** The proof is by induction on the length of the deduction of \( A \). We need to show that if there is a proof of \( A \) from \( \Gamma \), then for any interpretation \( I \) such that \( I(\gamma) = 1 \) for all \( \gamma \in \Gamma \), \( I(A) = 1 \).

**Base Case:** For a one-step deduction, we must have used the Given Rule, so that \( A \in \Gamma \). If the truth interpretation \( I \) has \( I(\gamma) = 1 \) for all \( \gamma \in \Gamma \), then of course \( I(A) = 1 \) since \( A \in \Gamma \).

**Induction:** Assume the theorem holds for all shorter deductions. Now proceed by cases on the other rules. We prove a few examples and leave the rest for the reader.

Suppose that the last step of the deduction is given by \( \lor \)-Introduction, so that \( A \) has the form \( B \lor C \). Without loss of generality, suppose we have derived \( B \) from \( \Gamma \) earlier in the proof. Suppose that \( I(\gamma) = 1 \) for all \( \gamma \in \Gamma \). Since the proof of \( \Gamma \vdash B \) is shorter than the given deduction of \( B \lor C \), by the inductive hypothesis, \( I(B) = 1 \). But then \( I(B \lor C) = 1 \) since \( I \) is an interpretation.

Suppose that the last step of the deduction is given by \&-Elimination. Suppose that \( I(\gamma) = 1 \) for all \( \gamma \in \Gamma \). Without loss of generality, \( A \) has been derived from a sentence of the form \( A \& B \), which has been derived from \( \Gamma \) in a strictly shorter proof. Since \( \Gamma \vdash A \& B \), it follows by inductive hypothesis that \( \Gamma \models A \& B \), and hence \( I(A \& B) = 1 \). Since \( I \) is an interpretation, it follows that \( I(A) = 1 \).

Suppose that the last step of the deduction is given by \( \to \)-Introduction. Then \( A \) has the form \( B \to C \). It follows that under the assumption of \( B \), we have derived \( C \) from \( \Gamma \). Thus \( \Gamma \cup \{B\} \vdash C \) in a strictly shorter proof. Suppose that \( I(\gamma) = 1 \) for all \( \gamma \in \Gamma \). We have two cases to consider.
Case 1: If $I(B) = 0$, it follows that $I(B \to C) = 1$.

Case 2: If $I(B) = 1$, then since $\Gamma \cup \{B\} \vdash C$, it follows that $I(C) = 1$. Then $I(B \to C) = 1$. In either case, the conclusion follows. 

Now we know that anything we can prove is true. We next consider the contrapositive of the Soundness Theorem.

**Definition 3.4.2.** A set $\Gamma$ of sentences is consistent if there is some sentence $A$ such that $\Gamma \not\vdash A$; otherwise $\Gamma$ is inconsistent.

**Lemma 3.4.3.** $\Gamma$ of sentences is inconsistent if and only if there is some sentence $A$ such that $\Gamma \vdash A$ and $\Gamma \vdash \neg A$.

**Proof.** Suppose first that $\Gamma$ is inconsistent. Then by definition, $\Gamma \vdash \phi$ for all formulas $\phi$ and hence $\Gamma \vdash A$ and $\Gamma \vdash \neg A$ for every sentence $A$.

Next suppose that, for some $A$, $\Gamma \vdash A$ and $\Gamma \vdash \neg A$. It follows by $\&$-Introduction that $\Gamma \vdash A \& \neg A$. By $\bot$-Introduction, $\Gamma \vdash \bot$. Then by $\bot$-Elimination, for each $\phi$, $\Gamma \vdash \phi$. Hence $\Gamma$ is inconsistent. 

**Proposition 3.4.4.** If $\Gamma$ is satisfiable, then it is consistent.

**Proof.** Assume that $\Gamma$ is satisfiable and let $I$ be an interpretation such that $I(\gamma) = 1$ for all $\gamma \in \Gamma$. Now suppose by way of contradiction that $\Gamma$ is not consistent. Then there is some sentence $A$ such that $\Gamma \vdash A$ and $\Gamma \vdash \neg A$. By the Soundness Theorem, $\Gamma \models A$ and $\Gamma \models \neg A$. But then $I(A) = 1$ and $I(\neg A) = 1$ which is impossible since $I$ is an interpretation. This contradiction demonstrates that $\Gamma$ is consistent.

In Section 3.5, we will prove the converse of the Soundness Theorem by showing that any consistent theory is satisfiable.

### 3.5 The Completeness Theorem

**Theorem 3.5.1.** (The Completeness Theorem, Version I) If $\Gamma \models A$, then $\Gamma \vdash A$.

**Theorem 3.5.2.** (The Completeness Theorem, Version II) If $\Gamma$ is consistent, then $\Gamma$ is satisfiable.

We will show that Version II implies Version I and then prove Version II. First we give alternate versions of the Compactness Theorem (Theorem 3.3.6).

**Theorem 3.5.3.** (Compactness Theorem, Version II). If every finite subset of $\Delta$ is consistent, then $\Delta$ is consistent.

**Proof.** We show the contrapositive. Suppose that $\Delta$ is not consistent. Then, for some $B$, $\Delta \vdash B \& \neg B$. It follows from Theorem 3.3.6 that $\Delta$ has a finite subset $\Delta_0$ such that $\Delta_0 \vdash B \& \neg B$. But then $\Delta_0$ is not consistent.

**Theorem 3.5.4.** (Compactness Theorem, Version III). Suppose that

(i) $\Delta = \bigcup_n \Delta_n$,

(ii) $\Delta_n \subseteq \Delta_{n+1}$ for every $n$, and

(iii) $\Delta_n$ is consistent for each $n$.

Then $\Delta$ is consistent.
3.5. THE COMPLETENESS THEOREM

Proof. Again we show the contrapositive. Suppose that $\Delta$ is not consistent. Then by Theorem 3.5.4, $\Delta$ has a finite, inconsistent subset $F = \{\delta_1, \delta_2, \ldots, \delta_k\}$. Since $\Delta = \bigcup_n \Delta_n$, there exists, for each $i \leq k$, some $n_i$ such that $\delta_i \in \Delta_{n_i}$. Letting $n = \max\{n_i : i \leq k\}$, it follows from the fact that the $\Delta$'s are inconsistent that $F \subseteq \Delta_n$. But then $\Delta_n$ is inconsistent. \hfill $\Box$

Next we prove a useful lemma.

**Lemma 3.5.5.** For any $\Gamma$ and $A$, $\Gamma \vdash A$ if and only if $\Gamma \cup \{\neg A\}$ is inconsistent.

**Proof.** Suppose first that $\Gamma \vdash A$. Then $\Gamma \cup \{\neg A\}$ proves both $A$ and $\neg A$ and is therefore inconsistent.

Suppose next that $\Gamma \cup \{\neg A\}$ is inconsistent. It follows from $\neg$-Introduction that $\Gamma \vdash \neg A$. Then by $\neg$-Elimination, $\Gamma \vdash A$. \hfill $\Box$

We are already in position to show that Version II of the Completeness Theorem implies Version I. We show the contrapositive of the statement of Version 1; that is, we show $\Gamma \not\models A$ implies $\Gamma \not\models \Gamma \cup \{\neg A\}$.

Suppose it is not the case that $\Gamma \models A$. Then by Lemma 3.5.5, $\Gamma \cup \{\neg A\}$ is consistent. Thus by Version II, $\Gamma \cup \{\neg A\}$ is satisfiable. Then it is not the case that $\Gamma \models A$.

We establish a few more lemmas.

**Lemma 3.5.6.** If $\Gamma$ is consistent, then for any $A$, either $\Gamma \cup \{A\}$ is consistent or $\Gamma \cup \{\neg A\}$ is consistent.

**Proof.** Suppose that $\Gamma \cup \{\neg A\}$ is inconsistent. Then by the previous lemma, $\Gamma \vdash A$. Then, for any $B$, $\Gamma \cup \{A\} \vdash B$ if and only if $\Gamma \vdash B$. Since $\Gamma$ is consistent, it follows that $\Gamma \cup \{A\}$ is also consistent. \hfill $\Box$

**Definition 3.5.7.** A set $\Delta$ of sentences is **maximally consistent** if it is consistent and for any sentence $A$, either $A \in \Delta$ or $\neg A \in \Delta$.

**Lemma 3.5.8.** Let $\Delta$ be maximally consistent.

1. For any sentence $A$, $\neg A \in \Delta$ if and only if $A \notin \Delta$.

2. For any sentence $A$, if $\Delta \vdash A$, then $A \in \Delta$.

**Proof.** (1) If $\neg A \in \Delta$, then $A \notin \Delta$ since $\Delta$ is consistent. If $A \notin \Delta$, then $\neg A \in \Delta$ since $\Delta$ is maximally consistent.

(2) Suppose that $\Delta \vdash A$ and suppose by way of contradiction that $A \notin \Delta$. Then by part (1), $\neg A \in \Delta$. But this contradicts the consistency of $\Delta$. \hfill $\Box$

**Proposition 3.5.9.** Let $\Delta$ be maximally consistent and define the function $I : \text{Sent} \rightarrow \{0, 1\}$ as follows. For each sentence $B$,

\[
I(B) = \begin{cases} 
1 & \text{if } B \in \Delta; \\
0 & \text{if } B \notin \Delta.
\end{cases}
\]

Then $I$ is a truth interpretation and $I(B) = 1$ for all $B \in \Delta$.

**Proof.** We need to show that $I$ preserves the four connectives: $\neg$, $\lor$, $\&$, and $\rightarrow$. We will show the first three and leave the last an exercise.

($\neg$): It follows from the definition of $I$ and Lemma 3.5.8 that $I(\neg A) = 1$ if and only if $\neg A \in \Delta$ if and only if $A \notin \Delta$ if and only if $I(A) = 0$.

($\lor$): Suppose that $I(A \lor B) = 1$. Then $A \lor B \in \Delta$. We argue by cases. If $A \in \Delta$, then clearly $\max[I(A), I(B)] = 1$. Now suppose that $A \notin \Delta$. Then by completeness, $\neg A \in \Delta$. It follows from Proposition 3.3.3(5) that $\Delta \vdash \neg B$. Hence $B \in \Delta$ by Lemma 3.5.8. Thus $\max[I(A), I(B)] = 1$.

Next suppose that $\max[I(A), I(B)] = 1$. Without loss of generality, $I(A) = 1$ and hence $A \in \Delta$. Then $\Delta \vdash A \lor B$ by $\lor$-Introduction, so that $A \lor B \in \Delta$ by Lemma 3.5.8 and hence $I(A \lor B) = 1$.

($\&$): Suppose that $I(A \& B) = 1$. Then $A \& B \in \Delta$. It follows from $\&$-Elimination that $\Delta \vdash A$ and $\Delta \vdash B$. Thus by Lemma 3.5.8, $A \in \Delta$ and $B \in \Delta$. Thus $I(A) = I(B) = 1$.

Next suppose that $I(A) = I(B) = 1$. Then $A \in \Delta$ and $B \in \Delta$. It follows from $\&$-Introduction that $\Delta \vdash A \& B$ and hence $A \& B \in \Delta$. Therefore $I(A \& B) = 1$. \hfill $\Box$
We now prove Version II of the Completeness Theorem.

**Proof of Theorem 3.5.2.** Let $\Gamma$ be a consistent set of propositional sentences. Let $A_0, A_1, \ldots$ be an enumeration of the set of sentences. We will define a sequence $\Delta_0 \subseteq \Delta_1 \subseteq \ldots$ and let $\Delta = \bigcup_n \Delta_n$.

We will show that $\Delta$ is a complete and consistent extension of $\Gamma$ and then define an interpretation $I = I_\Delta$ to show that $\Gamma$ is satisfiable.

$\Delta_0 = \Gamma$ and, for each $n$,

$$\Delta_{n+1} = \begin{cases} 
\Delta_n \cup \{A_n\}, & \text{if } \Delta_n \cup \{A_n\} \text{ is consistent} \\
\Delta_n \cup \{\neg A_n\}, & \text{otherwise.}
\end{cases}$$

It follows from the construction that, for each sentence $A_n$, either $A_n \in \Delta_{n+1}$ or $\neg A_n \in \Delta_{n+1}$. Hence $\Delta$ is complete. It remains to show that $\Delta$ is consistent.

**Claim 1:** For each $n$, $\Delta_n$ is consistent.

**Proof of Claim 1:** The proof is by induction. For the base case, we are given that $\Delta_0 = \Gamma$ is consistent. For the induction step, suppose that $\Delta_n$ is consistent. Then by Lemma 3.5.6, either $\Delta_n \cup \{A_n\}$ is consistent, or $\Delta_n \cup \{\neg A_n\}$ is consistent. In the first case, suppose that $\Delta_n \cup \{A_n\}$ is consistent. Then $\Delta_{n+1} = \Delta_n \cup \{A_n\}$ and hence $\Delta_{n+1}$ is consistent. In the second case, suppose that $\Delta_n \cup \{A_n\}$ is inconsistent. Then $\Delta_{n+1} = \Delta_n \cup \{\neg A_n\}$ and hence $\Delta_{n+1}$ is consistent by Lemma 3.5.6.

**Claim 2:** $\Delta$ is consistent.

**Proof of Claim 2:** This follows immediately from the Compactness Theorem Version III.

It now follows from Proposition 3.5.9 that there is a truth interpretation $I$ such that $I(\delta) = 1$ for all $\delta \in \Delta$. Since $\Gamma \subseteq \Delta$, this proves that $\Gamma$ is satisfiable.

We note the following consequence of the proof of the Completeness Theorem.

**Theorem 3.5.10.** Any consistent theory $\Gamma$ has a maximally consistent extension. □

### 3.6 Completeness, Consistency and Independence

For a given set of sentences $\Gamma$, we sometimes identify $\Gamma$ with the theory $\text{Th}(\Gamma) = \{B : \Gamma \vdash B\}$. Thus we can alternatively define $\Gamma$ to be consistent if there is no sentence $B$ such that $\Gamma \vdash B$ and $\Gamma \vdash \neg B$. Moreover, let us say that $\Gamma$ is complete if for every sentence $B$, either $\Gamma \vdash B$ or $\Gamma \vdash \neg B$ (Note that if $\Gamma$ is maximally consistent, it follows that $\Gamma$ is complete, but the converse need not hold. We say that a consistent set $\Gamma$ is independent if $\Gamma$ has no proper subset $\Delta$ such that $\text{Th}(\Delta) = \text{Th}(\Gamma)$; this means that $\Gamma$ is minimal among the sets $\Delta$ with $\text{Th}(\Delta) = \text{Th}(\Gamma)$.

For example, in the language $\mathcal{L}$ with three propositional variables $A, B, C$, the set $\{A, B, C\}$ is clearly independent and complete.

**Lemma 3.6.1.** $\Gamma$ is independent if and only if, for every $B \in \Gamma$, it is not the case that $\Gamma \setminus \{B\} \vdash B$.

**Proof.** Left to the reader. □

**Lemma 3.6.2.** A set $\Gamma$ of sentences is complete and consistent if and only if there is a unique interpretation $I$ satisfied by $\Gamma$.

**Proof.** Left to the reader. □

We conclude this chapter with several examples.

**Example 3.6.3.** Let $\mathcal{L} = \{A_0, A_1, \ldots\}$. 
1. The set \( \Gamma_0 = \{ A_0, A_0 \land A_1, A_1 \land A_2, \ldots \} \) is complete but not independent.

- It is complete since \( \Gamma_0 \vdash A_n \) for all \( n \), which determines the unique truth interpretation \( I \) where \( I(A_n) = 1 \) for all \( n \).
- It is not independent since, for each \( n \), \( (A_0 \land A_1 \land \cdots \land A_{n+1}) \to (A_0 \land \cdots \land A_n) \).

2. The set \( \Gamma_1 = \{ A_0, A_0 \to A_1, A_1 \to A_2, \ldots \} \) is complete and independent.

- It is complete since \( \Gamma_0 \vdash A_n \) for all \( n \), which determines the unique truth interpretation \( I \) where \( I(A_n) = 1 \) for all \( n \).
- To show that \( \Gamma_1 \) is independent, it suffice to show that, for each single formula \( A_n \to A_{n+1} \), it is not the case that \( \Gamma_1 \{ A_n \to A_{n+1} \} \vdash (A_n \to A_{n+1}) \). This is witnessed by the interpretation \( I \) where \( I(A_j) = 1 \) if \( j \leq n \) and \( I(A_j) = 0 \) if \( j > n \).

3. The set \( \Gamma_2 = \{ A_0 \lor A_1, A_2 \lor A_3, A_4 \lor A_5, \ldots \} \) is independent but not complete.

- It is not complete since there are many different interpretations satisfied by \( \Gamma_2 \). In particular, one interpretation could make \( A_n \) true if and only if \( n \) is odd, and another could make \( A_n \) true if and only if \( n \) is even.
- It is independent since, for each \( n \), we can satisfy every sentence of \( \Gamma_2 \) except \( A_{2n} \lor A_{2n+1} \) by the interpretation \( I \) where \( I(A_j) = 0 \) exactly when \( j = 2n \) or \( j = 2n+1 \).

**Exercises**

1. Prove that \( (A \to B) \) and \( ((\neg A) \lor B) \) are logically equivalent.

2. Construct a proof that \( \{ P, (\neg P) \} \vdash Q \).

3. Construct a proof that \( ((\neg A) \land (\neg B)) \vdash (\neg (A \lor B)) \).

4. Prove: \( \vdash ((\neg (A \lor (\neg A))) \to B) \).

5. Prove: \( \vdash (A \land (\neg A)) \to (\neg (A \lor (\neg A))) \).

6. Show that the Lindenbaum Algebra satisfies the DeMorgan Laws.

7. Every function \( TF : \{ 0, 1, 2, 3 \} \to \{ 0, 1 \} \) can be represented by means of an expression \( \varphi \) in \( \neg \) and \( \land \) and to propositional variables \( A, B \). Produce \( \varphi \) so that \( TF_\varphi \) has the values listed in the table below.

<table>
<thead>
<tr>
<th>( A )</th>
<th>( B )</th>
<th>( TF_\varphi )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>0</td>
</tr>
</tbody>
</table>

8. Prove that for any Boolean algebra \( B = (\mathcal{B}, \lor, \land, 0, 1) \) and any element \( g \) of \( \mathcal{B} \), the set \( F = \{ b \in \mathcal{B} \mid b \land g = g \} \) is a filter.
9. Investigate the following sets of formulas for satisfiability. For those that are satisfiable, give an interpretation which makes them all true. For those that are not satisfiable, show that a contradiction of the form $P \& (\neg P)$ can be derived from the set, by giving a proof.

(a) $A \rightarrow \neg (B \& C)$  
(b) $(A \lor B) \rightarrow (C \& D)$

(c) $(A \rightarrow B) \& (C \rightarrow D)$  
(d) $((A \rightarrow B) \& C) \& ((D \rightarrow B) \& E)$

(G $\rightarrow \neg (H \lor I)$  
(e) The contract is fulfilled if and only if the house is completed in February. If the house is completed in February, then we can move in March 1. If we can’t move in March 1, then we must pay rent for March. If the contract is not fulfilled, then we must pay rent for March. We will not pay rent for March. (Use $C$, $H$, $M$, $R$ for the various atomic propositions.)

(d) $((G \rightarrow \neg A) \& H) \rightarrow I$

(e) $\neg (\neg C \rightarrow E)$
Chapter 4

Predicate Logic

Propositional logic treats a basic part of the language of mathematics, building more complicated sentences from simple with connectives. However it is inadequate as it stands to express the richness of mathematics. Consider the axiom of the theory of Plane Geometry, PG, which expresses the fact that any two points belong to a line. We wrote that statement formally with two one-place predicates, $Pt$ for points and $Ln$ for lines, and one two-place predicate, $In$ for incidence as follows:

$$(\forall P, Q \in Pt)(\exists \ell \in Ln)((PIn \ell) \land (QIn \ell)).$$

This axiom includes predicates and quantifies certain elements. In order to test the truth of it, one needs to know how to interpret the predicates $Pt$, $Ln$ and $In$, and the individual elements $P$, $Q$, $\ell$. Notice that these elements are “quantified” by the quantifiers to “for every” and “there is . . . such that.” Predicate logic is an enrichment of propositional logic to include predicates, individuals and quantifiers, and is widely accepted as the standard language of mathematics.

4.1 The Language of Predicate Logic

The symbols of the language of the predicate logic are

1. logical connectives, $\neg$, $\lor$, $\land$, $\rightarrow$, $\leftrightarrow$;
2. the equality symbol $=$;
3. predicate letters $P_i$ for each natural number $i$;
4. function symbols $F_j$ for each natural number $j$;
5. constant symbols $c_k$ for each natural number $k$;
6. individual variables $v_\ell$ for each natural number $\ell$;
7. quantifier symbols $\exists$ (the existential quantifier) and $\forall$ (the universal quantifier); and
8. punctuation symbols (, ).

A predicate letter is intended to represent a relation. Thus each predicate letter $P$ is $n$-ary for some $n$, which means that we write $P(v_1, \ldots, v_n)$. Similarly, a function symbol also is $n$-ary for some $n$.

We make a few remarks on the quantifiers:

(a) $(\exists x)\phi$ is read “there exists an $x$ such that $\phi$ holds.”

(b) $(\forall x)\phi$ is read “for all $x$, $\phi$ holds.”
(c) \((\forall x)\theta\) may be thought of as an abbreviation for \((\neg(\exists x)(\neg\theta))\).

**Definition 4.1.1.** A countable first-order language is obtained by specifying a subset of the predicate letters, function symbols and constants.

One can also work with uncountable first-order languages, but aside from a few examples in Chapter 4, we will primarily work with countable first-order languages. An example of a first-order language is the language of arithmetic.

**Example 4.1.2.** The language of arithmetic is specified by \(\{<, +, \times, 0, 1\}\). Here < is a 2-place relation, + and \(\times\) are 2-place functions and 0, 1 are constants. Equality is a special 2-place relation that we will include in every language.

We now describe how first-order sentences are built up from a given language \(\mathcal{L}\).

**Definition 4.1.3.** The set of terms in a language \(\mathcal{L}\), denoted \(\text{Term}(\mathcal{L})\), is recursively defined by

1. each variable and constant is a term; and
2. if \(t_1, \ldots, t_n\) are terms and \(F\) is an \(n\)-place function symbol, then \(F(t_1, \ldots, t_n)\) is a term.

A constant term is a term with no variables.

**Definition 4.1.4.** Let \(\mathcal{L}\) be a first-order language. The collection of \(\mathcal{L}\)-formulas is defined by recursion. First, the set of atomic formulas, denoted \(\text{Atom}(\mathcal{L})\), consists of formulas of one of the following forms:

1. \(P(t_1, \ldots, t_n)\) where \(P\) is an \(n\)-place predicate letter and \(t_1, \ldots, t_n\) are terms; and
2. \(t_1 = t_2\) where \(t_1\) and \(t_2\) are terms.

The set of \(\mathcal{L}\)-formulas is closed under the following rules

3. If \(\phi\) and \(\theta\) are \(\mathcal{L}\)-formulas, then \((\phi \lor \theta)\) is an \(\mathcal{L}\)-formula. (Similarly, \((\phi \land \theta)\), \((\phi \to \theta)\), \((\phi \iff \theta)\), are \(\mathcal{L}\)-formulas.)
4. If \(\phi\) is an \(\mathcal{L}\)-formula, then \((\neg \phi)\) is an \(\mathcal{L}\)-formula.
5. If \(\phi\) is an \(\mathcal{L}\)-formula, then \((\exists v)\phi\) is an \(\mathcal{L}\)-formula (as is \((\forall v)\phi\)).

An example of an atomic formula in the language of arithmetic

\[0 + x = 0.\]

An example of a more complicated formula in the language of plane geometry is the statement that every element either has a point incident with it or is incident with some line.

\[(\forall v)(\exists x)((x\text{Inv }v) \lor (v\text{In }x)).\]

A variable \(v\) that occurs in a formula \(\phi\) becomes bound when it is placed in the scope of a quantifier, that is, \((\exists v)\) is placed in front of \(\phi\), and otherwise \(v\) is free. The concept of being free over-rides the concept of being bound in the sense that if a formula has both free and bound occurrences of a variable \(v\), then \(v\) occurs free in that formula. The formal definition of bound and free variables is given by recursion.

**Definition 4.1.5.** A variable \(v\) is free in a formula \(\phi\) if

1. \(\phi\) is atomic;
2. \(\phi\) is \((\psi \lor \theta)\) and \(v\) is free in whichever one of \(\psi\) and \(\theta\) in which it appears;
3. \(\phi\) is \((\neg \psi)\) and \(v\) is free in \(\psi\);
4. φ is \((\exists y)\psi\), \(v\) is free in \(\psi\) and \(y\) is not \(v\).

**Example 4.1.6.**

1. In the atomic formula \(x + 5 = 12\), the variable \(x\) is free.
2. In the formula \((\exists x)(x + 5 = 12)\), the variable \(x\) is bound.
3. In the formula \((\exists x)[(x \in \mathbb{R}) \& (|x - 5| = 10)]\), the variable \(x\) is bound.

We will refer to an \(L\)-formula with no free variables as an \(L\)-sentence.

### 4.2 Models and Interpretations

In propositional logic, we used truth tables and interpretations to consider the possible truth of complex statements in terms of their simplest components. In predicate logic, to consider the possible truth of complex statements that involve quantified variables, we need to introduce models with universes from which we can select the possible values for the variables.

**Definition 4.2.1.** Suppose that \(L\) is a first-order language with

(i) predicate symbols \(P_1, P_2, \ldots\),
(ii) function symbols \(F_1, F_2, \ldots\), and
(iii) constant symbols \(c_1, c_2, \ldots\).

Then an \(L\)-structure \(\mathfrak{A}\) consists of

(a) a nonempty set \(A\) (called the domain or universe of \(\mathfrak{A}\)),
(b) a relation \(P_i^{\mathfrak{A}}\) on \(A\) corresponding to each predicate symbol \(P_i\),
(c) a function \(F_i^{\mathfrak{A}}\) on \(A\) corresponding to each function symbol \(F_i\), and
(d) a element \(c_i^{\mathfrak{A}} \in A\) corresponding to each constant symbol \(c_i\).

Each relation \(P_i^{\mathfrak{A}}\) requires the same number of places as \(P_i\), so that \(P_i^{\mathfrak{A}}\) is a subset of \(A^r\) for some fixed \(r\) (called the arity of \(P_i\)). In addition, each function \(F_i^{\mathfrak{A}}\) requires the same number of places as \(F_i\), so that \(F_i^{\mathfrak{A}} : A^r \rightarrow A\) for some fixed \(r\) (called the arity of \(F_i\)).

**Definition 4.2.2.** Given a \(L\)-structure \(\mathfrak{A}\), an interpretation \(I\) into \(\mathfrak{A}\) is a function \(I\) from the variables and constants of \(L\) into the universe \(A\) of \(\mathfrak{A}\) that respects the interpretations of the symbols in \(L\).

In particular, we have

(i) for each constant symbol \(c_j\), \(I(c_j) = c_j^{\mathfrak{A}}\),
(ii) for each function symbol \(F_i\), if \(F_i\) has parity \(n\) and \(t_1, \ldots, t_n\) are terms such that \(I(t_1), I(t_2), \ldots I(t_n)\) have been defined, then

\[ I(F_i(t_1, \ldots, t_n)) = F_i^{\mathfrak{A}}(I(t_1), \ldots, I(t_n)).\]

For any interpretation \(I\) and any variable or constant \(x\) and for any element \(b\) of the universe, let \(I_{b/x}\) be the interpretation defined by

\[
I_{b/x}(z) = \begin{cases} 
  b & \text{if } z = x, \\
  I(z) & \text{otherwise.}
\end{cases}
\]
**Definition 4.2.3.** We define by recursion the relation that a structure $\mathfrak{A}$ satisfies a formula $\phi$ via an interpretation $I$ into $\mathfrak{A}$, denoted $\mathfrak{A} \models_I \phi$:

For atomic formulas, we have:

1. $\mathfrak{A} \models_I t = s$ if and only if $I(t) = I(s)$;

2. $\mathfrak{A} \models_I P_t(t_1, \ldots, t_n)$ if and only if $P^\mathfrak{A}_I(I(t_1), \ldots, I(t_n))$.

For formulas built up by the logical connectives we have:

3. $\mathfrak{A} \models_I (\phi \lor \theta)$ if and only if $\mathfrak{A} \models_I \phi$ or $\mathfrak{A} \models_I \theta$;

4. $\mathfrak{A} \models_I (\phi \land \theta)$ if and only if $\mathfrak{A} \models_I \phi$ and $\mathfrak{A} \models_I \theta$;

5. $\mathfrak{A} \models_I (\phi \rightarrow \theta)$ if and only if $\mathfrak{A} \not\models_I \phi$ or $\mathfrak{A} \models_I \theta$;

6. $\mathfrak{A} \models_I (\neg \phi)$ if and only if $\mathfrak{A} \not\models_I \phi$.

For formulas built up with quantifiers:

7. $\mathfrak{A} \models_I (\exists v)\phi$ if and only if there is an $a$ in $A$ such that $\mathfrak{A} \models_{I/a} \phi$;

8. $\mathfrak{A} \models_I (\forall v)\phi$ if and only if for every $a$ in $A$, $\mathfrak{A} \models_{I/a} \phi$.

If $\mathfrak{A} \models_I \phi$ for every interpretation $I$, we will suppress the subscript $I$, and simply write $\mathfrak{A} \models \phi$. In this case we say that $\mathfrak{A}$ is a model of $\phi$.

**Example 4.2.4.** Let $\mathcal{L}(GT)$ be the language of group theory, which uses the symbols $\{+, 0\}$. A structure for this language is $\mathfrak{A} = (\{0, 1, 2\}, +_{(\text{mod } 3)}, 0)$. Suppose we consider formulas of $\mathcal{L}(GT)$ which only have variables among $x_1, x_2, x_3, x_4$. Define an interpretation $I$ by $I(x_i) \equiv i \mod 3$ and $I(0) = 0$.

1. Claim: $\not\mathfrak{A} \models_I x_1 + x_2 = x_4$.
   We check this claim by computation. Note that $I(x_1) = 1, I(x_2) = 2, I(x_1 + x_2) = I(x_1) +_{\text{mod } 3} I(x_2) = 1 +_{\text{mod } 3} 2 = 0$. On the other hand, $I(x_4) = 1 \neq 0$, so $\not\mathfrak{A} \models_I x_1 + x_2 = x_4$.

2. Claim: $\mathfrak{A} \models_I (\exists x_2)(x_1 + x_2 = x_4)$
   Define $J = I_{0/2}$. As above check that $\mathfrak{A} \models_J x_1 + x_2 = x_4$. Then by the definition of the satisfaction of an existential formula, $\mathfrak{A} \models_I (\exists x_2)(x_1 + x_2 = x_4)$.

**Theorem 4.2.5.** For every $\mathcal{L}$-formula $\phi$, for all interpretations $I, J$, if $I$ and $J$ agree on all the variables free in $\phi$, then $\mathfrak{A} \models_I \phi$ if and only if $\mathfrak{A} \models_J \phi$.

**Proof.** Left to the reader. \qed

**Corollary 4.2.6.** If $\phi$ is an $\mathcal{L}$-sentence, then for all interpretations $I$ and $J$, we have $\mathfrak{A} \models_I \phi$ if and only if $\mathfrak{A} \models_J \phi$.

**Remark 4.2.7.** Thus for $\mathcal{L}$-sentences, we drop the subscript which indicates the interpretation of the variables, and we say simply $\mathfrak{A}$ models $\phi$.

**Definition 4.2.8.** Let $\phi$ be an $\mathcal{L}$-formula.

(i) $\phi$ is logically valid if $\mathfrak{A} \models_I \phi$ for every $\mathcal{L}$-structure $\mathfrak{A}$ and every interpretation $I$ into $\mathfrak{A}$.

(ii) $\phi$ is satisfiable if there is some $\mathcal{L}$-structure $\mathfrak{A}$ and some interpretation $I$ into $\mathfrak{A}$ such that $\mathfrak{A} \models_I \phi$.

(iii) $\phi$ is contradictory if $\phi$ is not satisfiable.
4.3. THE DEDUCTIVE CALCULUS

Definition 4.2.9. A \( \mathcal{L} \)-theory \( \Gamma \) is a set of \( \mathcal{L} \)-sentences. An \( \mathcal{L} \)-structure \( \mathfrak{A} \) is a model of an \( \mathcal{L} \)-theory \( \Gamma \) if and only if \( \mathfrak{A} \models \phi \) for all \( \phi \) in \( \Gamma \). In this case we also say that \( \Gamma \) is satisfiable.

Definition 4.2.10. For a set of \( \mathcal{L} \)-formulas \( \Gamma \) and an \( \mathcal{L} \)-formula \( \phi \), we write \( \Gamma \models \phi \) and say “\( \Gamma \) implies \( \phi \),” if for all \( \mathcal{L} \)-structures \( \mathfrak{A} \) and all \( \mathcal{L} \)-interpretations \( I \), if \( \mathfrak{A} \models \gamma \) for all \( \gamma \) in \( \Gamma \), then \( \mathfrak{A} \models \phi \).

Thus if \( \Gamma \) is an \( \mathcal{L} \)-theory and \( \phi \) an \( \mathcal{L} \)-sentence, then \( \Gamma \models \phi \) means every model of \( \Gamma \) is also a model of \( \phi \).

The following definition will be useful to us in the next section.

Definition 4.2.11. Given a term \( t \) and an \( \mathcal{L} \)-formula \( \phi \) with free variable \( x \), we write \( \phi[t/x] \) to indicate the result of substituting the term \( t \) for each free occurrence of \( x \) in \( \phi \).

Example 4.2.12. If \( \phi \) is the formula \( (\exists y)(y \neq x) \) is the formula, then \( \phi[y/x] \) is the formula \( (\exists y)(y \neq y) \), which we expect never to be true.

4.3 The Deductive Calculus

The Predicate Calculus is a system of axioms and rules which permit us to derive the true statements of predicate logic without the use of interpretations. The basic relation in the Predicate Calculus is the relation proves between a set \( \Gamma \) of \( \mathcal{L} \) formulas and an \( \mathcal{L} \)-formula \( \phi \), which formalizes the concept that \( \Gamma \) proves \( \phi \). This relation is denoted \( \Gamma \vdash \phi \). As a first step in defining this relation, we give a list of additional rules of deduction, which extend the list we gave for propositional logic.

Some of our rules of the predicate calculus require that we exercise some care in how we substitute variables into certain formulas. Let us say that \( \phi[t/x] \) is a legal substitution of \( t \) for \( x \) in \( \phi \) if no free occurrence of \( x \) in \( \phi \) occurs in the scope of a quantifier of any variable appearing in \( t \). For instance, if \( \phi \) has the form \( (\forall y)\phi(x,y) \), where \( x \) is free, I cannot legally substitute \( y \) in for \( x \), since then \( y \) would be bound by the universal quantifier.

10. (Equality rule) For any term \( t \), the formula \( t = t \) may be derived from \( \Gamma \) in one step.

11. (Term Substitution) For any terms \( t_1, t_2, \ldots, t_n, s_1, s_2, \ldots, s_n \), and any function symbol \( F \), if each of the sentences \( t_1 = s_1, t_2 = s_2, \ldots, t_n = s_n \) have been derived from \( \Gamma \), then we may derive \( F(t_1, t_2, \ldots, t_n) = F(s_1, s_2, \ldots, s_n) \) from \( \Gamma \) in one additional step.

12. (Atomic Formula Substitution) For any terms \( t_1, t_2, \ldots, t_n, s_1, s_2, \ldots, s_n \) and any atomic formula \( \phi \), if each of the sentences \( t_1 = s_1, t_2 = s_2, \ldots, t_n = s_n \), and \( \phi(t_1, t_2, \ldots, t_n) \), have been derived from \( \Gamma \), then we may derive \( \phi(s_1, s_2, \ldots, s_n) \) from \( \Gamma \) in one additional step.

13. (\( \forall \)-Elimination) For any term \( t \), if \( \phi[t/x] \) is a legal substitution and \( (\forall x)\phi \) has been derived from \( \Gamma \), then we may derive \( \phi[t/x] \) from \( \Gamma \) in one additional step.

14. (\( \exists \)-Elimination) To show that \( \Gamma \cup \{ (\exists x)\phi(x) \} \vdash \theta \), it suffices to show \( \Gamma \cup \{ \phi(y) \} \), where \( y \) is a new variable that does not appear free in any formula in \( \Gamma \) nor in \( \theta \).

15. (\( \forall \)-Introduction) Suppose that \( y \) does not appear free in any formula in \( \Gamma \), in any temporary assumption, nor in \( (\forall x)\phi \). If \( \phi[y/x] \) has been derived from \( \Gamma \), then we may derive \( (\forall x)\phi \) from \( \Gamma \) in one additional step.
16. (\exists\text{-Introduction}) If \( \phi[t/x] \) is a legal substitution and \( \phi[t/x] \) has been derived from \( \Gamma \), then we may derive \( (\exists x)\phi \) from \( \Gamma \) in one additional step.

We remark on three of the latter four rules. First, the reason for the restriction on substitution in \( \forall\text{-Elimination} \) is that we need to ensure that \( t \) does not contain any free variable that would be become bound when we substitute \( t \) for \( x \) in \( \phi \). For example, consider the formula \( (\forall x)(\exists y)x < y \) in the language of arithmetic. Let \( \phi \) be the formula \( (\exists y)x < y \), in which \( x \) is free but \( y \) is bound. Observe that if we substitute the term \( y \) for \( x \) in \( \phi \), the resulting formula is \( (\exists y)y < y \). Thus, from \( (\forall x)(\exists y)x < y \) we can derive, for instance, \( (\exists y)x < y \) or \( (\exists y)c < y \), but we cannot derive \( (\exists y)y < y \).

Second, the idea behind \( \exists\text{-Elimination} \) is this: Suppose in the course of my proof I have derived \( (\exists x)\phi(x) \). Informally, I would like to use the fact that \( \phi \) holds of some \( x \), but to do so, I need to refer to this object. So I pick an unused variable, say \( a \), and use this as a temporary name to stand for the object satisfying \( \phi \). Thus, I can write down \( \phi(a) \). Eventually in my proof, I will discard this temporary name (usually by \( \exists\text{-Introduction} \)).

Third, in \( \forall\text{-Introduction} \), if we think of the variable \( y \) as an arbitrary object, then when we show that \( y \) satisfies \( \phi \), we can conclude that \( \phi \) holds of every object. However, if \( y \) is free in a premise in \( \Gamma \) or a temporary assumption, it is not arbitrary. For example, suppose we begin with the statement \( (\exists x)(\forall z)(x + z = z) \) in the language of arithmetic and suppose we derive \( (\forall z)(y + z = z) \) by \( \exists\text{-Elimination} \) (where \( y \) is a temporary name). We are not allowed to apply \( \forall\text{-Introduction} \) here, for otherwise we could conclude \( (\forall x)(\forall z)(x + z = z) \), an undesirable conclusion.

**Definition 4.3.1.** The relation \( \Gamma \vdash \phi \) is the smallest subset of pairs \( (\Gamma, \phi) \) from \( \mathcal{P}(\text{Sent}) \times \text{Sent} \) that contains every pair \( (\Gamma, \phi) \) such that \( \phi \in \Gamma \) or \( \phi \) is \( t = t \) for some term \( t \), and which is closed under the 15 rules of deduction.

As in Propositional Calculus, to demonstrate that \( \Gamma \vdash \phi \), we construct a proof. The next proposition exhibits several proofs using the new axiom and rules of predicate logic.
4.3. THE DEDUCTIVE CALCULUS

**Proposition 4.3.2.**

1. \((\exists x)(x = x)\).

2. \((\forall x)(\forall y)[x = y \rightarrow y = x]\).

3. \((\forall x)(\forall y)(\forall z)[(x = y \& y = z) \rightarrow x = z]\).

4. \(((\forall x)\theta(x)) \rightarrow (\exists x)\theta(x)\).

5. \(((\exists x)(\forall y)\theta(x, y)) \rightarrow (\forall y)(\exists x)\theta(x, y)\).

6. (i) \((\exists x)[\phi(x) \lor \psi(x)] \vdash (\exists x)\phi(x) \lor (\exists x)\psi(x)\)

(ii) \((\exists x)\phi(x) \lor (\exists x)\psi(x) \vdash (\exists x)[\phi(x) \lor \psi(x)]\)

7. (i) \((\forall x)[\phi(x) \& \psi(x)] \vdash (\forall x)\phi(x) \& (\forall x)\psi(x)\)

(ii) \((\forall x)\phi(x) \& (\forall x)\psi(x) \vdash (\forall x)[\phi(x) \lor \psi(x)]\)

8. \((\exists x)[\phi(x) \lor \psi(x)] \vdash (\exists x)\phi(x) \& (\exists x)\psi(x)\)

9. \((\forall x)\phi(x) \lor (\forall x)\psi(x) \vdash (\forall x)[\phi(x) \lor \psi(x)]\)

10. \((\forall x)[\phi(x) \rightarrow \psi(f(x))] \rightarrow [(\exists x)\phi(x) \rightarrow (\exists x)\psi(x)]\).

**Proof.**

1. \((\exists x)(x = x)\)

   \begin{align*}
   1 & \quad x = x \quad \text{equality rule} \\
   2 & \quad (\exists x) x = x \quad \exists\text{-Introduction 1}
   \end{align*}

2. \((\forall x)(\forall y)[x = y \rightarrow y = x]\).

   \begin{align*}
   1 & \quad x = y \quad \text{temporary assumption} \\
   2 & \quad x = x \quad \text{equality rule} \\
   3 & \quad y = x \quad \text{term substitution 1,2} \\
   4 & \quad x = y \rightarrow y = x \quad \rightarrow\text{-Introduction 1-3} \\
   5 & \quad (\forall y)(x = y \rightarrow y = x) \quad \forall\text{-Introduction 4} \\
   6 & \quad (\forall x)(\forall y)(x = y \rightarrow y = x) \quad \forall\text{-Introduction 5}
   \end{align*}
5. \((\exists x)(\forall y)\theta(x, y) \vdash (\forall y)(\exists x)\theta(x, y)\).

1. \((\exists x)(\forall y)\theta(x, y)\) \hspace{1cm} \text{given rule}
2. \((\forall y)\theta(a, y)\) \hspace{1cm} \exists\text{-Elimination 1}
3. \(\theta(a, y)\) \hspace{1cm} \forall\text{-Elimination 2}
4. \((\exists x)\theta(x, y)\) \hspace{1cm} \exists\text{-Introduction 3}
5. \((\forall y)(\exists x)\theta(x, y)\) \hspace{1cm} \forall\text{-Introduction 4}

8. \((\exists x)[\phi(x) & \psi(x)] \vdash (\exists x)\phi(x) & (\exists x)\psi(x)\)

1. \((\exists x)[\phi(x) & \psi(x)]\) \hspace{1cm} \text{given rule}
2. \(\phi(a) & \psi(a)\) \hspace{1cm} \exists\text{-Elimination 1}
3. \(\phi(a)\) \hspace{1cm} &\text{-Elimination 2}
4. \((\exists x)\phi(x)\) \hspace{1cm} \exists\text{-Introduction 3}
5. \(\psi(a)\) \hspace{1cm} &\text{-Elimination 2}
6. \((\exists x)\psi(x)\) \hspace{1cm} \exists\text{-Introduction 5}
7. \((\exists x)\phi(x) & (\exists x)\psi(x)\) \hspace{1cm} &\text{-Introduction 4,6}

9. \((\forall x)\phi(x) \lor (\forall x)\psi(x) \vdash (\forall x)[\phi(x) \lor \psi(x)]\)

1. \((\forall x)\phi(x) \lor (\forall x)\psi(x)\) \hspace{1cm} \text{given rule}
2. \((\forall x)\phi(x)\) \hspace{1cm} \text{temporary assumption}
3. \(\phi(x)\) \hspace{1cm} \forall\text{-Elimination 2}
4. \(\phi(x) \lor \psi(x)\) \hspace{1cm} \lor\text{-Introduction 3}
5. \((\forall x)[\phi(x) \lor \psi(x)]\) \hspace{1cm} \forall\text{-Introduction 4}
6. \((\forall x)\psi(x)\) \hspace{1cm} \text{temporary assumption}
7. \(\psi(x)\) \hspace{1cm} \forall\text{-Elimination 6}
8. \(\phi(x) \lor \psi(x)\) \hspace{1cm} \lor\text{-Introduction 7}
9. \((\forall x)[\phi(x) \lor \psi(x)]\) \hspace{1cm} \forall\text{-Introduction 8}
10. \((\forall x)[\phi(x) \lor \psi(x)]\) \hspace{1cm} \lor\text{-Elimination 1-9}
4.4. Soundness Theorem for Predicate Logic

Our next goal is to prove the soundness theorem for predicate logic. First we will prove a lemma, which connects satisfaction of formulas with substituted variables to satisfaction with slightly modified interpretations of the original formulas.

**Lemma 4.4.1.** For every $\mathcal{L}$-formula $\phi$, every variable $x$, every term $t$, every structure $\mathcal{B}$ and every interpretation $I$ in $\mathcal{B}$, if no free occurrence of $x$ occurs in the scope of a quantifier over any variable appearing in $t$, then

$$\mathcal{B} \models_I \phi[t/x] \text{ if and only if } \mathcal{B} \models_{I_{b/x}} \phi$$

where $b = I(t)$.

**Proof.** Let $\mathcal{B} = (B, R_1, \ldots, f_1, \ldots, b_1, \ldots)$ be an $\mathcal{L}$-structure, and let $x, t, I$ be as above. We claim that for any term $r$, if $b = I(t)$, then $I(r[t/x]) = I_{b/x}(r)$. We prove this claim by induction on the term $r$.

- **Case** $r = a$ is a constant, then $r[t/x] = a$ so that $I(r[t/x]) = a \upharpoonright r = I_{b/x}(r)$.
- **Case** $r$ is a variable $y \neq x$, then $r[t/x] = y$ and $I_{b/x}(y) = I(y)$, so that $I(r[t/x]) = I(y) = I_{b/x}(r)$.
- **Case** $r = x$, then $r[t/x] = t$ and $I_{b/x}(x) = b$, so that $I(r[t/x]) = I(t) = b = I_{b/x}(r)$.
- **Case** Now assume the claim holds for terms $r_1, \ldots, r_n$ and let $r = f(r_1, \ldots, r_n)$ for some function symbol $f$. Then by induction $I(r_j[t/x]) = I_{b/x}(r_j)$ for $j = 1, 2, \ldots, n$. Then

$$r[t/x] = f(r_1[t/x], \ldots, r_n[t/x])$$

so

$$I_{b/x}(r) = f \upharpoonright r_{b/x}(r_1), \ldots, I_{b/x}(r_n)$$

$$= f \upharpoonright (I(r_1[t/x]), \ldots, I_{b/x}(r_n[t/x]))$$

$$= I(f(r_1[t/x], \ldots, r_n[t/x]))$$

$$= I(r[t/x]).$$

To prove the lemma, we proceed by induction on formulas.

10. $(\forall x)[\phi(x) \rightarrow \psi(f(x))] \vdash (\exists x)\phi(x) \rightarrow (\exists x)\psi(x)$.

<table>
<thead>
<tr>
<th>Step</th>
<th>Formula</th>
<th>Reason</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$(\forall x)[\phi(x) \rightarrow \psi(f(x))]$</td>
<td>given rule</td>
</tr>
<tr>
<td>2</td>
<td>$(\exists x)\phi(x)$</td>
<td>temporary assumption</td>
</tr>
<tr>
<td>3</td>
<td>$\phi(a)$</td>
<td>$\exists$-Elimination 2</td>
</tr>
<tr>
<td>4</td>
<td>$\phi(a) \rightarrow \psi(f(a))$</td>
<td>$\forall$-Elimination 1</td>
</tr>
<tr>
<td>5</td>
<td>$\psi(f(a))$</td>
<td>$\rightarrow$-Elimination 3,4</td>
</tr>
<tr>
<td>6</td>
<td>$(\exists x)\psi(x)$</td>
<td>$\exists$-Introduction 5</td>
</tr>
<tr>
<td>7</td>
<td>$(\exists x)\phi(x) \rightarrow (\exists x)\psi(x)$</td>
<td>$\rightarrow$-Introduction 2-6</td>
</tr>
</tbody>
</table>
• For an atomic formula \( \phi \) of the form \( s_1 = s_2 \), we have

\[ \mathcal{B} \models_I \phi[t/x] \iff \mathcal{B} \models_I s_1[t/x] = s_2[t/x] \]
\[ \iff I(s_1[t/x]) = I(s_2[t/x]) \]
\[ \iff I_{b/x}(s_1) = I_{b/x}(s_2) \text{ (by the claim)} \]
\[ \iff \mathcal{B} \models_{I_{b/x}} s_1 = s_2 \]
\[ \iff \mathcal{B} \models_{I_{b/x}} \phi. \]

• For an atomic formula \( \phi \) of the form \( P(r_1, \ldots, r_n) \), so that \( \phi[t/x] \) is \( P(r_1[t/x], \ldots, r_n[t/x]) \), we have

\[ \mathcal{B} \models_I \phi[t/x] \iff \mathcal{B} \models_I P(r_1[t/x], \ldots, r_n[t/x]) \]
\[ \iff P^\mathcal{B}(I(r_1[t/x]), \ldots, I(r_n[t/x])) \]
\[ \iff P^\mathcal{B}(I_{b/x}(r_1), \ldots, I_{b/x}(r_n)) \text{ (by the claim)} \]
\[ \iff \mathcal{B} \models_{I_{b/x}} P(r_1, \ldots, r_n) \]
\[ \iff \mathcal{B} \models_{I_{b/x}} \phi. \]

• The inductive step for \( \mathcal{L} \)-formulas is straightforward except for formulas of the form \( \forall y \phi \): Let \( \psi \) be \( \forall y \phi \), where the Lemma holds for the formula \( \phi \). Then

\[ \mathcal{B} \models_I \psi[t/x] \iff \mathcal{B} \models_I \forall y \phi[t/x] \]
\[ \iff \mathcal{B} \models_{I_{b/y}} \phi[t/x] \text{ (for each } a \in B) \]
\[ \iff \mathcal{B} \models_{(I(t_1), \ldots, I(t_n))} \phi \text{ (by the inductive hypothesis)} \]
\[ \iff \mathcal{B} \models_{(I_{b/y}(t_1), \ldots, I_{b/y}(t_n))} \phi \text{ (for each } a \in B) \]
\[ \iff \mathcal{B} \models_{I_{b/y}} \forall y \phi \]
\[ \iff \mathcal{B} \models_{I_{b/y}} \psi. \]

\[ \square \]

**Theorem 4.4.2** (Soundness Theorem of Predicate Logic). If \( \Gamma \vdash \phi \), then \( \Gamma \models \phi \).

*Proof.* As in the proof of the soundness theorem for propositional logic, the proof is again by induction on the length of the deduction of \( \phi \). We need to show that if there is a proof of \( \phi \) from \( \Gamma \), then for any structure \( \mathcal{A} \) and any interpretation \( I \) into \( \mathcal{A} \), if \( \mathcal{A} \models_I \gamma \) for all \( \gamma \in \Gamma \), then \( \mathcal{A} \models_I \phi \). The arguments for the rules from Propositional Logic carry over here, so we just need to verify the result holds for the new rules.

Suppose the result holds for all formulas obtained in proofs of length strictly less than \( n \) lines.

• (Equality rule) Suppose the last line of a proof of length \( n \) with premises \( \Gamma \) is \( t = t \) for some term \( t \). Suppose \( \mathcal{A} \models_I \Gamma \). Then since \( I(t) = I(t) \), we have \( \mathcal{A} \models_I t = t \).

• (Term substitution) Suppose the last line of a proof of length \( n \) with premises \( \Gamma \) is \( F(s_1, \ldots, s_n) = F(t_1, \ldots, t_n) \), obtained by term substitution. Then we must have established \( s_1 = t_1, \ldots, s_n = t_n \) earlier in the proof. By the inductive hypothesis, we must have \( \Gamma \models s_1 = t_1, \ldots, \Gamma \models s_n = t_n \). Suppose that \( \mathcal{A} \models_I \gamma \) for every \( \gamma \in \Gamma \). Then \( I(s_i) = I(t_i) \) for \( i = 1, \ldots, n \). So

\[ I(F(s_1, \ldots, s_n)) = F^\mathcal{A}(I(s_1), \ldots, I(s_n)) \]
\[ = F^\mathcal{A}(I(t_1), \ldots, I(t_n)) \]
\[ = I(F(t_1, \ldots, t_n)). \]

Hence \( \mathcal{A} \models_I F(s_1, \ldots, s_n) = F(t_1, \ldots, t_n) \).
• (Atomic formula substitution) The argument is similar to the previous one and is left to the reader.

• (\(\forall\)-Elimination) For any term \(t\), if \(\phi[t/x]\) is a legal substitution and \((\forall x)\phi\) has been derived from \(\Gamma\), then we may derive \(\phi[t/x]\) from \(\Gamma\) in one additional step.

Suppose that the last line of a proof of length \(n\) with premises \(\Gamma\) is \(\phi[t/x]\), obtained by \(\forall\)-Elimination. Thus, we must have derived \(\forall x\phi(x)\) earlier in the proof. Let \(\mathcal{A} \models \gamma\) for every \(\gamma \in \Gamma\). Then by the inductive hypothesis, we have \(\Gamma \models \forall x\phi(x)\), which implies that \(\mathcal{A} \models \phi[x/x] \phi(a)\) for every \(a \in A\). If \(I(t) = b\), then since \(\mathcal{A} \models \phi[y/y] \phi(x)\), by Lemma 4.4.1 we have \(\mathcal{A} \models \phi[t/x]\). Since \(\mathcal{A}\) and \(I\) were arbitrary, we can conclude that \(\Gamma \models \phi[t/x]\).

• (\(\exists\)-Elimination) To show that \(\Gamma \cup \{\exists x \phi(x)\} \vdash \theta\), it suffices to show \(\Gamma \cup \{\phi[y/x]\} \vdash \theta\), where \(y\) is a new variable that does not appear free in any formula in \(\Gamma\) nor in \(\theta\).

Suppose that the last line of a proof of length \(n\) with premises \(\Gamma\) is given by \(\exists\)-Elimination. Then \(\Gamma \vdash \exists x \phi(x)\) in less than \(n\) lines and \(\Gamma \cup \{\phi[y/x]\} \vdash \theta\) in less than \(n\) lines. Let \(\mathcal{A} \models \gamma\) for every \(\gamma \in \Gamma\). Then by the inductive hypothesis, we have \(\Gamma \models \exists x \phi(x)\), which implies that \(\mathcal{A} \models \phi(x)\) for some \(b \in A\). Let \(J = I_{b/y}\), so that \(J(\gamma) = b\). It follows that \(\mathcal{A} \models \gamma\), since \(I = J\) except on possibly \(y\) and \(y\) does not appear free in \(\phi\). Then by Lemma 4.4.1, \(\mathcal{A} \models \gamma\), and hence \(\mathcal{A} \models \gamma\). It follows that \(\mathcal{A} \models \gamma\), since \(I = J\) except on possibly \(y\) and \(y\) does not appear free in \(\theta\). Since \(\mathcal{A}\) and \(I\) were arbitrary, we can conclude that \(\Gamma \cup \{\exists x \phi(x)\} \vdash \theta\).

• (\(\forall\)-Introduction) Suppose that \(y\) does not appear free in any formula in \(\Gamma\), in any temporary assumption, nor in \(\forall x \phi\). If \(\phi[y/x]\) has been derived from \(\Gamma\), then we may derive \(\forall x \phi\) from \(\Gamma\) in one additional step.

Suppose that the last line of a proof of length \(n\) with premises \(\Gamma\) is \((\forall x)\phi(x)\), obtained by \(\forall\)-Introduction. Thus, we must have derived \(\phi[y/x]\) from \(\Gamma\) earlier in the proof (where \(y\) satisfies the necessary conditions described above). Let \(\mathcal{A} \models \gamma\) for every \(\gamma \in \Gamma\). Since \(y\) does not appear free in \(\Gamma\), then for any \(a \in A\), \(\mathcal{A} \models \gamma\). For an arbitrary \(a \in A\), let \(J = I_{a/y}\), so that \(J(\gamma) = a\). By the inductive hypothesis, we have \(\Gamma \models \phi[y/x]\), which implies that \(\mathcal{A} \models \phi[y/x]\). Then by Lemma 4.4.1, \(\mathcal{A} \models \phi[y/x]\). Since \(I_{a/y} = J_{a/y}\) except on possibly \(y\), which does not appear free in \(\phi\), we have \(\mathcal{A} \models \phi[a/y] \phi\). As \(a\) was arbitrary, we have shown \(\mathcal{A} \models \phi[a/y] \phi\) for every \(a \in A\). Hence \(\mathcal{A} \models \forall x \phi(x)\). Since \(\mathcal{A}\) and \(I\) were arbitrary, we can conclude that \(\Gamma \models \forall x \phi(x)\).

• (\(\exists\)-Introduction) If \(\phi[t/x]\) is a legal substitution and \(\phi[t/x]\) has been derived from \(\Gamma\), then we may derive \((\exists x) \phi\) from \(\Gamma\) in one additional step.

Suppose that the last line of a proof of length \(n\) with premises \(\Gamma\) is \((\exists x)\phi(x)\), obtained by \(\exists\)-Introduction. Thus, we must have derived \(\phi[t/x]\) from \(\Gamma\) earlier in the proof, where \(t\) is some term. Let \(\mathcal{A} \models \gamma\) for every \(\gamma \in \Gamma\). Then \(I(t) = a\) for some \(a \in A\). Since \(\Gamma \vdash \phi[t/x]\) in less than \(n\) lines, by the inductive hypothesis, it follows that \(\mathcal{A} \models \gamma\). Then by Lemma 4.4.1, \(\mathcal{A} \models \gamma\), which implies that \(\mathcal{A} \models \exists x \phi(x)\). Since \(\mathcal{A}\) and \(I\) were arbitrary, we can conclude that \(\Gamma \models \exists x \phi(x)\).
Chapter 5

Models for Predicate Logic

5.1 Models

In this chapter, we will prove the completeness theorem for predicate logic by showing how to build a model for a consistent first-order theory. We will also discuss several consequences of the compactness theorem for first-order logic and consider several relations that hold between various models of a given first-order theory, namely isomorphism and elementary equivalence.

5.2 The Completeness Theorem for Predicate Logic

Fix a first-order theory $\mathcal{L}$. For convenience, we will assume that our $\mathcal{L}$-formulas are built up only using $\neg$, $\lor$, and $\exists$. We will also make use of the following key facts (the proofs of which we will omit):

1. If $A$ is a tautology in propositional logic, then if we replace each instance of each propositional variable in $A$ with an $\mathcal{L}$-formula, the resulting $\mathcal{L}$-formula is true in all $\mathcal{L}$-structures.

2. For any $\mathcal{L}$-structure $\mathcal{A}$ and any interpretation $I$ into $\mathcal{A},$

$$\mathcal{A} \models I(\forall x)\phi \iff \mathcal{A} \models I(\exists x)(\neg \phi).$$

We will also use the following analogues of results we proved in Chapter 2, the proofs of which are the same:

**Lemma 5.2.1.** Let $\Gamma$ be an $\mathcal{L}$-theory.

1. If $\Gamma$ is not consistent, then $\Gamma \vdash \phi$ for every $\mathcal{L}$-sentence $\phi$.

2. For an $\mathcal{L}$-sentence $\phi$, $\Gamma \vdash \phi$ if and only if $\Gamma \cup \{\neg \phi\}$ is inconsistent.

3. If $\Gamma$ is consistent, then for any $\mathcal{L}$-sentence $\phi$, either $\Gamma \cup \{\phi\}$ is consistent or $\Gamma \cup \{\neg \phi\}$ is consistent.

The following result, known as the Constants Theorem, plays an important role in the proof of the completeness theorem.

**Theorem 5.2.2 (Constants Theorem).** Let $\Gamma$ be an $\mathcal{L}$-theory. If $\Gamma \vdash \phi(c)$ and $c$ does not appear in $\Gamma$, then $\Gamma \vdash (\forall x)\phi(x)$.

**Proof.** Given a proof of $\phi(c)$ from $\Gamma$, let $v$ be a variable not appearing in $\Gamma$. If we replace every instance of $c$ with $v$ in the proof of $\phi(c)$, we have a proof of $\phi(v)$ from $\Gamma$. Then by $\forall$-Introduction, we have $\Gamma \vdash (\forall x)\phi(x)$.

Gödel’s completeness theorem can be articulated in two ways, which we will prove are equivalent:
Theorem 5.2.3 (Completeness theorem, Version 1). For any $\mathcal{L}$-theory $\Gamma$ and any $\mathcal{L}$-sentence $\phi$,

$$\Gamma \models \phi \Rightarrow \Gamma \vdash \phi.$$ 

Theorem 5.2.4 (Completeness theorem, Version 2). Every consistent theory has a model.

We claim that the two versions are equivalent.

**Proof of claim.** First, suppose that every consistent theory has a model, and suppose further that $\Gamma \models \phi$. If $\Gamma$ is not consistent, then $\Gamma$ proves every sentence, and hence $\Gamma \vdash \phi$. If, however, $\Gamma$ is consistent, we have two cases to consider. If $\Gamma \cup \{\neg \phi\}$ is inconsistent, then by Lemma 5.2.1(2), it follows that $\Gamma \vdash \phi$. In the case that $\Gamma \cup \{\neg \phi\}$ is consistent, by the second version of the completeness theorem, there is some $\mathcal{L}'$-structure $\mathcal{A}$ such that $\mathcal{A} \models \Gamma \cup \{\neg \phi\}$, from which it follows that $\mathcal{A} \models \Gamma$ and $\mathcal{A} \models \neg \phi$. But we have assumed that $\Gamma \models \phi$, and hence $\mathcal{A} \models \phi$, which is impossible. Thus, if $\Gamma$ is consistent, it follows that $\Gamma \cup \{\neg \phi\}$ is inconsistent.

For the other direction, suppose the first version of the completeness theorem holds and let $\Gamma$ be an arbitrary $\mathcal{L}$-theory. Suppose $\Gamma$ has no model. Then vacuously, $\Gamma \models \neg (\phi \lor \neg \phi)$, where $\phi$ is the sentence $(\forall x) x = x$. It follows from the first version of the completeness theorem that $\Gamma \vdash \neg (\phi \lor \neg \phi)$, and hence $\Gamma$ is inconsistent.

We now turn to the proof of the second version of completeness theorem. As in the proof of the completeness theorem for propositional logic, we will use the compactness theorem, which comes in several forms (just as it did in with propositional logic).

**Theorem 5.2.5.** Let $\Gamma$ be an $\mathcal{L}$-theory.

1. For an $\mathcal{L}$-sentence $\phi$, if $\Gamma \vdash \phi$, there is some finite $\Gamma_0 \subseteq \Gamma$, $\Gamma_0 \vdash \phi$.

2. If every finite $\Gamma_0 \subseteq \Gamma$ is consistent, then $\Gamma$ is consistent.

3. If $\Gamma = \bigcup_n \Gamma_n$ is , $\Gamma_n \subseteq \Gamma_{n+1}$ for every $n$, and each $\Gamma_n$ is consistent, then $\Gamma$ is consistent.

As in the case of propositional logic, (1) follows by induction on proof length, while (2) follows directly from (1) and (3) follows directly from (2).

Our strategy for proving the completeness theorem is as follows. Given $\Gamma$, we want to extend it to a maximally consistent collection of $\mathcal{L}'$-formulas, like the proof of the completeness theorem for propositional logic. The problem that we now encounter (that did not occur in the propositional case) is that it is unclear how to make sentences of the form $(\exists x) \theta$.

The solution to this problem, due to Henkin, is to extend the language $\mathcal{L}$ to a language $\mathcal{L}'$ by adding new constants $c_0, c_1, c_2, \ldots$, which we will use to witness the truth of existential sentences.

Hereafter, let us assume that $\mathcal{L}'$ is countably infinite (which is not a necessary restriction), so that we will only need to add countably many new constants to our language. Using these constants, we will build a model of $\Gamma$, where the universe of our model consists of certain equivalence classes on the set of all $\mathcal{L}'$-terms with no variables (the so-called Herbrand universe of $\mathcal{L}'$). The model will satisfy a collection $\Delta \supseteq \Gamma$ that is maximally consistent and Henkin complete, which means that for each $\mathcal{L}'$-formula $\theta(v)$ with exactly one free variable $v$, if $(\exists v) \theta(v)$ is in $\Delta$, then there is some constant $c$ in our language such that $\theta(c)$ is in $\Delta$.

**Proof of Theorem 5.2.4.** Let $\phi_0, \phi_1, \ldots$ be an enumeration of all $\mathcal{L}'$-sentences. We define a sequence $\Gamma = \Delta_1 \subseteq \Delta_0 \subseteq \Delta_1 \subseteq \ldots$ such that for each $n \in \mathbb{N}$,

$$\Delta_{2n} = \begin{cases} \Delta_{2n-1} \cup \{\phi_n\} & \text{if } \Delta_{2n-1} \cup \{\phi_n\} \text{ is consistent}, \\ \Delta_{2n-1} \cup \{\neg \phi_n\} & \text{otherwise} \end{cases}$$

and

$$\Delta_{2n+1} = \begin{cases} \Delta_{2n} \cup \{\theta(c_m)\} & \text{if } \phi_n \text{ is of the form } (\exists v) \theta(v) \text{ and is in } \Delta_{2n}, \\ \Delta_{2n} & \text{otherwise} \end{cases}$$
where $c_m$ is the first constant in our list of new constants that has not appeared in $\Delta_{2n}$. Then we define $\Delta = \cup_n \Delta_n$.

We now prove a series of claims.

**Claim 1:** $\Delta$ is complete (that is, for every $\mathcal{L}'$-sentence $\phi$, either $\phi \in \Delta$ or $\neg \phi \in \Delta$).

**Proof of Claim 1:** This follows immediately from the construction.

**Claim 2:** Each $\Delta_k$ is consistent.

**Proof of Claim 2:** We prove this claim by induction. First, $\Delta_{-1} = \Gamma$ is consistent by assumption. Now suppose that $\Delta_k$ is consistent. If $k = 2n$ for some $n$, then clearly $\Delta_k$ is consistent, since if $\Delta_{2n-1} \cup \{\phi_n\}$ is consistent, then we set $\Delta_k = \Delta_{2n-1} \cup \{\phi_n\}$, and if not, then by Lemma 5.2.1(3), $\Delta_{2n-1} \cup \{\neg \phi_n\}$ is consistent, and so we set $\Delta_k = \Delta_{2n-1} \cup \{\neg \phi_n\}$.

If $k = 2n + 1$ for some $n$, then if $\phi_n$ is not of the form $(\exists \bar{v}) \theta(\bar{v})$ or if it is but it is not in $\Delta_{2n}$, then $\Delta_{2n+1} = \Delta_{2n}$ is consistent by induction. If $\phi_n$ is of the form $(\exists \bar{v}) \theta(\bar{v})$ and is in $\Delta_{2n}$, then let $c = c_m$ be the first constant not appearing in $\Delta_{2n}$. Suppose that $\Delta_k = \Delta_{2k+1} = \Delta_{2n} \cup \{\theta(c)\}$ is not consistent. Then by Lemma 5.2.1(2), $\Delta_{2n} \vdash \neg \theta(c)$. Then by the Constants Theorem, $\Delta_{2n} \vdash (\forall \bar{x}) \neg \theta(\bar{x})$. But since $\phi_n$ is the formula $(\exists \bar{v}) \theta(\bar{v})$ and is in $\Delta_{2n}$, it follows that $\Delta_{2n}$ is inconsistent, contradicting our inductive hypothesis. Thus $\Delta_{k} = \Delta_{2n+1}$ is consistent.

**Claim 3:** $\Delta = \cup_n \Delta_n$ is consistent.

**Proof of Claim 3:** This follows from the third version of the compactness theorem.

**Claim 4:** $\Delta$ is Henkin complete (that is, for each $\mathcal{L}'$-formula $\theta(\bar{v})$ with exactly one free variable and $(\exists \bar{v}) \theta(\bar{v}) \in \Delta$, then $\theta(c) \in \Delta$ for some constant $c$).

**Proof of Claim 4:** Suppose that $(\exists \bar{v}) \theta(\bar{v}) \in \Delta$. Then there is some $n$ such that $(\exists \bar{v}) \theta(\bar{v})$ is the formula $\phi_n$. Since $\Delta_{2n-1} \cup \{\phi_n\} \subseteq \Delta$ is consistent, $(\exists \bar{v}) \theta(\bar{v}) \in \Delta_{2n}$. Then by construction, $\theta(c) \in \Delta_{2n+1}$ for some constant $c$.

Our final task is to build a model $\mathcal{A}$ such that $\mathcal{A} \models \Delta$, from which it will follow that $\mathcal{A} \models \Gamma$ (since $\Gamma \subseteq \Delta$). We define an equivalence relation on the Herbrand universe of $\mathcal{L}'$ (i.e., the set of constant $\mathcal{L}'$-terms, or equivalently, the $\mathcal{L}'$-terms that contain no variables). For constant terms $s$ and $t$, we define

$$s \sim t \iff s = t \in \Delta.$$

**Claim 5:** $\sim$ is an equivalence relation.

**Proof of Claim 5:**

- Every sentence of the form $t = t$ must be in $\Delta$ since $\Delta$ is complete, so $\sim$ is reflexive.
- If $s = t \in \Delta$, then $t = s$ must also be in $\Delta$ since $\Delta$ is complete, so $\sim$ is symmetric.
- If $r = s, s = t \in \Delta$, then $r = t$ must also be in $\Delta$ since $\Delta$ is complete, so $\sim$ is transitive.

For a constant term $s$, let $[s]$ denote the equivalence class of $s$. Then we define an $\mathcal{L}'$-structure as follows:

(i) $A = \{[t] : t$ is a constant term of $\mathcal{L}'\}$;

(ii) for each function symbol $f$ of the language $\mathcal{L}$, we define

$$f^\mathcal{A}([t_1], \ldots, [t_n]) = [f(t_1, \ldots, t_n)],$$

where $n$ is the arity of $f$;
(iii) for each predicate symbol $P$ of the language $\mathcal{L}'$, we define
\[ P^\mathcal{A}([t_1], \ldots, [t_n]) \text{ if and only if } P(t_1, \ldots, t_n) \in \Delta, \]
where $n$ is the arity of $P$; and

(iv) for each constant symbol $c$ of the language $\mathcal{L}'$, we define
\[ c^\mathcal{A} = [c]. \]

Claim 6: $\mathcal{A} = (A, f, \ldots, P, \ldots, c, \ldots)$ is well-defined.

Proof of Claim 6: We have to show in particular that the interpretation of function symbols and predicate symbols in $\mathcal{A}$ is well-defined. Suppose that $s_1 = t_1, \ldots, s_n = t_n \in \Delta$ and
\[ f^\mathcal{A}([t_1], \ldots, [t_n]) = [f(t_1, \ldots, t_n)]. \tag{5.1} \]

By our first assumption, it follows that $\Delta \models s_i = t_i$ for $i = 1, \ldots, n$. Then by term substitution, $\Delta \models f(s_1, \ldots, s_n) = f(t_1, \ldots, t_n)$, and so $f(s_1, \ldots, s_n) = f(t_1, \ldots, t_n) \in \Delta$. It follows that
\[ [f(s_1, \ldots, s_n)] = [f(t_1, \ldots, t_n)]. \tag{5.2} \]

Combining (5.1) and (5.2) yields
\[ f^\mathcal{A}([t_1], \ldots, [t_n]) = [f(t_1, \ldots, t_n)] = [f(s_1, \ldots, s_n)] = f^\mathcal{A}([s_1], \ldots, [s_n]). \]

A similar argument shows that the interpretation of predicate symbols is well-defined.

Claim 7: Let $I$ be an interpretation into $\mathcal{A}$. Then $I(t) = [t]$ for every constant term $t$.

Proof of Claim 7: We verify this inductively for constant symbols and then for function symbols applied to constant terms.

- Suppose $t$ is a constant symbol $c$. Then $I(c) = c^\mathcal{A} = [c]$.
- Suppose that $t$ is the term $f(t_1, \ldots, t_n)$ for constant symbol $f$ and constant terms $t_1, \ldots, t_n$, where $I(t_i) = [t_i]$ for $i = 1, \ldots, n$. Then
\[ I(f(t_1, \ldots, t_n)) = f^\mathcal{A}(I(t_1), \ldots, I(t_n)) = f^\mathcal{A}([t_1], \ldots, [t_n]) = [f(t_1, \ldots, t_n)]. \]

Claim 8: $\mathcal{A} \models \Delta$. We verify this by proving that for every interpretation $I$ into $\mathcal{A}$ and every $\mathcal{L}'$-sentence $\phi$, $\mathcal{A} \models_I \phi$ if and only if $\phi \in \Delta$.

- If $\phi$ is $s = t$ for some terms $s, t$, then
\[ \mathcal{A} \models_I s = t \iff I(s) = I(t) \]
\[ \iff [s] = [t] \]
\[ \iff s \sim t \]
\[ \iff s = t \in \Delta. \]

- If $\phi$ is $P(t_1, \ldots, t_n)$ for some predicate symbol $P$, then
\[ \mathcal{A} \models_I P(t_1, \ldots, t_n) \iff P^\mathcal{A}(I(t_1), \ldots, I(t_n)) \]
\[ \iff P^\mathcal{A}([t_1], \ldots, [t_n]) \]
\[ \iff P(t_1, \ldots, t_n) \in \Delta. \]
5.3. CONSEQUENCES OF THE COMPLETENESS THEOREM

• If \( \phi \) is \( \neg \psi \) for some \( \mathcal{L}' \)-sentence \( \psi \), then
  \[
  \mathcal{A} \models \neg \psi \iff \mathcal{A} \not\models \psi \\
  \iff \psi \notin \Delta \\
  \iff \neg \psi \in \Delta.
  \]

• If \( \phi \) is \( \psi \lor \theta \) for some \( \mathcal{L}' \)-sentences \( \psi \) and \( \theta \), then
  \[
  \mathcal{A} \models \psi \lor \theta \iff \mathcal{A} \models \psi \lor \theta \\
  \iff \psi \in \Delta \lor \theta \in \Delta \\
  \iff \psi \lor \theta \in \Delta.
  \]

• If \( \phi \) is \( (\exists v) \theta(v) \) for some \( \mathcal{L}' \)-formula \( \theta \) with one free variable \( v \), then
  \[
  \mathcal{A} \models (\exists v) \theta(v) \iff \mathcal{A} \models (\exists v) \theta(v) \\
  \iff \mathcal{A} \models (\exists v) \theta(v) \text{ for some } b \in A \\
  \iff \mathcal{A} \models (\exists v) \theta(v) \text{ where } b = [c] \\
  \iff \theta(c) \in \Delta \\
  \iff (\exists v) \theta(v) \in \Delta.
  \]

Since \( \mathcal{A} \models \Delta \), it follows that \( \mathcal{A} \models \Gamma \). Note that \( \mathcal{A} \) is an \( \mathcal{L}' \)-structure while \( \Gamma \) is only an \( \mathcal{L} \)-theory (as it does not contain any expression involving any of the additional constants). Then let \( \mathcal{A}^* \) be the \( \mathcal{L} \)-structure with the same universe as \( \mathcal{A} \) and the same interpretations of the function symbols and predicate symbols, but without interpreting the constants symbols that are in \( \mathcal{L}' \setminus \mathcal{L} \) (the so-called reduct of \( \mathcal{A} \)). Then clearly \( \mathcal{A}^* \models \Gamma \), and the proof is complete.

5.3 Consequences of the completeness theorem

The same consequences we derived from the Soundness and Completeness Theorem for Propositional Logic apply now to Predicate Logic with basically the same proofs.

**Theorem 5.3.1.** For any set of sentences \( \Gamma \), \( \Gamma \) is satisfiable if and only if \( \Gamma \) is consistent.

**Theorem 5.3.2.** If \( \Sigma \) is a consistent theory, then \( \Sigma \) is included in some complete, consistent theory.

We also have an additional version of the compactness theorem, which is the most common formulation of compactness.

**Theorem 5.3.3 (Compactness Theorem for Predicate Logic).**
An \( \mathcal{L} \)-theory \( \Gamma \) is satisfiable if and only if every finite subset of \( \Gamma \) is satisfiable.

**Proof.** (\( \Rightarrow \)) If \( \mathcal{A} \models \Gamma \), then it immediately follows that \( \mathcal{A} \models \Gamma_0 \) for any finite \( \Gamma_0 \subseteq \Gamma \).

(\( \Leftarrow \)) Suppose that \( \Gamma \) is not satisfiable. By the completeness theorem, \( \Gamma \) is not consistent. Then \( \Gamma \vdash \phi \& \neg \phi \) for some \( \mathcal{L} \)-sentence \( \phi \). Then by the first formulation of the compactness theorem there is some finite \( \Gamma_0 \subseteq \Gamma \) such that \( \Gamma_0 \vdash \phi \& \neg \phi \). It follows that \( \Gamma_0 \) is not satisfiable.

We now consider two applications of the compactness theorem, the first yielding a model of arithmetic with infinite natural numbers and the second yielding a model of the real numbers with infinitesimals.
Example 5.3.4. Let $\mathcal{L} = \{+, \times, <, 0, 1\}$ be the language of arithmetic, and let $\Gamma = Th(\mathbb{N})$, the set of $\mathcal{L}$-sentences true in the standard model of arithmetic. Let us expand $\mathcal{L}$ to $\mathcal{L}'$ by adding a new constant $c$ to our language. We extend $\Gamma$ to an $\mathcal{L}'$-theory $\Gamma'$ by adding all sentences of the form

$$\psi_n : c > 1 + \ldots + 1 \text{ } (n \text{ times})$$

We claim that every finite $\Gamma'_0 \subseteq \Gamma'$ is satisfiable. Given any finite $\Gamma'_0 \subseteq \Gamma'$, $\Gamma'_0$ consists of at most finitely many sentences from $\Gamma$ and at most finitely many sentences of the form $\psi_1$. It follows that

$$\Gamma'_0 \subseteq \Gamma \cup \{\psi_{n_1}, \psi_{n_2}, \ldots, \psi_{n_k}\}$$

for some $n_1, n_2, \ldots, n_k \in \mathbb{N}$, where these latter sentences assert that $c$ is larger than each of the values $n_1, n_2, \ldots, n_k$. Let $n = \max\{n_1, \ldots, n_k\}$ then let $\mathcal{A}_f = (\mathbb{N}, +, \times, <, 0, 1, n)$, so that $c^{\mathcal{A}_f} = n$ and hence $\mathcal{A}_f \models \Gamma'_0$. Then by the compactness theorem, there is some $\mathcal{L}'$-structure $\mathcal{B}$ such that $\mathcal{B} \models \Gamma'$. In the universe of $\mathcal{B}$, we have objects that behave exactly like $0, 1, 2, 3, \ldots$ (in a sense we will make precise shortly), but the interpretation of $c$ in $\mathcal{B}$ satisfies $c^{\mathcal{B}} > n$ for every $n \in \mathbb{N}$ and hence behaves like an infinite natural number. We will write the universe of $\mathcal{B}$ as $\mathbb{N}^*$.

Example 5.3.5. Let $\mathcal{L}$ consist of

- an $n$-ary function symbol $F_f$ for every $f : \mathbb{R}^n \to \mathbb{R}$;
- an $n$-ary predicate symbol $P_A$ for every $A \subseteq \mathbb{R}^n$; and
- a constant symbol $c_r$ for every $r \in \mathbb{R}$.

Let $\mathcal{B}$ be the $\mathcal{L}$-structure with universe $\mathbb{R}$ satisfying

- $F^\mathcal{B}_f = f$ for every function symbol $F_f$;
- $P^\mathcal{B}_A = A$ for every predicate symbol $P_A$; and
- $c^\mathcal{B}_r = r$ for every constant symbol $c_r$.

Let us expand $\mathcal{L}$ to $\mathcal{L}'$ by adding a new constant $d$ to our language. We extend $\Gamma$ to an $\mathcal{L}'$-theory $\Gamma'$ by adding all sentences of the form

$$\theta_r : c_0 < d < c_r$$

for $r \in \mathbb{R}^>0$. As in the previous example, every finite $\Gamma'_0 \subseteq \Gamma'$ is satisfiable. Hence by the compactness theorem, $\Gamma'$ is satisfiable. Let $\mathcal{A} \models \Gamma$. The universe of $\mathcal{A}$ contains a copy of $\mathbb{R}$ (in a sense we will make precise shortly). In addition, $d^\mathcal{A}$ is infinitesimal object. For every real number in $\mathcal{A}$, $0 < d^\mathcal{A} < r$ holds. We will write the universe of $\mathcal{A}$ as $\mathbb{R}^*$.

Now we consider a question that was not appropriate to consider in the context of propositional logic, namely, what are the sizes of models of a given theory? Our main theorem is a consequence of the proof of the Completeness Theorem. We proved the Completeness Theorem only in the case of a countable language $L$, and we built a countable model (which was possibly finite). By using a little care (and some set theory), one can modify steps (1) and (2) for an uncountable language to define by transfinite recursion a theory $\Delta$ and prove by transfinite induction that $\Delta$ has the desired properties. The construction leads to a model whose size is at most the size of the language with which one started. Thus we have:

**Theorem 5.3.6** (Löwenheim-Skolem Theorem). If $\Gamma$ is an $\mathcal{L}$-theory with an infinite model, then $\Gamma$ has a model of size $\kappa$ for every infinite $\kappa$ with $|L| \leq \kappa$. 

5.4. ISOMORPHISM AND ELEMENTARY EQUIVALENCE

Proof Sketch. First we add $\kappa$ new constant symbols $\langle d_\alpha \mid \alpha < \kappa \rangle$ to our language $\mathcal{L}$. Next we expand $\Gamma$ to $\Gamma'$ by adding formulas that say $d_\alpha \neq d_\beta$ for the different constants:

$$\Gamma' = \Gamma \cup \{ \neg d_\alpha = d_\beta : \alpha < \beta < \kappa \}.$$ 

Since $\Gamma$ has an infinite model, each finite $\Gamma_0' \subseteq \Gamma'$ has a model. Hence by the compactness theorem, $\Gamma'$ has a model. By the soundness theorem, $\Gamma'$ is consistent. Then use the proof of the completeness theorem to define a model $\mathcal{B}'$ of $\Gamma'$ the universe of which has size $|B| \leq \kappa$. Since $\mathcal{B}' \models d_\alpha \neq d_\beta$ for $\alpha \neq \beta$, there are at least $\kappa$ many elements. Thus $|B| = \kappa$ and so $\Gamma'$ has a model $\mathcal{B}'$ of the desired cardinality. Let $\mathcal{B}$ be the reduct of $\mathcal{B}'$ obtained by removing the new constant symbols from our expanded language. Then $\mathcal{B}$ is a model of the desired size for $\Gamma$. \hfill \Box

5.4 Isomorphism and elementary equivalence

We conclude this chapter with one last set of definitions and examples.

Definition 5.4.1. Given $\mathcal{L}$-structures $\mathcal{A}$ and $\mathcal{B}$, a bijection $H : \mathcal{A} \to \mathcal{B}$ is an isomorphism if it satisfies:

1. For every constant $c \in \mathcal{L}$, $H(c^\mathcal{A}) = c^\mathcal{B}$.
2. For every $k$-ary predicate symbol $P \in \mathcal{L}$ and every $a_1, \ldots, a_k \in A$,

$$P^\mathcal{A}(a_1, \ldots, a_k) \iff P^\mathcal{B}(H(a_1), \ldots, H(a_k)),$$

3. For every $k$-ary function symbol $F \in \mathcal{L}$ and every $a_1, \ldots, a_k \in A$,

$$H(F^\mathcal{A}(a_1, \ldots, a_k)) = F^\mathcal{B}(H(a_1), \ldots, H(a_k)).$$

Furthermore, $\mathcal{A}$ and $\mathcal{B}$ are isomorphic, denoted $\mathcal{A} \cong \mathcal{B}$, if there exists an isomorphism between $\mathcal{A}$ and $\mathcal{B}$.

Example 5.4.2. The ordered group $(\mathbb{R}, +, <)$ of real numbers under addition is isomorphic to the ordered group $(\mathbb{R}^{\geq 0}, +, <)$ of positive real numbers under multiplication under the mapping $H(x) = 2^x$. The key observation here is that $H(x + y) = 2^{x+y} = 2^x \cdot 2^y = H(x) \cdot H(y)$.

We compare the relation of isomorphism with the following relation between models.

Definition 5.4.3. $\mathcal{L}$-structures $\mathcal{A}$ and $\mathcal{B}$ are elementarily equivalent, denoted $\mathcal{A} \equiv \mathcal{B}$, if for any $\mathcal{L}$-sentence $\phi$,

$$\mathcal{A} \models \phi \iff \mathcal{B} \models \phi.$$ 

How do the relations of $\cong$ and $\equiv$ compare? First, we have the following theorem.

Theorem 5.4.4. If $\mathcal{A}$ and $\mathcal{B}$ are $\mathcal{L}$-structures satisfying $\mathcal{A} \cong \mathcal{B}$, then $\mathcal{A} \equiv \mathcal{B}$.

The proof is by induction on the complexity of $\mathcal{L}$-sentences. The converse of this theorem does not hold, as shown by the following example.

Example 5.4.5. $(\mathbb{Q}, \leq)$ and $(\mathbb{R}, \leq)$, both models of the theory of dense linear orders without endpoints, are elementarily equivalent, which follows from the fact that the theory of dense linear orders without endpoints is complete (which we will prove in Chapter 6). Note, however, that these structures are not isomorphic, since they have different cardinalities.

We conclude this chapter with one last set of definitions and examples.

Definition 5.4.6. Let $\mathcal{A}$ and $\mathcal{B}$ be $\mathcal{L}$-structures with corresponding domains $A \subseteq B$. 


CHAPTER 5. MODELS FOR PREDICATE LOGIC

1. \( \mathcal{A} \) is a submodel of \( \mathcal{B} \) (\( \mathcal{A} \subseteq \mathcal{B} \)) if the following are satisfied:
   (a) for each constant \( c \in \mathcal{L} \), \( c^\mathcal{A} = c^\mathcal{B} \);
   (b) for each \( n \)-ary function symbol \( f \in \mathcal{L} \) and each \( a_1, \ldots, a_n \in \mathcal{A} \),
       \( f^\mathcal{A}(a_1, \ldots, a_n) = f^\mathcal{B}(a_1, \ldots, a_n) \);
   (c) for each \( n \)-ary relation symbol \( R \in \mathcal{L} \) and each \( a_1, \ldots, a_n \in \mathcal{A} \),
       \( R^\mathcal{A}(a_1, \ldots, a_n) \iff R^\mathcal{B}(a_1, \ldots, a_n) \).

2. \( \mathcal{A} \) is an elementary submodel of \( \mathcal{B} \) (written \( \mathcal{A} \preceq \mathcal{B} \)) if
   (a) \( \mathcal{A} \) is a submodel of \( \mathcal{B} \);
   (b) for each \( \mathcal{L} \)-formula \( \phi(x_1, \ldots, x_n) \) and each \( a_1, \ldots, a_n \in \mathcal{A} \),
       \( \mathcal{A} \models \phi(a_1, \ldots, a_n) \iff \mathcal{B} \models \phi(a_1, \ldots, a_n) \).

Example 5.4.7. Consider the rings \((\mathbb{Z}, 0, 1, +, \cdot) \subseteq (\mathbb{Q}, 0, 1, +, \cdot) \subseteq (\mathbb{R}, 0, 1, +, \cdot)\).  
• \((\mathbb{Z}, 0, 1, +, \cdot) \) is a submodel of \((\mathbb{Q}, 0, 1, +, \cdot) \) and \((\mathbb{Q}, 0, 1, +, \cdot) \) is a submodel of \((\mathbb{R}, 0, 1, +, \cdot) \).
• \((\mathbb{Z}, 0, 1, +, \cdot) \) is not an elementary submodel of \((\mathbb{Q}, 0, 1, +, \cdot) \), since \( \mathbb{Q} \models (\exists x)x + x = 1 \) which is false in \( \mathbb{Z} \).
• Neither \((\mathbb{Z}, 0, 1, +, \cdot) \) nor \((\mathbb{Q}, 0, 1, +, \cdot) \) is an elementary submodel of \((\mathbb{R}, 0, 1, +, \cdot) \) since \( \mathbb{R} \models (\exists x)x \cdot x = 2 \), which is false in both \( \mathbb{Z} \) and \( \mathbb{Q} \).

Example 5.4.8. The following elementary submodel relations hold:
• \((\mathbb{Q}, \leq) \preceq (\mathbb{R}, \leq) \)
• \((\mathbb{N}, 0, 1, +, \cdot) \preceq (\mathbb{N}^*, 0, 1, +, \cdot) \).
• \((\mathbb{R}, 0, 1, +, \cdot) \preceq (\mathbb{R}^*, 0, 1, +, \cdot) \).

The latter two items in the previous example justify the claims that the natural numbers are contained in models of non-standard arithmetic and that the real numbers are contained in models of non-standard analysis.
We conclude with one last example.

Example 5.4.9. \( \mathbb{Z}_3 = \{0, 1, 2\} \) with addition modulo 3 is not a submodel of \( \mathbb{Z}_6 = \{0, 1, 2, 3, 4, 5\} \) with addition modulo 6 because they have different addition functions: In \( \mathbb{Z}_3 \), \( 2 + 2 = 1 \) whereas in \( \mathbb{Z}_6 \), \( 2 + 2 = 4 \). However, \( \mathbb{Z}_3 \) is isomorphic to the subgroup of \( \mathbb{Z}_6 \) consisting of \( \{0, 2, 4\} \).
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Computability Theory
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Bibliography