Foundations of Mathematics

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Chapter 1

Introduction

The first problem in the study of the foundations of mathematics is to determine the nature of mathematics. That is, what are mathematicians supposed to do?

For the sake of discussion, let us consider four kinds of mathematical activity.

The first and most natural activity is computation. That is, the creation and application of algorithms to use in the solution of mathematical or scientific problems. For example, we learn algorithms for addition and multiplication of integers in elementary school. In college we may learn the Euclidean Algorithm, which is used to compute the least common denominator of two positive integers. It is non-trivial to prove that the Euclidean Algorithm actually works. In this course we will consider the concept of algorithms and the question of whether certain problems may or may not be solvable by an algorithm.

The second kind of mathematical activity consists of discovering properties of natural mathematical structures such as the integers, the real line and Euclidean geometry, in somewhat the same way that a physicist discovers properties of the universe of matter and energy. That is, by experiment and by thought. Thus we have commutative law of addition, the density of the ordering of the real line and the incidence axioms for points and lines. I am thinking in particular here of properties which cannot be derived from previous principles but which will be taken as axioms or definitions of our structures. This can also include the discovery of complicated theorems which it is hoped will follow from previously accepted principles. In this course we will discuss the Incompleteness Theorem of Godel, which implies that there will always be new properties of the natural numbers $\{0, 1, 2, ...\}$ with addition and multiplication remaining to be discovered.

The third kind of mathematical activity consists of deriving (or proving) theorems from a given set of axioms, usually those abstracted from the study of the first kind. These theorems will now apply to a whole family of models, those which satisfy the axioms. For example, one may discover that all reasonably small positive integers may be obtained as a sum of 5 or fewer squares and conjecture that this property is true of all positive integers. Now it remains to prove the conjecture, using known properties of the integers. Most of the time a proof will apply to more than one mathematical structure. For example, we have in finite group theory Lagrange's Theorem that the order of a subgroup divides the number of elements of the group. This applies to any structure which satisfies the axioms of group theory. In this course we will consider the concept of mathematical proof. The Completeness Theorem of Godel tells us that any true theorem can be proved (eventually).

The fourth kind of mathematical activity consists of constructing new models. As an example, we have the constuction of the various finite groups, culminating in the monster simple group recently found by Griess. Most of the models are based on the natural structures of the integers and the reals, but the powerful ideas of set theory have led to many models which could not have been found otherwise. This fourth kind of mathematics includes demonstrating the independence of the axioms found in the first kind and also includes the "give a counterexample" part of the standard mathematical question: "Prove or give a counterexample." In this course we will consider the concept of models and use models to show that various conjectures are independent. The most famous result of this kind is the independence of the parallel Postulate of Euclidean Geometry. A more recent example is Cohen's model in which the Continuum Hypothesis is false.

The study of the foundations of mathematics is sometimes called meta- mathematics. The primary tool in this study is mathematical logic. In particular, mathematical logic provides the formal language of mathematics, in which theorems are stated. Therefore we begin with the propositional and predicate calculus and the notions of truth and models.

Set Theory and Mathematical Logic compose the foundation of pure mathematics. Using the axioms of set theory, we can construct our universe of discourse, beginning with the natural numbers, moving on with sets and functions over the natural numbers, integers, rationals and real numbers, and eventually developing the transfinite ordinal and cardinal numbers. Mathematical logic provides the language of higher mathematics which allow one to frame the definitions, lemmas, theorems and conjectures which form the every day work of mathematicians. The axioms and rules of deduction set up the system in which we can prove our conjectures, thus turning them into theorems.

A separate volume on set theory begins with a chapter introducing the axioms of set theory, including a brief review of the notions of sets, functions, relations, intersections, unions, complements and their connection with elementary logic. The second chapter introduces the notion of cardinality, including finite versus infinite, and countable versus uncountable sets. We define the Von Neumann natural numbers $\omega = \{0, 1, 2, \dots\}$ in the context of set theory. The methods of recursive and inductive definability over the natural numbers are used to define operations including addition and multiplication on the natural numbers. These methods are also used to define the transitive closure of a set A as the closure of A under the union operator and to define the hereditarily finite sets as the closure of 0 under the power set operator. The notion of a model of set theory is introduced. Conditions are given under which a given set A can satisfy certain of the axioms, such as the union axiom, the power set axiom, and so on. It is shown that the hereditarily finite sets satisfy all axioms except for the Axiom of Infinity.

Several topics covered here are not typically found in a standard textbook.

An effort is made to connect foundations with the usual mathematics major topics of algebra, analysis, geometry and topology. Thus we have chapters on Boolean algebras, on non-standard analysis, and on the foundations of geometry. There is an introduction to descriptive set theory, including cardinality of sets of real numbers. The topics of inductive and recursive definability plays an important role in all areas of logic, including set theory, computability theory, and proof theory. As part of the material on the axioms of set theory, we consider models of various subsets of the axioms, as an introduction to consisitency and independence. Our development of computability theory begins with the study of finite state automata and is enhanced by an introduction to algorithmic randomness, the preeminent topic in computability in recent times. This additional material gives the instructor options for creating a course which provides the basic elements of set theory and logic, as well as making a solid connection with many other areas of mathematics.

Chapter 2

Foundations of Geometry

2.1 Introduction

Plane geometry is an area of mathematics that has been studied since ancient times. The roots of the word *geometry* are the Greek words *ge* meaning "earth" and *metria* meaning "measuring". This name reflects the computational approach to geometric problems that had been used before the time of Euclid, (ca. 300 B.C.), who introduced the axiomatic approach in his book, *Elements*. He was attempting to capture the reality of the space of our universe through this abstraction. Thus the theory of geometry was an attempt to capture the essence of a particular model.

Euclid did not limit himself to plane geometry in the *Elements*, but also included chapters on algebra, ratio, proportion and number theory. His book set a new standard for the way mathematics was done. It was so systematic and encompassing that many earlier mathematical works were discarded and thereby lost for historians of mathematics.

We start with a discussion of the foundations of plane geometry because it gives an accessible example of many of the questions of interest.

2.2 Axioms of plane geometry

Definition 2.2.1. The theory of **Plane Geometry**, PG, has two one-place predicates, *Pt* and *Ln*, to distinguish the two kinds of objects in plane geometry, and a binary *incidence* relation, *In*, to indicate that a point is *on* or *incident* with a line.

By an abuse of notation, write $P \in Pt$ for $Pt(P)$ and $\ell \in Ln$ for $Ln(\ell)$.

There are five axioms in the theory PG:

(A_0) (Everything is either a point or line, but not both; only points are on lines.)

$$(\forall x)((x \in Pt \vee x \in Ln) \,\&\, \neg(x \in Pt \,\&\, x \in Ln)) \,\&\, (\forall x, y)(xIny \rightarrow (x \in Pt \,\&\, y \in Ln)).$$

(A_1) (Any two points belong to a line.)

$$(\forall P, Q \in Pt)(\exists \ell \in Ln)(PIn\ell \,\&\, QIn\ell).$$

(A_2) (Every line has at least two points.)

$$(\forall \ell \in Ln)(\exists P, Q \in Pt)(PIn\ell \,\&\, QIn\ell \,\&\, P \neq Q).$$

(A_3) (Two lines intersect in at most one point.)

$$(\forall \ell, g \in Ln)(\forall P, Q \in Pt)((\ell \neq g \,\&\, P, QIn\ell \,\&\, P, QIng) \rightarrow P = Q).$$

(A_4) (There are four points no three on the same line.) $(\exists P_0, P_1, P_2, P_3 \in Pt)(P_0 \neq P_1 \neq P_2 \neq P_3$ & $P_2 \neq P_0 \neq P_3 \neq P_1$ & $(\forall \ell \in Ln)($

$\neg(P_0 In\ell$ & $P_1 In\ell$ & $P_2 In\ell)$ &

$\neg(P_0 In\ell$ & $P_1 In\ell$ & $P_3 In\ell)$ &

$\neg(P_0 In\ell$ & $P_2 In\ell$ & $P_3 In\ell)$ &

$\neg(P_1 In\ell$ & $P_2 In\ell$ & $P_3 In\ell))$.

The axiom labeled 0 simply says that our objects have the types we intend, and is of a different character than the other axioms. In addition to these axioms, Euclid had one that asserted the existence of circles of arbitrary center and arbitrary radius, and one that asserted that all right angles are equal. He also had another axiom for points and lines, called the parallel postulate, which he attempted to show was a consequence of the other axioms.

Definition 2.2.2. Two lines are *parallel* if there is no point incident with both of them.

Definition 2.2.3. For $n \geq 0$, the *n-parallel postulate*, P_n, is the following statement:

(P_n) For any line ℓ and any point Q not on the line ℓ, there are n lines parallel to ℓ through the point Q.

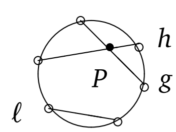
P_1 is the familiar parallel postulate.

For nearly two thousand years, people tried to prove what Euclid had conjectured. Namely, they tried to prove that P_1 was a consequence of the other axioms. In the 1800's, models of the other axioms were produced which were not models of P_1.

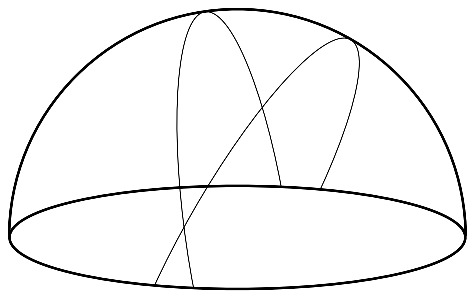
2.3 Non-Euclidean models

Nikolai Lobachevski (1793-1856), a Russian mathematician, and Janos Bolyai (1802-1860), a Hungarian mathematician, both produced models of the other axioms together with the parallel postulate P_∞, that there are infinitely many lines parallel to a given line through a given point. This geometry is known as *Lobachevskian Geometry*. It is enough to assume $P_{\geq 2}$ together with the circle and angle axioms to get P_∞.

Example 2.3.1. (A model for Lobachevskian Geometry): Fix a circle, C, in a Euclidean plane. The points of the geometry are the interior points of C. The lines of the geometry are the intersection of lines of the Euclidean plane with the interior of the circle. Given any line ℓ of the geometry and any point Q of the geometry which is not on ℓ, every Euclidean line through Q which intersects ℓ on or outside of C gives rise to a line of the geometry which is parallel to ℓ.



Example 2.3.2. (A model for Riemannian Geometry): Fix a sphere, S, in Euclidean 3-space. The points of the geometry may be thought of as either the points of the upper half of the sphere, or as equivalence classes consisting of the pairs of points on opposite ends of diameters of the sphere (antipodal points). If one chooses to look at the points as coming from the upper half of the sphere, one must take care to get exactly one from each of the equivalence classes. The lines of the geometry are the intersection of the great circles with the points. Since any two great circles meet in two antipodal points, every pair of lines intersects. Thus this model satisfies P_0.



Bernhard Riemann (1826-1866), a German mathematician, was a student of Karl Gauss (1777-1855), who is regarded as the greatest mathematician of the nineteenth century. Gauss made contributions in the areas of astronomy, geodesy and electricity as well as mathematics. While Gauss considered the possibility of non-Euclidean geometry, he never published anything about the subject.

2.4 Finite geometries

Next we turn to finite geometries, ones with only finitely many points and lines. To get the theory of the finite projective plane of order q, denoted PG(q), in addition to the five axioms given above, we add two more:

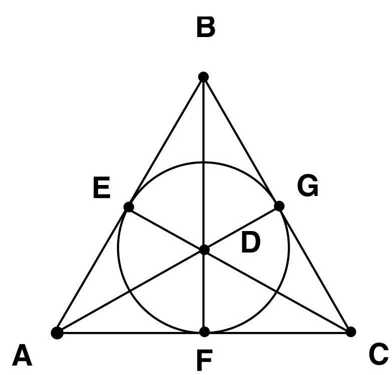
($A_5(q)$) Every line contains exactly $q+1$ points.

($A_6(q)$) Every point lies on exactly $q+1$ lines.

The first geometry we look at is the finite projective plane of order 2, PG(2), also known as the Fano Plane.

Theorem 2.4.1. *The theory PG(2) consisting of PG together with the two axioms $(5)_2$ and $(6)_2$ determines a finite geometry of seven points and seven lines, called the Fano plane.*

Proof. See Exercise 5 to prove from the axioms and Exercise 4 that the following diagram gives a model of PG(2) and that any model must have the designated number of points and lines. □



Next we construct a different model of this finite geometry using a vector space. The vector space underlying the construction is the vector space of dimension three over the field of two elements, $Z_2 = \{0, 1\}$. The points of the geometry are one dimensional subspaces. Since a one-dimensional subspace of Z_2 has exactly two triples in it, one of which is the triple $(0,0,0)$, we identify the points with the triples of 0's and 1's that are not all zero. The lines of the geometry are the two dimensional subspaces. The incidence relation is determined by a point is on a line if the one dimensional subspace is a subspace of the two dimensional subspace. Since a two dimensional subspace is picked out as the orthogonal complement of a one- dimensional subspace, each two dimensional subspace is identified with the non-zero triple, and to test if point (i,j,k) is on line $[\ell,m,n]$, one tests the condition

$$i\ell + jm + kn \equiv 0 \pmod 2.$$

There are exactly $2^3 = 8$ ordered triples of 0's and 1's, of which one is the all zero vector. Thus the ordered triples pick out the correct number of points and lines. The following array gives the incidence relation, and allows us to check that there are three points on every line and three lines through every point.

In	[1,0,0]	[0,1,0]	[0,0,1]	[1,1,0]	[1,0,1]	[0,1,1]	[1,1,1]
(1,0,0)	0	1	1	0	0	1	0
(0,1,0)	1	0	1	0	1	0	0
(0,0,1)	1	1	0	1	0	0	0
(1,1,0)	0	0	1	1	0	0	1
(1,0,1)	0	1	0	0	1	0	1
(0,1,1)	1	0	0	0	0	1	1
(1,1,1)	0	0	0	1	1	1	0

The vector space construction works over other finite fields as well. The next bigger example is the projective geometry of order 3, PG(3). The points are the one dimensional subspaces of the vector space of dimension 3 over the field of three elements, $Z_3 = \{0,1,2\}$. This vector space has $3^3 - 1 = 27 - 1 = 26$ non-zero vectors. Each one dimensional subspace has two non-zero elements, so there are $26/2 = 13$ points in the geometry. As above, the lines are the orthogonal or perpendicular complements of the subspaces that form the lines, so there are also 13 of them. The test for incidence is similar to the one above, except that one must work $\pmod 3$ rather than $\pmod 2$.

This construction works for each finite field. In each case the order of the projective geometry is the size of the field.

The next few lemmas list a few facts about projective planes.

Lemma 2.4.2. *In any model of PG(q), any two lines intersect in a point.*

Lemma 2.4.3. *In any model of PG(q) there are exactly* $q^2 + q + 1$ *points.*

Lemma 2.4.4. *In any model of PG(q) there are exactly* $q^2 + q + 1$ *lines.*

For models built using the vector space construction over a field of q elements, it is easy to compute the number of points and lines as $\frac{q^3-1}{q-1} = q^2 + q + 1$. However there are non-isomorphic projective planes of the same order. For a long time four non-isomorphic planes of order nine were known, each obtained by a variation on the above vector space construction. Recently it has been shown with the help of a computer that there are exactly four non-isomorphic planes of order nine.

Since the order of a finite field is always a prime power, the models discussed so far all have prime power order. Much work has gone into the search for models of non-prime power order. A well-publicized result showed that there were no projective planes of order 10. This proof required many hours of computer time.

2.5 Exercises

1. Translate Axioms $(5)_q$ and $(6)_q$ into the formal language.

2. Label the three points of intersection of the lines in the illustration of Riemannian geometry which form a triangle above the "equator".

3. Define an isomorphism between the two models of $PG(2)$.

4. Prove from the axioms that any two lines in $PG(q)$ must intersect in a point. (Hint: Show that if g and h do not intersect and P is incident with g, then P is on at least one more line than h has points.)

5. Construct a model for $PG(2)$ starting with four non-collinear points A, B, C and D and denoting the additional point on the line $\overline{AB}$ by E, the additional point on the line $\overline{AC}$ by F, and the additional point on the line $\overline{BC}$ by G. Use the axioms and exercise 1.4 to justify the construction.

6. Prove from the axioms that in any model of PG(q) there are exactly $q^2 + q + 1$ points.

7. Prove from the axioms that in any model of PG(q) there are exactly $q^2 + q + 1$ lines.

8. List the 13 one-dimensional subspaces of $\mathbb{Z}_3^3$ by giving one generator of each. (Hint: Proceeding lexicographically, four of them begin with "0" and the other nine begin with "1".) These are the points of $PG(3)$. Identify the 13 two dimensional subspaces of $\mathbb{Z}_3^3$ as orthogonal complements of these one-dimensional spaces. These are the lines of $PG(3)$. For each line, list the four points on the line.

9. Show that the axioms 1,2,3,4 for Plane Geometry are independent by constructing models which satisfy exactly 3 of the axioms. (There are 4 possible cases here.)

Chapter 3

Propositional Logic

3.1 The basic definitions

Propositional logic concerns relationships between sentences built up from primitive proposition symbols with logical connectives.

The symbols of the language of predicate calculus are

1. Logical connectives: $\neg$, $\&$, $\vee$, $\rightarrow$, $\leftrightarrow$
2. Punctuation symbols: (,)
3. Propositional variables: $A_0, A_1, A_2, \ldots$.

A propositional variable is intended to represent a proposition which can either be true or false. Restricted versions, $\mathscr{L}$, of the language of propositional logic can be constructed by specifying a subset of the propositional variables. In this case, let $\mathrm{PVar}(\mathscr{L})$ denote the propositional variables of $\mathscr{L}$.

Definition 3.1.1. The collection of *sentences*, denoted $Sent(\mathscr{L})$, of a propositional language $\mathscr{L}$ is defined by recursion.

1. The basis of the set of sentences is the set $\mathrm{PVar}(\mathscr{L})$ of propositional variables of $\mathscr{L}$.
2. The set of sentences is closed under the following production rules:
   (a) If A is a sentence, then so is $(\neg A)$.
   (b) If A and B are sentences, then so is $(A \& B)$.
   (c) If A and B are sentences, then so is $(A \vee B)$.
   (d) If A and B are sentences, then so is $(A \rightarrow B)$.
   (e) If A and B are sentences, then so is $(A \leftrightarrow B)$.

Notice that as long as $\mathscr{L}$ has at least one propositional variable, then $Sent(\mathscr{L})$ is infinite. When there is no ambiguity, we will drop parentheses.

In order to use propositional logic, we would like to give meaning to the propositional variables. Rather than assigning specific propositions to the propositional variables and then determining their truth or falsity, we consider truth interpretations.

Definition 3.1.2. A *truth interpretation* for a propositional language $\mathscr{L}$ is a function

$$I : \mathrm{PVar}(\mathscr{L}) \to \{0,1\}.$$

If $I(A_i) = 0$, then the propositional variable A_i is considered represent a false proposition under this interpretation. On the other hand, if $I(A_i) = 1$, then the propositional variable A_i is considered to represent a true proposition under this interpretation.

There is a unique way to extend the truth interpretation to all sentences of $\mathscr{L}$ so that the interpretation of the logical connectives reflects how these connectives are normally understood by mathematicians.

Definition 3.1.3. Define an extension of a truth interpretation $I : \mathrm{PVar}(\mathscr{L}) \to \{0,1\}$ for a propositional language to the collection of all sentences of the language by recursion:

1. On the basis of the set of sentences, $\mathrm{PVar}(\mathscr{L})$, the truth interpretation has already been defined.
2. The definition is extended to satisfy the following closure rules:
   (a) If $I(A)$ is defined, then $I(\neg A) = 1 - I(A)$.
   (b) If $I(A)$ and $I(B)$ are defined, then $I(A \,\&\, B) = I(A) \cdot I(B)$.
   (c) If $I(A)$ and $I(B)$ are defined, then $I(A \vee B) = max\{I(A), I(B)\}$.
   (d) If $I(A)$ and $I(B)$ are defined, then

$$I(A \to B) = \begin{cases} 0 & \text{if } I(A) = 1 \text{ and } I(B) = 0, \\ 1 & \text{otherwise.} \end{cases} \tag{3.1}$$

   (e) If $I(A)$ and $I(B)$ are defined, then $I(A \leftrightarrow B) = 1$ if and only if $I(A) = I(B)$.

Intuitively, tautologies are statements which are always true, and contradictions are ones which are never true. These concepts can be defined precisely in terms of interpretations.

Definition 3.1.4. A sentence φ is a *tautology* for a propositional language $\mathscr{L}$ if every truth interpretation I has value 1 on φ, $I(\varphi) = 1$. φ is a *contradiction* if every truth interpretation I has value 0 on φ, $I(\varphi) = 0$. Two sentences φ and ψ are *logically equivalent*, in symbols $\varphi \Longleftrightarrow \psi$, if every truth interpretation I takes the same value on both of them, $I(\varphi) = I(\psi)$. A sentence φ is *satisfiable* if there is some truth interpretation I with $I(\varphi) = 1$.

The notion of logical equivalence is an equivalence relation; that is, it is a reflexive, symmetric and transitive relation. The equivalence classes given by logical equivalence are infinite for non-trivial languages (i.e., those languages containing at least one propositional variable). However, if the language has only finitely many propositional variables, then there are only finitely many equivalence classes.

Notice that if $\mathscr{L}$ has n propositional variables, then there are exactly $d = 2^n$ truth interpretations, which we may list as $\mathscr{I} = \{I_0, I_1, \ldots, I_{d-1}\}$. Since each I_i maps the truth values 0 or 1 to each of the n propositional variables, we can think of each truth interpretation as a function from the set $\{0, \ldots, n-1\}$ to the set $\{0,1\}$. The collection of such functions can be written as $\{0,1\}^n$, which can also be interpreted as the collection of binary strings of length n.

Each sentence φ gives rise to a function $TF_\varphi : \mathscr{I} \to \{0,1\}$ defined by $TF_\varphi(I_i) = I_i(\varphi)$. Informally, TF_φ lists the column under φ in a truth table. Note that for any two sentences φ and ψ, if $TF_\varphi = TF_\psi$ then φ and ψ are logically equivalent. Thus there are exactly $2^d = 2^{2^n}$ many equivalence classes.

Lemma 3.1.5. *The following pairs of sentences are logically equivalent as indicated by the metalogical symbol* $\Longleftrightarrow$*:*

1. $\neg\neg A \iff A$.
2. $\neg A \vee \neg B \iff \neg(A \,\&\, B)$.
3. $\neg A \,\&\, \neg B \iff \neg(A \vee B)$.
4. $A \to B \iff \neg A \vee B$.
5. $A \leftrightarrow B \iff (A \to B) \,\&\, (B \to A)$.

Proof. Each of these statements can be proved using a truth table, so from one example the reader may do the others. Notice that truth tables give an algorithmic approach to questions of logical equivalence.

	A	B	$(\neg A)$	$(\neg B)$	$((\neg A) \vee (\neg B))$	$(A \,\&\, B)$	$(\neg(A \,\&\, B))$
I_0	0	0	1	1	1	0	1
I_1	1	0	0	1	1	0	1
I_2	0	1	1	0	1	0	1
I_3	1	1	0	0	0	1	0
					↑		↑

□

Using the above equivalences, one could assume that $\neg$ and $\vee$ are primitive connectives, and define the others in terms of them. The following list gives three pairs of connectives each of which is sufficient to get all our basic list:

$$\neg, \vee$$
$$\neg, \,\&$$
$$\neg, \to$$

In logic, the word "theory" has a technical meaning, and refers to any set of statements, whether meaningful or not.

Definition 3.1.6. A set Γ of sentences in a language $\mathscr{L}$ is *satisfiable* if there is some interpretation I with $I(\varphi) = 1$ for all $\varphi \in \Gamma$. A set of sentences Γ *logically implies* a sentence φ, in symbols, $\Gamma \models \varphi$ if for every interpretation I, if $I(\psi) = 1$ for all $\psi \in \Gamma$, then $I(\varphi) = 1$. A *(propositional) theory* in a language $\mathscr{L}$ is a set of sentences $\Gamma \subseteq Sent(\mathscr{L})$ which is closed under logical implication.

Notice that a theory as a set of sentences matches with the notion of the theory of plane geometry as a set of axioms. In studying that theory, we developed several models. The interpretations play the role here that models played in that discussion. Here is an example of the notion of logical implication defined above.

Lemma 3.1.7. $\{(A \,\&\, B), (\neg C)\} \models (A \vee B)$.

3.2 Disjunctive Normal Form Theorem

In this section we will show that the language of propositional calculus is sufficient to represent every possible truth function.

Definition 3.2.1.

1. A *literal* is either a propositional variable A_i or its negation $\neg A_i$.
2. A *conjunctive clause* is a conjunction of literals and a *disjunctive clause* is a disjunction of literals. We will assume in each case that each propositional variable occurs at most once.
3. A propositional sentence is in *disjunctive normal form* if it is a disjunction of conjunctive clauses and it is in *conjunctive normal form* if it is a conjunction of disjunctive clauses.

Lemma 3.2.2.

(i) For any conjunctive clause $C = \phi(A_1, \dots, A_n)$*, there is a unique interpretation* $I_C : \{A_1, \dots, A_n\} \to \{0,1\}$ *such that* $I_C(\phi) = 1$.

(ii) Conversely, for any interpretation $I : \{A_1, \dots, A_n\} \to \{0,1\}$*, there is a unique conjunctive clause* C_I *(up to permutation of literals) such that* $I(C_I) = 1$ *and for any interpretation* $J \neq I$, $J(C_I) = 0$.

Proof. (i) Let

$$B_i = \begin{cases} A_i & \text{if } C \text{ contains } A_i \text{ as a conjunct} \\ \neg A_i & \text{if } C \text{ contains } \neg A_i \text{ as a conjunct} \end{cases}.$$

It follows that $C = B_1 \,\&\, \dots \,\&\, B_n$. Now let $I_C(A_i) = 1$ if and only if $A_i = B_i$. Then clearly $I(B_i) = 1$ for $i = 1, 2, \dots, n$ and therefore $I_C(C) = 1$. To show uniqueness, if $J(C) = 1$ for some interpretation J, then $\phi(B_i) = 1$ for each i and hence $J = I_C$.

(ii) Let

$$B_i = \begin{cases} A_i & \text{if } I(A_i) = 1 \\ \neg A_i & \text{if } I(A_i) = 0 \end{cases}.$$

Let $C_I = B_1 \,\&\, \dots \,\&\, B_n$. As above $I(C_I) = 1$ and $J(C_I) = 1$ implies that $J = I$.

It follows as above that I is the unique interpretation under which C_I is true. We claim that C_I is the unique conjunctive clause with this property. Suppose not. Then there is some conjunctive clause C' such that $I(C') = 1$ and $C' \neq C_I$. This implies that there is some literal A_i in C' and $\neg A_i$ in C_I (or vice versa). But $I(C') = 1$ implies that $I(A_i) = 1$ and $I(C_I) = 1$ implies that $I(\neg A_i) = 1$, which is clearly impossible. Thus C_I is unique. □

Here is the Disjunctive Normal Form Theorem.

Theorem 3.2.3. *For any truth function* $F : \{0,1\}^n \to \{0,1\}$*, there is a sentence* ϕ *in disjunctive normal form such that* $F = TF_\phi$.

Proof. Let $I_1, I_2, \dots, I_k$ be the interpretations in $\{0,1\}^n$ such that $F(I_i) = 1$ for $i = 1, \dots, k$. For each i, let $C_i = C_{I_i}$ be the conjunctive clauses guaranteed to hold by the previous lemma. Now let $\phi = C_1 \lor C_2 \lor \dots \lor C_k$. Then for any interpretation I,

$$\begin{aligned} TF_\phi(I) = 1 &\text{ if and only if } I(\phi) = 1 \text{ (by definition)} \\ &\text{ if and only if } I(C_i) = 1 \text{ for some } i = 1, \dots, k \\ &\text{ if and only if } I = I_i \text{ for some } i \text{ (by the previous lemma)} \\ &\text{ if and only if } F(I) = 1 \text{ (by the choice of } I_1, \dots, I_k) \end{aligned}$$

Hence $TF_\phi = F$ as desired. □

Example 3.2.4. Suppose that we want a formula $\phi(A_1, A_2, A_3)$ such that $I(\phi) = 1$ only for the three interpretations $(0,1,0)$, $(1,1,0)$ and $(1,1,1)$. Then

$$\phi = (\neg A_1 \,\&\, A_2 \,\&\, \neg A_3) \lor (A_1 \,\&\, A_2 \,\&\, \neg A_3) \lor (A_1 \,\&\, A_2 \,\&\, A_3).$$

It follows that the connectives $\neg, \&, \lor$ are sufficient to express all truth functions. By the deMorgan laws (2,3 of Lemma 2.5) $\neg, \lor$ are sufficient and $\neg, \land$ are also sufficient.

3.3 Proofs

One of the basic tasks that mathematicians do is proving theorems. This section develops the Propositional Calculus, which is a system rules of inference for propositional languages. With it one formalizes the notion of proof. Then one can ask questions about what can be proved, what cannot be proved, and how the notion of proof is related to the notion of interpretations.

The basic relation in the Propositional Calculus is the relation *proves* between a set, Γ of sentences and a sentence B. A more long-winded paraphrase of the relation "Γ proves B" is "there is a proof of B using what ever hypotheses are needed from Γ". This relation is denoted $X \vdash Y$, with the following abbreviations for special cases:

Formal Version:	$\Gamma \vdash \{B\}$	$\{A\} \vdash B$	$\emptyset \vdash B$
Abbreviation:	$\Gamma \vdash B$	$A \vdash B$	$\vdash B$

Let $\bot$ be a new symbol that we will add to our propositional language. The intended interpretation of $\bot$ is 'falsehood,' akin to asserting a contradiction.

Definition 3.3.1. A *formal proof* or derivation of a propositional sentence ϕ from a collection of propositional sentences Γ is a finite sequence of propositional sentences terminating in ϕ where each sentence in the sequence is either in Γ or is obtained from sentences occurring earlier in the sequence by means of one of the following rules.

1. (Given rule) Any $B \in \Gamma$ may be derived from Γ in one step.

2. (&-Elimination) If $(A \,\&\, B)$ has been derived from Γ then either of A or B may be derived from Γ in one further step.

3. ($\vee$-Elimination) If $(A \vee B)$ has been derived from Γ, under the further assumption of A we can derive C from Γ, and under the further assumption of B we can derive C from Γ, then we can derive C from Γ in one further step.

4. ($\rightarrow$-Elimination) If $(A \rightarrow B)$ and A have been derived from Γ, then B can be derived from Γ in one further step.

5. ($\bot$-Elimination) If $\bot$ has been deduced from Γ, then we can derive any sentence A from Γ in one further step.

6. ($\neg$-Elimination) If $\neg\neg A$ has been deduced from Γ, then we can derive A from Γ in one further step.

7. (&-Introduction) If A and B have been derived from Γ, then $(A \,\&\, B)$ may be derived from Γ in one further step.

8. ($\vee$-Introduction) If A has been derived from Γ, then either of $(A \vee B)$, $(B \vee A)$ may be derived from Γ in one further step.

9. ($\rightarrow$-Introduction) If under the assumption of A we can derive B from Γ, then we can derive $A \rightarrow B$ from Γ in one further step.

10. ($\bot$-Introduction) If $(A \,\&\, \neg A)$ has been deduced from Γ, then we can derive $\bot$ from Γ in one further step.

11. ($\neg$-Introduction) If $\bot$ has been deduced from Γ and A, then we can derive $\neg A$ from Γ in one further step.

The relation $\Gamma \vdash A$ can now be defined to hold if there is a formal proof of A from Γ that uses the rules given above. The symbol $\vdash$ is sometimes called a *(single) turnstile*. Here is a more precise, formal definition.

Definition 3.3.2. The relation $\Gamma \vdash B$ is the smallest subset of pairs (Γ, B) from $\mathscr{P}(Sent) \times Sent$ which contains every pair (Γ, B) such that $B \in \Gamma$ and is closed under the above rules of deduction.

We now provide some examples of proofs.

Proposition 3.3.3. *For any sentences A, B, C*

1. $\vdash A \to A$
2. $A \to B \vdash \neg B \to \neg A$
3. $\{A \to B, B \to C\} \vdash A \to C$
4. $A \vdash A \vee B$ *and* $A \vdash B \vee A$
5. $\{A \vee B, \neg A\} \vdash B$
6. $A \vee A \vdash A$
7. $A \vdash \neg\neg A$
8. $A \vee B \vdash B \vee A$ *and* $A \,\&\, B \vdash B \,\&\, A$
9. $(A \vee B) \vee C \vdash A \vee (B \vee C)$ *and* $A \vee (B \vee C) \vdash (A \vee B) \vee C$
10. $(A \,\&\, B) \,\&\, C \vdash A \,\&\, (B \,\&\, C)$ *and* $A \,\&\, (B \,\&\, C) \vdash (A \,\&\, B) \,\&\, C$
11. $A \,\&\, (B \vee C) \vdash (A \,\&\, B) \vee (A \,\&\, C)$ *and* $(A \,\&\, B) \vee (A \,\&\, C) \vdash A \,\&\, (B \vee C$
12. $A \vee (B \,\&\, C) \vdash (A \vee B) \,\&\, (A \vee C)$ *and* $(A \vee B) \,\&\, (A \vee C) \vdash A \vee (B \,\&\, C)$
13. $\neg(A \,\&\, B) \vdash \neg A \vee \neg B$ *and* $\neg A \vee \neg B \vdash \neg(A \,\&\, B)$
14. $\neg(A \vee B) \vdash \neg A \,\&\, \neg B$ *and* $\neg A \,\&\, \neg B \vdash \neg(A \vee B)$
15. $\neg A \vee B \vdash A \to B$ *and* $A \to B \vdash \neg A \vee B$
16. $\vdash A \vee \neg A$

We give brief sketches of some of these proofs to illustrate the various methods.

Proof.

1. $\vdash A \to A$

1	$\quad A$	Assumption
2	$\quad A$	Given
3	$A \to A$	$\to$-Introduction (1-2)

3. $\{A \to B, B \to C\} \vdash A \to C$

1	$A \to B$	Given
2	$B \to C$	Given
3	│ A	Assumption
4	│ B	$\to$-Elimination 1,3
5	│ C	$\to$-Elimination 2,4
6	$A \to C$	$\to$-Introduction 3-5

4. $A \vdash A \lor B$ and $A \vdash B \lor A$

1	A	Given
2	$A \lor B$	$\lor$-Introduction 1

1	A	Given
2	$B \lor A$	$\lor$-Introduction 1

5. $\{A \lor B, \neg A\} \vdash B$

1	$A \lor B$	Given
2	$\neg A$	Given
3	│ A	Assumption
4	│ $A \,\&\, \neg A$	& -Introduction 2,3
5	│ $\bot$	$\bot$-Introduction 4
6	│ B	$\bot$-Elimination 5
7	│ B	Assumption
8	│ B	Given
9	B	$\lor$-Elimination 1-8

6. $A \lor A \vdash A$

1	$A \lor A$	Given
2	│ A	Assumption
3	│ A	Given
4	│ A	Assumption
5	│ A	Given
6	A	$\lor$-Elimination 1-5

7. $A \vdash \neg\neg A$

1	A	Given
2	$\quad \neg A$	Assumption
3	$\quad A \,\&\, \neg A$	&-Introduction 1,2
4	$\quad \bot$	$\bot$-Introduction 3
5	$\neg\neg A$	$\neg$-Introduction 1-4

8. $A \vee B \vdash B \vee A$ and $A \,\&\, B \vdash B \,\&\, A$

1	$A \vee B$	Given
2	$\quad A$	Assumption
3	$\quad B \vee A$	$\vee$-Introduction 2
4	$\quad B$	Assumption
5	$\quad B \vee A$	$\vee$-Introduction 2
6	$B \vee A$	$\vee$-Elimination 1-5

1	$A \,\&\, B$	Given
2	A	&-Elimination
3	B	&-Elimination
4	$B \,\&\, A$	&-Introduction 2-3

10. $(A \,\&\, B) \,\&\, C \vdash A \,\&\, (B \,\&\, C)$ and $A \,\&\, (B \,\&\, C) \vdash (A \,\&\, B) \,\&\, C$

1	$(A \,\&\, B) \,\&\, C$	Given
2	$A \,\&\, B$	&-Elimination 1
3	A	&-Elimination 2
4	B	&-Elimination 2
5	C	&-Elimination 1
6	$B \,\&\, C$	&-Introduction 4,5
7	$A \,\&\, (B \,\&\, C)$	&-Introduction 3,6

13. $\neg(A \,\&\, B) \vdash \neg A \vee \neg B$ and $\neg A \vee \neg B \vdash \neg(A \,\&\, B)$

1	$\neg(A \mathbin{\&} B)$	Given
2	$A \vee \neg A$	Item 6
3	$\mid \neg A$	Assumption
4	$\mid \neg A \vee \neg B$	$\vee$-Introduction 4
5	$\mid A$	Assumption
6	$\mid B \vee \neg B$	Item 6
7	$\mid \mid \neg B$	Assumption
8	$\mid \mid \neg A \vee \neg B$	$\vee$-Introduction 7
9	$\mid \mid B$	Assumption
10	$\mid \mid A \mathbin{\&} B$	& -Introduction 5,9
11	$\mid \mid \bot$	$\bot$-Introduction 1,10
12	$\mid \mid \neg A \vee \neg B$	Item 5
13	$\mid \neg A \vee \neg B$	$\vee$-Elimination 6-12
14	$\neg A \vee \neg B$	$\vee$-Elimination 2-13

1	$\neg A \vee \neg B$	Given
2	$\mid A \mathbin{\&} B$	Assumption
3	$\mid A$	& -Elimination 2
4	$\mid \neg\neg A$	Item 8, 3
5	$\mid \neg B$	Disjunctive Syllogism 1,4
6	$\mid B$	& -Elimination 2
7	$\mid \bot$	$\bot$-Introduction 5,6
8	$\neg(A \mathbin{\&} B)$	Proof by Contradiction 2-7

15. $\neg A \vee B \vdash A \rightarrow B$ and $A \rightarrow B \vdash \neg A \vee B$

1	$\neg A \vee B$	Given
2	$\mid A$	Assumption
3	$\mid \neg\neg A$	Item 8, 2
4	$\mid B$	Item 5 1,3
5	$A \rightarrow B$	$\rightarrow$-Introduction 2-4

1	$A \rightarrow B$	Given
2	$\mid \neg(\neg A \vee B)$	Assumption
3	$\mid \neg\neg A \,\&\, \neg B$	Item 14, 1
4	$\mid \neg\neg A$	&-Introduction 3
5	$\mid A$	¬-Elimination 4
6	$\mid B$	→-Elimination 1,5
7	$\mid \neg B$	&-Introduction 3
8	$\mid B \,\&\, \neg B$	&-Introduction 6,7
9	$\mid \bot$	⊥-Introduction 8
10	$\neg A \vee B$	⊥-Elimination 2-9

16. $\vdash A \vee \neg A$

1	$\mid \neg(A \vee \neg A)$	Assumption
2	$\mid \neg A \,\&\, \neg\neg A$	Item 14, 1
3	$\mid \bot$	⊥-Introduction 3
4	$\neg A \vee A$	¬∨-Rule (1-2)

□

The following general properties about $\vdash$ will be useful when we prove the soundness and completeness theorems.

Lemma 3.3.4. *For any sentences A and B, if $\Gamma \vdash A$ and $\Gamma \cup \{A\} \vdash B$, then $\Gamma \vdash B$.*

Proof. $\Gamma \cup \{A\} \vdash B$ implies $\Gamma \vdash A \rightarrow B$ by →-Introduction. Combining this latter fact with the fact that $\Gamma \vdash A$ yields $\Gamma \vdash B$ by →-Elimination. □

Lemma 3.3.5. *If $\Gamma \vdash B$ and $\Gamma \subseteq \Delta$, then $\Delta \vdash B$.*

Proof. This follows by induction on proof length. For the base case, if B follows from Γ on the basis of the Given Rule, then it must be the case that $B \in \Gamma$. Since $\Gamma \subset \Delta$ it follows that $B \in \Delta$ and hence $\Delta \vdash B$ by the Given Rule.

If the final step in the proof of B from Γ is made on the basis of any one of the rules, then we may assume by the induction hypothesis that the other formulas used in these deductions follow from Δ (since they follow from Γ). We will look at two cases and leave the rest to the reader.

Suppose that the last step comes by →-Elimination, where we have derived $A \rightarrow B$ and A from Γ earlier in the proof. Then we have $\Gamma \vdash A \rightarrow B$ and $\Gamma \vdash B$. By the induction hypothesis, $\Delta \vdash A$ and $\Delta \vdash A \rightarrow B$. Hence $\Delta \vdash B$ by →-Elimation.

Suppose that the last step comes from &-Elimination, where we have derived $A \,\&\, B$ from Γ earlier in the proof. Since $\Gamma \vdash A \,\&\, B$, by inductive hypothesis it follows that $\Delta \vdash A \,\&\, B$. Hence $\Delta \vdash B$ by &-elimination. □

Next we prove a version of the Compactness Theorem for our deduction system.

Theorem 3.3.6. *If $\Gamma \vdash B$, then there is a finite set $\Gamma_0 \subseteq \Gamma$ such that $\Gamma_0 \vdash B$.*

Proof. Again we argue by induction on proofs. For the base case, if B follows from Γ on the basis of the Given Rule, then $B \in \Gamma$ and we can let $\Gamma_0 = \{B\}$.

If the final step in the proof of B from Γ is made on the basis of any one of the rules, then we may assume by the induction hypothesis that the other formulas used in these deductions follow from some finite $\Gamma_0 \subseteq \Gamma$. We will look at two cases and leave the rest to the reader.

Suppose that the last step of the proof comes by $\vee$-Introduction, so that B is of the form $C \vee D$. Then, without loss of generality, we can assume that we derived C from Γ earlier in the proof. Thus $\Gamma \vdash C$. By the induction hypothesis, there is a finite $\Gamma_0 \subseteq \Gamma$ such that $\Gamma_0 \vdash C$. Hence by $\vee$-Introduction, $\Gamma_0 \vdash C \vee D$.

Suppose that the last step of the proof comes by $\vee$-Elimination. Then earlier in the proof

(i) we have derived some formula $C \vee D$ from Γ,

(ii) under the assumption of C we have derived B from Γ, and

(iii) under the assumption of D we have derived B from Γ.

Thus, $\Gamma \vdash C \vee D$, $\Gamma \cup \{C\} \vdash B$, and $\Gamma \cup \{D\} \vdash B$. Then by assumption, by the induction hypothesis, there exist finite sets Γ_0, Γ_1, and Γ_2 of Γ such that $\Gamma_0 \vdash C \vee D$, $\Gamma_1 \cup \{C\} \vdash B$ and $\Gamma_2 \cup \{D\} \vdash B$. By Lemma 3.3.5,

(i) $\Gamma_0 \cup \Gamma_1 \cup \Gamma_2 \vdash C \vee D$

(ii) $\Gamma_0 \cup \Gamma_1 \cup \Gamma_2 \cup \{C\} \vdash B$

(iii) $\Gamma_0 \cup \Gamma_1 \cup \Gamma_2 \cup \{D\} \vdash B$

Thus by $\vee$-Elimination, we have $\Gamma_0 \cup \Gamma_1 \cup \Gamma_2 \vdash B$. Since $\Gamma_0 \cup \Gamma_1 \cup \Gamma_2$ is finite and $\Gamma_0 \cup \Gamma_1 \cup \Gamma_2 \subseteq \Gamma$, the result follows. □

3.4 The Soundness Theorem

We now determine the precise relationship between $\vdash$ and $\models$ for propositional logic. Our first major theorem says that if one can prove something in A from a theory Γ, then Γ logically implies A.

Theorem 3.4.1 (Soundness Theorem). *If $\Gamma \vdash A$, then $\Gamma \models A$.*

Proof. The proof is by induction on the length of the deduction of A. We need to show that if there is a proof of A from Γ, then for any interpretation I such that $I(\gamma) = 1$ for all $\gamma \in \Gamma$, $I(A) = 1$.

(Base Case): For a one-step deduction, we must have used the Given Rule, so that $A \in \Gamma$. If the truth interpretation I has $I(\gamma) = 1$ for all $\gamma \in \Gamma$, then of course $I(A) = 1$ since $A \in \Gamma$.

(Induction): Assume the theorem holds for all shorter deductions. Now proceed by cases on the other rules. We prove a few examples and leave the rest for the reader.

Suppose that the last step of the deduction is given by $\vee$-Introduction, so that A has the form $B \vee C$. Without loss of generality, suppose we have derived B from Γ earlier in the proof. Suppose that $I(\gamma) = 1$ for all $\gamma \in \Gamma$. Since the proof of $\Gamma \vdash B$ is shorter than the given deduction of $B \vee C$, by the inductive hypothesis, $I(B) = 1$. But then $I(B \vee C) = 1$ since I is an interpretation.

Suppose that the last step of the deduction is given by &-Elimination. Suppose that $I(\gamma) = 1$ for all $\gamma \in \Gamma$. Without loss of generality A has been derived from a sentence of the form $A \,\&\, B$, which has been derived from Γ in a strictly shorter proof. Since $\Gamma \vdash A \,\&\, B$, it follows by inductive hypothesis that $\Gamma \models A \,\&\, B$, and hence $I(A \,\&\, B) = 1$. Since I is an interpretation, it follows that $I(A) = 1$.

Suppose that the last step of the deduction is given by $\rightarrow$-Introduction. Then A has the form $B \rightarrow C$. It follows that under the assumption of B, we have derived C from Γ. Thus $\Gamma \cup \{B\} \vdash C$ in a strictly shorter proof. Suppose that $I(\gamma) = 1$ for all $\gamma \in \Gamma$. We have two cases to consider.

Case 1: If $I(B) = 0$, it follows that $I(B \to C) = 1$.

Case 2: If $I(B) = 1$, then since $\Gamma \cup \{B\} \vdash C$,

it follows that $I(C) = 1$. Then $I(B \to C) = 1$. In either case, the conclusion follows.

□

Now we know that anything we can prove is true. We next consider the contrapositive of the Soundness Theorem.

Definition 3.4.2. A set Γ of sentences is *consistent* if there is some sentence A such that $\Gamma \nvdash A$; otherwise Γ is *inconsistent*.

Lemma 3.4.3. *Γ of sentences is* inconsistent *if and only if there is some sentence A such that $\Gamma \vdash A$ and $\Gamma \vdash \neg A$.*

Proof. Suppose first that Γ is inconsistent. Then by definition, $\Gamma \vdash \phi$ for all formulas ϕ and hence $\Gamma \vdash A$ and $\Gamma \vdash \neg A$ for *every* sentence A.

Next suppose that, for some A, $\Gamma \vdash A$ and $\Gamma \vdash \neg A$. It follows by &-Introduction that $\Gamma \vdash A \,\&\, \neg A$. By $\bot$-Introduction, $\Gamma \vdash \bot$. Then by $\bot$-Elimination, for each ϕ, $\Gamma \vdash \phi$. Hence Γ is inconsistent. □

Proposition 3.4.4. *If Γ is satisfiable, then it is consistent.*

Proof. Assume that Γ is satisfiable and let I be an interpretation such that $I(\gamma) = 1$ for all $\gamma \in \Gamma$. Now suppose by way of contradiction that Γ is *not* consistent. Then there is some sentence A such that $\Gamma \vdash A$ and $\Gamma \vdash \neg A$. By the Soundness Theorem, $\Gamma \models A$ and $\Gamma \models \neg A$. But then $I(A) = 1$ and $I(\neg A) = 1$ which is impossible since I is an interpretation. This contradiction demonstrates that Γ is consistent. □

In Section 3.5, we will prove the converse of the Soundness Theorem by showing that any consistent theory is satisfiable.

3.5 The Completeness Theorem

Theorem 3.5.1. *(The Completeness Theorem, Version I) If $\Gamma \models A$, then $\Gamma \vdash A$.*

Theorem 3.5.2. *(The Completeness Theorem, Version II) If Γ is consistent, then Γ is satisfiable.*

We will show that Version II implies Version I and then prove Version II. First we give alternate versions of the Compactness Theorem (Theorem 3.3.6).

Theorem 3.5.3. *(Compactness Theorem, Version II). If every finite subset of Δ is consistent, then Δ is consistent.*

Proof. We show the contrapositive. Suppose that Δ is not consistent. Then, for some B, $\Delta \vdash B \,\&\, \neg B$. It follows from Theorem 3.3.6 that Δ has a finite subset Δ_0 such that $\Delta_0 \vdash B \,\&\, \neg B$. But then Δ_0 is not consistent. □

Theorem 3.5.4. *(Compactness Theorem, Version III). Suppose that*

(i) $\Delta = \bigcup_n \Delta_n$,

(ii) $\Delta_n \subseteq \Delta_{n+1}$ *for every n, and*

(iii) Δ_n *is consistent for each n.*

Then Δ is consistent.

Proof. Again we show the contrapositive. Suppose that Δ is not consistent. Then by Theorem 3.5.4, Δ has a finite, inconsistent subset $F = \{\delta_1, \delta_2, \ldots, \delta_k\}$. Since $\Delta = \bigcup_n \Delta_n$, there exists, for each $i \leq k$, some n_i such that $\delta_i \in \Delta_i$. Letting $n = \max\{n_i : i \leq k\}$, it follows from the fact that the Δ_j's are inconsistent that $F \subseteq \Delta_n$. But then Δ_n is inconsistent. □

Next we prove a useful lemma.

Lemma 3.5.5. *For any* Γ *and* A, $\Gamma \vdash A$ *if and only if* $\Gamma \cup \{\neg A\}$ *is inconsistent.*

Proof. Suppose first that $\Gamma \vdash A$. Then $\Gamma \cup \{\neg A\}$ proves both A and $\neg A$ and is therefore inconsistent.

Suppose next that $\Gamma \cup \{\neg A\}$ is inconsistent. It follows from $\neg$-Introduction that $\Gamma \vdash \neg\neg A$. Then by $\neg$-Elimination, $\Gamma \vdash A$. □

We are already in position to show that Version II of the Completeness Theorem implies Version I. We show the contrapositive of the statement of Version 1; that is, we show $\Gamma \nvdash A$ implies $\Gamma \not\models A$. Suppose it is not the case that $\Gamma \vdash A$. Then by Lemma 3.5.5, $\Gamma \cup \{\neg A\}$ is consistent. Thus by Version II, $\Gamma \cup \{\neg A\}$ is satisfiable. Then it is not the case that $\Gamma \models A$.

We establish a few more lemmas.

Lemma 3.5.6. *If* Γ *is consistent, then for any* A, *either* $\Gamma \cup \{A\}$ *is consistent or* $\Gamma \cup \{\neg A\}$ *is consistent.*

Proof. Suppose that $\Gamma \cup \{\neg A\}$ is inconsistent. Then by the previous lemma, $\Gamma \vdash A$. Then, for any B, $\Gamma \cup \{A\} \vdash B$ if and only if $\Gamma \vdash B$. Since Γ is consistent, it follows that $\Gamma \cup \{A\}$ is also consistent. □

Definition 3.5.7. A set Δ of sentences is *maximally consistent* if it is consistent and for any sentence A, either $A \in \Delta$ or $\neg A \in \Delta$.

Lemma 3.5.8. *Let* Δ *be maximally consistent.*

1. *For any sentence* A, $\neg A \in \Delta$ *if and only if* $A \notin \Delta$.
2. *For any sentence* A, *if* $\Delta \vdash A$, *then* $A \in \Delta$.

Proof. (1) If $\neg A \in \Delta$, then $A \notin \Delta$ since Δ is consistent. If $A \notin \Delta$, then $\neg A \in \Delta$ since Δ is maximally consistent.

(2) Suppose that $\Delta \vdash A$ and suppose by way of contradiction that $A \notin \Delta$. Then by part (1), $\neg A \in \Delta$. But this contradicts the consistency of Δ. □

Proposition 3.5.9. *Let* Δ *be maximally consistent and define the function* $I : Sent \to \{0, 1\}$ *as follows. For each sentence* B,

$$I(B) = \begin{cases} 1 & \text{if } B \in \Delta; \\ 0 & \text{if } B \notin \Delta. \end{cases}$$

Then I *is a truth interpretation and* $I(B) = 1$ *for all* $B \in \Delta$.

Proof. We need to show that I preserves the four connectives: $\neg$, $\vee$, &, and $\rightarrow$. We will show the first three and leave the last an exercise.

($\neg$): It follows from the definition of I and Lemma 3.5.8 that $I(\neg A) = 1$ if and only if $\neg A \in \Delta$ if and only if $A \notin \Delta$ if and only if $I(A) = 0$.

($\vee$): Suppose that $I(A \vee B) = 1$. Then $A \vee B \in \Delta$. We argue by cases. If $A \in \Delta$, then clearly $\max\{I(A), I(B)\} = 1$. Now suppose that $A \notin \Delta$. Then by completeness, $\neg A \in \Delta$. It follows from Proposition 3.3.3(5) that $\Delta \vdash B$. Hence $B \in \Delta$ by Lemma 3.5.8. Thus $\max\{I(A), I(B)\} = 1$.

Next suppose that $\max\{I(A), I(B)\} = 1$. Without loss of generality, $I(A) = 1$ and hence $A \in \Delta$. Then $\Delta \vdash A \vee B$ by $\vee$-Introduction, so that $A \vee B \in \Delta$ by Lemma 3.5.8 and hence $I(A \vee B) = 1$.

(&): Suppose that $I(A \,\&\, B) = 1$. Then $A \,\&\, B \in \Delta$. It follows from &-Elimination that $\Delta \vdash A$ and $\Delta \vdash B$. Thus by Lemma 3.5.8, $A \in \Delta$ and $B \in \Delta$. Thus $I(A) = I(B) = 1$.

Next suppose that $I(A) = I(B) = 1$. Then $A \in \Delta$ and $B \in \Delta$. It follows from &-Introduction that $\Delta \vdash A \,\&\, B$ and hence $A \,\&\, B \in \Delta$. Therefore $I(A \,\&\, B) = 1$. □

We now prove Version II of the Completeness Theorem.

Proof of Theorem 3.5.2. Let Γ be a consistent set of propositional sentences. Let $A_0, A_1, \dots$ be an enumeration of the set of sentences. We will define a sequence $\Delta_0 \subseteq \Delta_1 \subseteq \dots$ and let $\Delta = \bigcup_n \Delta_n$. We will show that Δ is a complete and consistent extension of Γ and then define an interpretation $I = I_\Delta$ to show that Γ is satisfiable.

$\Delta_0 = \Gamma$ and, for each n,

$$\Delta_{n+1} = \begin{cases} \Delta_n \cup \{A_n\}, & \text{if } \Delta_n \cup \{A_n\} \text{ is consistent} \\ \Delta_n \cup \{\neg A_n\}, & \text{otherwise.} \end{cases}$$

It follows from the construction that, for each sentence A_n, either $A_n \in \Delta_{n+1}$ or $\neg A_n \in \Delta_{n+1}$. Hence Δ is complete. It remains to show that Δ is consistent.

Claim 1: For each n, Δ_n is consistent.

Proof of Claim 1: The proof is by induction. For the base case, we are given that $\Delta_0 = \Gamma$ is consistent. For the induction step, suppose that Δ_n is consistent. Then by Lemma 3.5.6, either $\Delta_n \cup \{A_n\}$ is consistent, or $\Delta_n \cup \{\neg A_n\}$ is consistent. In the first case, suppose that $\Delta_n \cup \{A_n\}$ is consistent. Then $\Delta_{n+1} = \Delta_n \cup \{A_n\}$ and hence Δ_{n+1} is consistent. In the second case, suppose that $\Delta_n \cup \{A_n\}$ is inconsistent. Then $\Delta_{n+1} = \Delta_n \cup \{\neg A_n\}$ and hence Δ_{n+1} is consistent by Lemma 3.5.6.

Claim 2: Δ is consistent.

Proof of Claim 2: This follows immediately from the Compactness Theorem Version III.

It now follows from Proposition 3.5.9 that there is a truth interpretation I such that $I(\delta) = 1$ for all $\delta \in \Delta$. Since $\Gamma \subseteq \Delta$, this proves that Γ is satisfiable. □

We note the following consequence of the proof of the Completeness Theorem.

Theorem 3.5.10. *Any consistent theory Γ has a maximally consistent extension.* □

3.6 Completeness, Consistency and Independence

For a given set of sentences Γ, we sometimes identify Γ with the theory $Th(\Gamma) = \{B : \Gamma \vdash B\}$. Thus we can alternatively define Γ to be consistent if there is no sentence B such that $\Gamma \vdash B$ and $\Gamma \vdash \neg B$. Moreover, let us say that Γ is complete if for every sentence B, either $\Gamma \vdash B$ or $\Gamma \vdash \neg B$ (Note that if Γ is maximally consistent, it follows that Γ is complete, but the converse need not hold. We say that a consistent set Γ is *independent* if Γ has no proper subset Δ such that $Th(\Delta) = Th(\Gamma)$; this means that Γ is minimal among the sets Δ with $Th(\Delta) = Th(\Gamma)$.

For example, in the language $\mathscr{L}$ with three propositional variables A, B, C, the set $\{A, B, C\}$ is clearly independent and complete.

Lemma 3.6.1. *Γ is independent if and only if, for every $B \in \Gamma$, it is not the case that $\Gamma \setminus \{B\} \vdash B$.*

Proof. Left to the reader. □

Lemma 3.6.2.

A set Γ of sentences is complete and consistent if and only if there is a unique interpretation I satisfied by Γ.

Proof. Left to the reader. □

We conclude this chapter with several examples.

Example 3.6.3. Let $\mathscr{L} = \{A_0, A_1, \dots\}$.

1. The set $\Gamma_0 = \{A_0, A_0 \,\&\, A_1, A_1 \,\&\, A_2, \ldots\}$ is complete but not independent.
   - It is complete since $\Gamma_0 \vdash A_n$ for all n, which determines the unique truth interpretation I where $I(A_n) = 1$ for all n.
   - It is not independent since, for each n, $(A_0 \,\&\, A_1 \cdots \,\&\, A_{n+1}) \rightarrow (A_0 \,\&\, \cdots \,\&\, A_n)$.

2. The set $\Gamma_1 = \{A_0, A_0 \rightarrow A_1, A_1 \rightarrow A_2, \ldots\}$ is complete and independent.
   - It is complete since $\Gamma_0 \vdash A_n$ for all n, which determines the unique truth interpretation I where $I(A_n) = 1$ for all n.
   - To show that Γ_1 is independent, it suffice to show that, for each single formula $A_n \rightarrow A_{n+1}$, it is not the case that $\Gamma_1 \setminus \{A_n \rightarrow A_{n+1}\} \vdash (A_n \rightarrow A_{n+1})$. This is witnessed by the interpetation I where $I(A_j) = 1$ if $j \leq n$ and $I(A_j) = 0$ if $j > n$.

3. The set $\Gamma_2 = \{A_0 \vee A_1, A_2 \vee A_3, A_4 \vee A_5, \ldots\}$ is independent but not complete.
   - It is not complete since there are many different interpretations satisfied by Γ_2. In particular, one interpretation could make A_n true if and only if n is odd, and another could make A_n true if and only if n is even.
   - It is independent since, for each n, we can satisfy every sentence of Γ_2 *except* $A_{2n} \vee A_{2n+1}$ by the interpretation I where $I(A_j) = 0$ exactly when $j = 2n$ or $j = 2n+1$.

Exercises

1. Prove that $(A \rightarrow B)$ and $((\neg A) \vee B)$ are logically equivalent.

2. Construct a proof that $\{P, (\neg P)\} \vdash Q$.

3. Construct a proof that $((\neg A) \,\&\, (\neg B)) \vdash (\neg(A \vee B))$.

4. Prove: $\vdash ((\neg(A \vee (\neg A))) \rightarrow B)$.

5. Prove: $\vdash (A \,\&\, (\neg A)) \rightarrow \neg(A \vee (\neg A))$.

6. Show that the Lindenbaum Algebra satisfies the DeMorgan Laws.

7. Every function $TF : \{0,1,2,3\} \rightarrow \{0,1\}$ can be represented by means of an expression φ in $\neg$ and and to propositional variables A, B. Produce φ so that TF_φ has the values listed in the table below.

	A	B	TF_φ
I_0	0	0	0
I_1	1	0	1
I_2	0	1	1
I_3	1	1	0

8. Prove that for any Boolean algebra $\mathfrak{B} = (\mathscr{B}, \vee, \wedge, 0, 1)$ and any element g of $\mathscr{B}$, the set $F = \left\{ b \in \mathscr{B} \mid b \wedge g = g \right\}$ is a filter.

9. Investigate the following sets of formulas for satisfiability. For those that are satisfiable, give an interpretation which makes them all true. For those that are not satisfiable, show that a contradiction of the form $P \,\&\, (\neg P)$ can be derived from the set, by giving a proof.

(a)
$A \to \neg(B \,\&\, C)$
$(D \vee E) \to G$
$G \to \neg(H \vee I)$
$\neg C \,\&\, E \,\&\, H$

(b)
$(A \vee B) \to (C \,\&\, D)$
$(D \vee E) \to G$
$A \vee \neg G$

(c)
$(A \to B) \,\&\, (C \to D)$
$(B \to D) \,\&\, (\neg C \to A)$
$(E \to G) \,\&\, (G \to \neg D)$
$\neg E \to E$

(d)
$((A \to B) \,\&\, C) \,\&\, ((D \to B) \,\&\, E)$
$((G \to \neg A) \,\&\, H) \to I$
$\neg(\neg C \to E)$

(e) The contract is fulfilled if and only if the house is completed in February. If the house is completed in February, then we can move in March 1. If we can't move in March 1, then we must pay rent for March. If the contract is not fulfilled, then we must pay rent for March. We will not pay rent for March. (Use C, H, M, R for the various atomic propositions.)

Chapter 4

Predicate Logic

Propositional logic treats a basic part of the language of mathematics, building more complicated sentences from simple with connectives. However it is inadequate as it stands to express the richness of mathematics. Consider the axiom of the theory of **Plane Geometry, PG**, which expresses the fact that any two points belong to a line. We wrote that statement formally with two one-place predicates, *Pt* for points and *Ln* for lines, and one two-place predicate, *In* for incidence as follows:

$$(\forall P, Q \in Pt)(\exists \ell \in Ln)((P In \ell) \,\&\, (Q In \ell)).$$

This axiom includes predicates and quantifies certain elements. In order to test the truth of it, one needs to know how to interpret the predicates *Pt*, *Ln* and *In*, and the individual elements P, Q, ℓ. Notice that these elements are "quantified" by the quantifiers to "for every" and "there is ... such that." Predicate logic is an enrichment of propositional logic to include predicates, individuals and quantifiers, and is widely accepted as the standard language of mathematics.

4.1 The Language of Predicate Logic

The symbols of the language of the predicate logic are

1. logical connectives, $\neg$, $\vee$, $\&$, $\rightarrow$, $\leftrightarrow$;
2. the equality symbol $=$;
3. predicate letters P_i for each natural number i;
4. function symbols F_j for each natural number j;
5. constant symbols c_k for each natural number k;
6. individual variables v_ℓ for each natural number ℓ;
7. quantifier symbols $\exists$ (the *existential* quantifier) and $\forall$ (the *universal* quantifer); and
8. punctuation symbols (,).

A predicate letter is intended to represent a relation. Thus each predicate letter P is n-ary for some n, which means that we write $P(v_1, \dots, v_n)$. Similarly, a function symbol also is n-ary for some n.

We make a few remarks on the quantifiers:

(a) $(\exists x)\phi$ is read "there exists an x such that ϕ holds."

(b) $(\forall x)\phi$ is read "for all x, ϕ holds."

(c) $(\forall x)\,\theta$ may be thought of as an abbreviation for $(\neg(\exists x)(\neg\theta))$.

Definition 4.1.1. A countable first-order language is obtained by specifying a subset of the predicate letters, function symbols and constants.

One can also work with uncountable first-order languages, but aside from a few examples in Chapter 4, we will primarily work with countable first-order languages. An example of a first-order language is the language of arithmetic.

Example 4.1.2. The *language of arithmetic* is specified by $\{<, +, \times, 0, 1\}$. Here $<$ is a 2-place relation, $+$ and $\times$ are 2-place functions and 0, 1 are constants. Equality is a special 2-place relation that we will include in every language.

We now describe how first-order sentences are built up from a given language $\mathscr{L}$.

Definition 4.1.3. The set of *terms* in a language $\mathscr{L}$, denoted *Term*($\mathscr{L}$), is recursively defined by

1. each variable and constant is a term; and
2. if $t_1, \ldots, t_n$ are terms and F is an n-place function symbol, then $F(t_1, \ldots, t_n)$ is a term.

A *constant term* is a term with no variables.

Definition 4.1.4. Let $\mathscr{L}$ be a first-order language. The collection of $\mathscr{L}$-formulas is defined by recursion. First, the set of atomic formulas, denoted *Atom*($\mathscr{L}$), consists of formulas of one of the following forms:

1. $P(t_1, \ldots, t_n)$ where P is an n-place predicate letter and $t_1, \ldots, t_n$ are terms; and
2. $t_1 = t_2$ where t_1 and t_2 are terms.

The set of $\mathscr{L}$-formulas is closed under the following rules

3. If ϕ and θ are $\mathscr{L}$-formulas, then $(\phi \vee \theta)$ is an $\mathscr{L}$-formula. (Similarly, $(\phi \,\&\, \theta)$, $(\phi \to \theta)$, $(\phi \leftrightarrow \theta)$, are $\mathscr{L}$-formulas.)
4. If ϕ is an $\mathscr{L}$-formula, then $(\neg\phi)$ is an $\mathscr{L}$-formula.
5. If ϕ is an $\mathscr{L}$-formula, then $(\exists v)\phi$ is an $\mathscr{L}$-formula (as is $(\forall v)\phi$).

An example of an atomic formula in the language of arithmetic

$$0 + x = 0.$$

An example of a more complicated formula in the language, of plane geometry is the statement that every element either has a point incident with it or is incident with some line.

$$(\forall v)(\exists x)((xInv) \vee (vInx)).$$

A variable v that occurs in a formula ϕ becomes *bound* when it is placed in the scope of a quantifier, that is, $(\exists v)$ is placed in front of ϕ, and otherwise v is *free*. The concept of being *free* over-rides the concept of being *bound* in the sense that if a formula has both free and bound occurrences of a variable v, then v occurs free in that formula. The formal definition of bound and free variables is given by recursion.

Definition 4.1.5. A variable v is *free* in a formula ϕ if

1. ϕ is atomic;
2. ϕ is $(\psi \vee \theta)$ and v is free in whichever one of ψ and θ in which it appears;
3. ϕ is $(\neg\psi)$ and v is free in ψ;

4. ϕ is $(\exists y)\psi$, v is free in ψ and y is not v.

Example 4.1.6.

1. In the atomic formula $x+5=12$, the variable x is free.
2. In the formula $(\exists x)(x+5=12)$, the variable x is bound.
3. In the formula $(\exists x)[(x \in \Re^+) \,\&\, (|x-5|=10)]$, the variable x is bound.

We will refer to an $\mathscr{L}$-formula with no free variables as an *$\mathscr{L}$-sentence.*

4.2 Models and Interpretations

In propositional logic, we used truth tables and interpretations to consider the possible truth of complex statements in terms of their simplest components. In predicate logic, to consider the possible truth of complex statements that involve quantified variables, we need to introduce models with universes from which we can select the possible values for the variables.

Definition 4.2.1. Suppose that $\mathscr{L}$ is a first-order language with

(i) predicate symbols $P_1, P_2, \dots,$

(ii) function symbols $F_1, F_2, \dots,$ and

(iii) constant symbols $c_1, c_2, \dots.$

Then an *$\mathscr{L}$-structure* $\mathfrak{A}$ consists of

(a) a nonempty set A (called the *domain* or *universe* of $\mathfrak{A}$),

(b) a relation $P_i^{\mathfrak{A}}$ on A corresponding to each predicate symbol P_i,

(c) a function $F_i^{\mathfrak{A}}$ on A corresponding to each function symbol F_i, and

(d) a element $c_i^{\mathfrak{A}} \in A$ corresponding to each constant symbol c_i.

Each relation $P_i^{\mathfrak{A}}$ requires the same number of places as P_i, so that $P_i^{\mathfrak{A}}$ is a subset of A^r for some fixed r (called the *arity* of R_i.) In addition, each function $F_i^{\mathfrak{A}}$ requires the same number of places as F_i, so that $F_i^{\mathfrak{A}} : A^r \to A$ for some fixed r (called the *arity* of R_j).

Definition 4.2.2. Given a $\mathscr{L}$-structure $\mathfrak{A}$, an *interpretation I into* $\mathfrak{A}$ is a function I from the variables and constants of $\mathscr{L}$ into the universe A of $\mathfrak{A}$ that respects the interpretations of the symbols in $\mathscr{L}$. In particular, we have

(i) for each constant symbol c_j, $I(c_j) = c_j^{\mathfrak{A}}$,

(ii) for each function symbol F_i, if F_i has parity n and $t_1, \dots, t_n$ are terms such that $I(t_1), I(t_2), \dots I(t_n)$ have been defined, then

$$I(F_i(t_1, \dots, t_n)) = F_i^{\mathfrak{A}}(I(t_1), \dots, I(t_n)).$$

For any interpretation I and any variable or constant x and for any element b of the universe, let $I_{b/x}$ be the interpretation defined by

$$I_{b/x}(z) = \begin{cases} b & \text{if } z = x, \\ I(z) & \text{otherwise.} \end{cases}$$

Definition 4.2.3. We define by recursion the relation that a structure $\mathfrak{A}$ *satisfies* a formula ϕ via an interpretation I into $\mathfrak{A}$, denoted n $\mathfrak{A} \models_I \phi$:

For atomic formulas, we have:

1. $\mathfrak{A} \models_I t = s$ if and only if $I(t) = I(s)$;
2. $\mathfrak{A} \models_I P_i(t_1, \ldots, t_n)$ if and only if $P_i^{\mathfrak{A}}(I(t_1), \ldots, I(t_n))$.

For formulas built up by the logical connectives we have:

3. $\mathfrak{A} \models_I (\phi \vee \theta)$ if and only if $\mathfrak{A} \models_I \phi$ or $\mathfrak{A} \models_I \theta$;
4. $\mathfrak{A} \models_I (\phi \;\&\; \theta)$ if and only if $\mathfrak{A} \models_I \phi$ and $\mathfrak{A} \models_I \theta$;
5. $\mathfrak{A} \models_I (\phi \rightarrow \theta)$ if and only if $\mathfrak{A} \not\models_I \phi$ or $\mathfrak{A} \models_I \theta$;
6. $\mathfrak{A} \models_I (\neg\phi)$ if and only if $\mathfrak{A} \not\models_I \phi$.

For formulas built up with quantifiers:

7. $\mathfrak{A} \models_I (\exists v)\phi$ if and only if there is an a in A such that $\mathfrak{A} \models_{I_{a/x}} \phi$;
8. $\mathfrak{A} \models_I (\forall v)\phi$ if and only if for every a in A, $\mathfrak{A} \models_{I_{a/x}} \phi$.

If $\mathfrak{A} \models_I \phi$ for every interpretation I, we will suppress the subscript I, and simply write $\mathfrak{A} \models \phi$. In this case we say that $\mathfrak{A}$ is a *model* of ϕ.

Example 4.2.4. Let $\mathscr{L}(GT)$ be the language of group theory, which uses the symbols $\{+, 0\}$. A structure for this language is $\mathfrak{A} = (\{0, 1, 2\}, +_{(\text{mod } 3)}, 0)$. Suppose we consider formulas of $\mathscr{L}(GT)$ which only have variables among x_1, x_2, x_3, x_4. Define an interpretation I by $I(x_i) \equiv i \mod 3$ and $I(0) = 0$.

1. Claim: $\mathfrak{A} \not\models_I x_1 + x_2 = x_4$.
   We check this claim by computation. Note that $I(x_1) = 1$, $I(x_2) = 2$, $I(x_1 + x_2) = I(x_1) +_{mod3} I(x_2) = 1 +_{mod3} 2 = 0$. On the other hand, $I(x_4) = 1 \neq 0$, so $\mathfrak{A} \not\models_I x_1 + x_2 = x_4$.
2. Claim: $\mathfrak{A} \models (\exists x_2)(x_1 + x_2 = x_4)$
   Define $J = I_{0/x_2}$. As above check that $\mathfrak{A} \models_J x_1 + x_2 = x_4$. Then by the definition of the satisfaction of an existential formula, $\mathfrak{A} \models_I (\exists x_2)(x_1 + x_2 = x_4)$.

Theorem 4.2.5. *For every $\mathscr{L}$-formula ϕ, for all interpretations I, J, if I and J agree on all the variables free in ϕ, then $\mathfrak{A} \models_I \phi$ if and only if $\mathfrak{A} \models_J \phi$.*

Proof. Left to the reader. □

Corollary 4.2.6. *If ϕ is an $\mathscr{L}$-sentence, then for all interpretations I and J, we have $\mathfrak{A} \models_I \phi$ if and only if $\mathfrak{A} \models_J \phi$.*

Remark 4.2.7. *Thus for $\mathscr{L}$-sentences, we drop the subscript which indicates the interpretation of the variables, and we say simply $\mathfrak{A}$ models ϕ.*

Definition 4.2.8. Let ϕ be an $\mathscr{L}$-formula.

(i) ϕ is *logically valid* if $\mathfrak{A} \models_I \phi$ for every $\mathscr{L}$-structure $\mathfrak{A}$ and every interpretation I into $\mathfrak{A}$.

(ii) ϕ is *satisfiable* if there is some $\mathscr{L}$-structure $\mathfrak{A}$ and some interpretation I into $\mathfrak{A}$ such that $\mathfrak{A} \models_I \phi$.

(iii) ϕ is *contradictory* if ϕ is not satisfiable.

Definition 4.2.9. A $\mathscr{L}$-theory Γ is a set of $\mathscr{L}$-sentences. An $\mathscr{L}$-structure $\mathfrak{A}$ is a *model* of an $\mathscr{L}$-theory Γ if and only if $\mathfrak{A} \models \phi$ for all ϕ in Γ. In this case we also say that Γ is *satisfiable*.

Definition 4.2.10. For a set of $\mathscr{L}$-formulas Γ and an $\mathscr{L}$-formula ϕ, we write $\Gamma \models \phi$ and say "Γ *implies* ϕ," if for all $\mathscr{L}$-structures $\mathfrak{A}$ and for all $\mathscr{L}$-interpretations I, if $\mathfrak{A} \models_I \gamma$ for all γ in Γ, then $\mathfrak{A} \models_I \phi$.

Thus if Γ is an $\mathscr{L}$-theory and ϕ an $\mathscr{L}$-sentence, then $\Gamma \models \phi$ means every model of Γ is also a model of ϕ.

The following definition will be useful to us in the next section.

Definition 4.2.11. Given a term t and an $\mathscr{L}$-formula ϕ with free variable x, we write $\phi[t/x]$ to indicate the result of substituting the term t for each free occurrence of x in ϕ.

Example 4.2.12. If ϕ is the formula $(\exists y)(y \neq x)$ is the formula, then $\phi[y/x]$ is the formula $(\exists y)(y \neq y)$, which we expect never to be true.

4.3 The Deductive Calculus

The Predicate Calculus is a system of axioms and rules which permit us to derive the true statements of predicate logic without the use of interpretations. The basic relation in the Predicate Calculus is the relation *proves* between a set Γ of $\mathscr{L}$ formulas and an $\mathscr{L}$-formula ϕ, which formalizes the concept that Γ proves ϕ. This relation is denoted $\Gamma \vdash \phi$. As a first step in defining this relation, we give a list of additional rules of deduction, which extend the list we gave for propositional logic.

Some of our rules of the predicate calculus require that we exercise some care in how we substitute variables into certain formulas. Let us say that $\phi[t/x]$ is a *legal substitution* of t for x in ϕ if no free occurrence of x in ϕ occurs in the scope of a quantifier of any variable appearing in t. For instance, if ϕ has the form $(\forall y)\phi(x, y)$, where x is free, I cannot legally substitute y in for x, since then y would be bound by the universal quantifier.

10. (Equality rule) For any term t, the formula $t = t$ may be derived from Γ is one step.

11. (Term Substitution) For any terms $t_1, t_2, \ldots, t_n$, $s_1, s_2, \ldots, s_n$, and any function symbol F, if each of the sentences $t_1 = s_1, t_2 = s_2, \ldots, t_n = s_n$ have been derived from Γ, then we may derive $F(t_1, t_2, \ldots, t_n) = F(s_1, s_2, \ldots, s_n)$ from Γ in one additional step.

12. (Atomic Formula Substitution) For any terms $t_1, t_2, \ldots, t_n, s_1, s_2, \ldots, s_n$ and any atomic formula ϕ, if each of the sentences $t_1 = s_1, t_2 = s_2, \ldots, t_n = s_n$, and $\phi(t_1, t_2, \ldots, t_n)$, have been derived from Γ, then we may derive $\phi(s_1, s_2, \ldots, s_n)$ from Γ in one additional step.

13. ($\forall$-Elimination) For any term t, if $\phi[t/x]$ is a legal substitution and $(\forall x)\phi$ has been derived from Γ, then we may derive $\phi[t/x]$ from Γ in one additional step.

14. ($\exists$-Elimination) To show that $\Gamma \cup \{(\exists x)\phi(x)\} \vdash \theta$, it suffices to show $\Gamma \cup \{\phi(y)\}$, where y is a new variable that does not appear free in any formula in Γ nor in θ.

15. ($\forall$-Introduction) Suppose that y does not appear free in any formula in Γ, in any temporary assumption, nor in $(\forall x)\phi$. If $\phi[y/x]$ has been derived from Γ, then we may derive $(\forall x)\phi$ from Γ in one additional step.

16. ($\exists$-Introduction) If $\phi[t/x]$ is a legal substitution and $\phi[t/x]$ has been derived from Γ, then we may derive $(\exists x)\phi$ from Γ in one additional step.

We remark on three of the latter four rules. First, the reason for the restriction on substitution in $\forall$-Elimination is that we need to ensure that t does not contain any free variable that would be become bound when we substitute t for x in ϕ. For example, consider the formula $(\forall x)(\exists y)x < y$ in the language of arithmetic. Let ϕ be the formula $(\exists y)x < y$, in which x is free but y is bound. Observe that if we substitute the term y for x in ϕ, the resulting formula is $(\exists y)y < y$. Thus, from $(\forall x)(\exists y)x < y$ we can derive, for instance, $(\exists y)x < y$ or $(\exists y)c < y$, but we cannot derive $(\exists y)y < y$.

Second, the idea behind $\exists$-Elimination is this: Suppose in the course of my proof I have derived $(\exists x)\phi(x)$. Informally, I would like to use the fact that ϕ holds of some x, but to do so, I need to refer to this object. So I pick an unused variable, say a, and use this as a temporary name to stand for the object satisfying ϕ. Thus, I can write down $\phi(a)$. Eventually in my proof, I will discard this temporary name (usually by $\exists$-Introduction).

Third, in $\forall$-Introduction, if we think of the variable y as an arbitrary object, then when we show that y satisfies ϕ, we can conclude that ϕ holds of *every* object. However, if y is free in a premise in Γ or a temporary assumption, it is not arbitrary. For example, suppose we begin with the statement $(\exists x)(\forall z)(x + z = z)$ in the language of arithmetic and suppose we derive $(\forall z)(y + z = z)$ by $\exists$-Elimination (where y is a temporary name). We are *not* allowed to apply $\forall$-Introduction here, for otherwise we could conclude $(\forall x)(\forall z)(x + z = z)$, an undesirable conclusion.

Definition 4.3.1. The relation $\Gamma \vdash \phi$ is the smallest subset of pairs (Γ, ϕ) from $\mathscr{P}(Sent) \times Sent$ that contains every pair (Γ, ϕ) such that $\phi \in \Gamma$ or ϕ is $t = t$ for some term t, and which is closed under the 15 rules of deduction.

As in Propositional Calculus, to demonstrate that $\Gamma \vdash \phi$, we construct a proof. The next proposition exhibits several proofs using the new axiom and rules of predicate logic.

Proposition 4.3.2.

1. $(\exists x)(x = x)$.
2. $(\forall x)(\forall y)[x = y \rightarrow y = x]$.
3. $(\forall x)(\forall y)(\forall z)[(x = y \;\;\&\;\; y = z) \rightarrow x = z]$.
4. $((\forall x)\theta(x)) \rightarrow (\exists x)\theta(x)$.
5. $((\exists x)(\forall y)\theta(x, y)) \rightarrow (\forall y)(\exists x)\theta(x, y)$.
6. *(i)* $(\exists x)[\phi(x) \vee \psi(x)] \vdash (\exists x)\phi(x) \vee (\exists x)\psi(x)$

   (ii) $(\exists x)\phi(x) \vee (\exists x)\psi(x) \vdash (\exists x)[\phi(x) \vee \psi(x)]$
7. *(i)* $(\forall x)[\phi(x) \,\&\, \psi(x)] \vdash (\forall x)\phi(x) \,\&\, (\forall x)\psi(x)$

   (ii) $(\forall x)\phi(x) \,\&\, (\forall x)\psi(x) \vdash (\forall x)[\phi(x) \vee \psi(x)]$
8. $(\exists x)[\phi(x) \,\&\, \psi(x)] \vdash (\exists x)\phi(x) \,\&\, (\exists x)\psi(x)$
9. $(\forall x)\phi(x) \vee (\forall x)\psi(x) \vdash (\forall x)[\phi(x) \vee \psi(x)]$
10. $(\forall x)[\phi(x) \rightarrow \psi(f(x))] \rightarrow [(\exists x)\phi(x) \;\rightarrow\; (\exists x)\psi(x)]$.

Proof. 1. $(\exists x)(x = x)$

1	$x = x$	equality rule
2	$(\exists x)\, x = x$	$\exists$-Introduction 1

2. $(\forall x)(\forall y)[x = y \rightarrow y = x]$.

1	$\quad x = y$	temporary assumption
2	$\quad x = x$	equality rule
3	$\quad y = x$	term substitution 1,2
4	$x = y \rightarrow y = x$	$\rightarrow$-Introduction 1-3
5	$(\forall y)(x = y \rightarrow y = x)$	$\forall$-Introduction 4
6	$(\forall x)(\forall y)(x = y \rightarrow y = x)$	$\forall$-Introduction 5

5. $(\exists x)(\forall y)\theta(x,y) \vdash (\forall y)(\exists x)\theta(x,y)$.

1	$(\exists x)(\forall y)\theta(x,y)$	given rule
2	$(\forall y)\theta(a,y)$	∃-Elimination 1
3	$\theta(a,y)$	∀-Elimination 2
4	$(\exists x)\,\theta(x,y)$	∃-Introduction 3
5	$(\forall y)(\exists x)\,\theta(x,y)$	∀-Introduction 4

8. $(\exists x)[\phi(x) \,\&\, \psi(x)] \vdash (\exists x)\phi(x) \,\&\, (\exists x)\psi(x)$

1	$(\exists x)[\phi(x) \,\&\, \psi(x)]$	given rule
2	$\phi(a) \,\&\, \psi(a)$	∃-Elimination 1
3	$\phi(a)$	&-Elimination 2
4	$(\exists x)\phi(x)$	∃-Introduction 3
5	$\psi(a)$	&-Elimination 2
6	$(\exists x)\psi(x)$	∃-Introduction 5
7	$(\exists x)\phi(x) \,\&\, (\exists x)\psi(x)$	&-Introduction 4,6

9. $(\forall x)\,\phi(x) \vee (\forall x)\,\psi(x) \vdash (\forall x)[\phi(x) \vee \psi(x)]$

1	$(\forall x)\,\phi(x) \vee (\forall x)\,\psi(x)$	given rule
2	$\mid\ (\forall x)\,\phi(x)$	temporary assumption
3	$\mid\ \phi(x)$	∀-Elimination 2
4	$\mid\ \phi(x) \vee \psi(x)$	∨-Introduction 3
5	$\mid\ (\forall x)[\phi(x) \vee \psi(x)]$	∀-Introduction 4
6	$\mid\ (\forall x)\,\psi(x)$	temporary assumption
7	$\mid\ \psi(x)$	∀-Elimination 6
8	$\mid\ \phi(x) \vee \psi(x)$	∨-Introduction 7
9	$\mid\ (\forall x)[\phi(x) \vee \psi(x)]$	∀-Introduction 8
10	$(\forall x)[\phi(x) \vee \psi(x)]$	∨-Elimination 1-9

10. $(\forall x)[\phi(x) \to \psi(f(x))] \vdash (\exists x)\phi(x) \ \to\ (\exists x)\psi(x)$.

1	$(\forall x)[\phi(x) \to \psi(f(x))]$	given rule
2	$\quad$ | $(\exists x)\phi(x)$	temporary assumption
3	$\quad$ | $\phi(a)$	$\exists$-Elimination 2
4	$\quad$ | $\phi(a) \to \psi(f(a))$	$\forall$-Elimination 1
5	$\quad$ | $\psi(f(a))$	$\to$-Elimination 3,4
6	$\quad$ | $(\exists x)\psi(x)$	$\exists$-Introduction 5
7	$(\exists x)\phi(x) \ \to\ (\exists x)\psi(x)$	$\to$-Introduction 2-6

□

4.4 Soundness Theorem for Predicate Logic

Our next goal is to proof the soundness theorem for predicate logic. First we will prove a lemma, which connects satisfaction of formulas with substituted variables to satisfaction with slightly modified interpretations of the original formulas.

Lemma 4.4.1. *For every $\mathscr{L}$-formula ϕ, every variable x, every term t, every structure $\mathfrak{B}$ and every interpretation I in $\mathfrak{B}$, if no free occurrence of x occurs in the scope of a quantifier over any variable appearing in t, then*

$$\mathfrak{B} \models_I \phi[t/x] \text{ if and only if } \mathfrak{B} \models_{I_{b/x}} \phi$$

where $b = I(t)$.

Proof. Let $\mathfrak{B} = (B, R_1, \ldots, f_1, \ldots, b_1, \ldots)$ be an $\mathscr{L}$-structure, and let x, t, and I be as above. We claim that for any term r, if $b = I(t)$, then $I(r[t/x]) = I_{b/x}(r)$. We prove this claim by induction on the term r.

- If $r = a$ is a constant, then $r[t/x] = a$ so that $I(r[t/x]) = a^{\mathscr{B}} = I_{b/x}(r)$.
- If r is a variable $y \neq x$, then $r[t/x] = y$ and $I_{b/x}(y) = I(y)$, so that $I(r[t/x]) = I(y) = I_{b/x}(r)$.
- If $r = x$, then $r[t/x] = t$ and $I_{b/x}(x) = b$, so that $I(r[t/x]) = I(t) = b = I_{b/x}(r)$.
- Now assume the claim holds for terms $r_1, \ldots, r_n$ and let $r = f(r_1, \ldots, r_n)$ for some function symbol f. Then by induction $I(r_j[t/x]) = I_{b/x}(r_j)$ for $j = 1, 2, \ldots, n$. Then

  $$r[t/x] = f(r_1[t/x], \ldots, r_n[t/x])$$

  so

  $$\begin{aligned} I_{b/x}(r) &= f^{\mathscr{B}}(I_{b/x}(r_1), \ldots, I_{b/x}(r_n)) \\ &= f^{\mathscr{B}}(I(r_1[t/x]), \ldots, I(r_n[t/x])) \\ &= I(f(r_1[t/x], \ldots, r_n[t/x]) \\ &= I(r[t/x]). \end{aligned}$$

To prove the lemma, we proceed by induction on formulas.

- For an atomic formula ϕ of the form $s_1 = s_2$, we have

$$\begin{aligned}\mathscr{B} \models_I \phi[t/x] &\Leftrightarrow \mathscr{B} \models_I s_1[t/x] = s_2[t/x]\\ &\Leftrightarrow I(s_1[t/x]) = I(s_2[t/x])\\ &\Leftrightarrow I_{b/x}(s_1) = I_{b/x}(s_2) \text{ (by the claim)}\\ &\Leftrightarrow \mathscr{B} \models_{I_{b/x}} s_1 = s_2\\ &\Leftrightarrow \mathscr{B} \models_{I_{b/x}} \phi.\end{aligned}$$

- For an atomic formula ϕ of the form $P(r_1, \ldots, r_n)$, so that $\phi[t/x]$ is $P(r_1[t/x], \ldots, r_n[t/x])$, we have

$$\begin{aligned}\mathscr{B} \models_I \phi[t/x] &\Leftrightarrow \mathscr{B} \models_I P(r_1[t/x], \ldots, r_n[t/x])\\ &\Leftrightarrow P^{\mathscr{B}}(I(r_1[t/x]), \ldots, I(r_n[t/x]))\\ &\Leftrightarrow P^{\mathscr{B}}(I_{b/x}(r_1), \ldots, I_{b/x}(r_n)) \text{ (by the claim)}\\ &\Leftrightarrow \mathscr{B} \models_{I_{b/x}} P(r_1, \ldots, r_n)\\ &\Leftrightarrow \mathscr{B} \models_{I_{b/x}} \phi.\end{aligned}$$

- The inductive step for $\mathscr{L}$-formulas is straightforward except for formulas of the form $\forall y \phi$: Let ψ be $\forall y \phi$, where the Lemma holds for the formula ϕ. Then

$$\begin{aligned}\mathscr{B} \models_I \psi[t/x] &\Leftrightarrow \mathscr{B} \models_I \forall y \phi[t/x]\\ &\Leftrightarrow \mathscr{B} \models_{I_{a/y}} \phi[t/x] \text{ (for each } a \in B)\\ &\Leftrightarrow \mathscr{B} \models_{(I_{a/y})_{b/x}} \phi \text{ (by the inductive hypothesis)}\\ &\Leftrightarrow \mathscr{B} \models_{(I_{b/x})_{a/y}} \phi \text{ (for each } a \in B)\\ &\Leftrightarrow \mathscr{B} \models_{I_{b/x}} \forall y \phi\\ &\Leftrightarrow \mathscr{B} \models_{I_{b/x}} \psi.\end{aligned}$$

□

Theorem 4.4.2 (Soundness Theorem of Predicate Logic)**.** *If* $\Gamma \vdash \phi$*, then* $\Gamma \models \phi$.

Proof. As in the proof of the soundness theorem for propositional logic, the proof is again by induction on the length of the deduction of ϕ. We need to show that if there is a proof of ϕ from Γ, then for any structure $\mathscr{A}$ and any interpretation I into $\mathscr{A}$, if $\mathscr{A} \models_I \gamma$ for all $\gamma \in \Gamma$, then $\mathscr{A} \models_I \phi$. The arguments for the rules from Propositional Logic carry over here, so we just need to verify the result holds for the new rules.

Suppose the result holds for all formulas obtained in proofs of length strictly less than n lines.

- (Equality rule) Suppose the last line of a proof of length n with premises Γ is $t = t$ for some term t. Suppose $\mathscr{A} \models_I \Gamma$. Then since $I(t) = I(t)$, we have $\mathscr{A} \models t = t$.

- (Term substitution) Suppose the last line of a proof of length n with premises Γ is $F(s_1, \ldots, s_n) = F(t_1, \ldots, t_n)$, obtained by term substitution. Then we must have established $s_1 = t_1, \ldots, s_n = t_n$ earlier in the proof. By the inductive hypothesis, we must have $\Gamma \models s_1 = t_1, \ldots \Gamma \models s_n = t_n$. Suppose that $\mathscr{A} \models_I \gamma$ for every $\gamma \in \Gamma$. Then $I(s_i) = I(t_i)$ for $i = 1, \ldots n$. So

$$\begin{aligned}I(F(s_1, \ldots, s_n)) &= F^{\mathscr{A}}(I(s_1), \ldots, I(s_n))\\ &= F^{\mathscr{A}}(I(t_1), \ldots, I(t_n))\\ I(F(I(s_1), \ldots, I(s_n))\end{aligned}$$

Hence $\mathscr{A} \models_I F(s_1, \ldots, s_n) = F(t_1, \ldots, t_n)$.

- (Atomic formula substitution) The argument is similar to the previous one and is left to the reader.

- ($\forall$-Elimination) *For any term t, if $\phi[t/x]$ is a legal substitution and $(\forall x)\phi$ has been derived from Γ, then we may derive $\phi[t/x]$ from Γ in one additional step.*

  Suppose that the last line of a proof of length n with premises Γ is $\phi[t/x]$, obtained by $\forall$-Elimination. Thus, we must have derived $\forall x\phi(x)$ earlier in the proof. Let $\mathscr{A} \models_I \gamma$ for every $\gamma \in \Gamma$. Then by the inductive hypothesis, we have $\Gamma \models \forall x\phi(x)$, which implies that $\mathscr{A} \models_{I_{a/x}} \phi(x)$ for every $a \in A$. If $I(t) = b$, then since $\mathscr{A} \models_{I_{b/x}} \phi(x)$, by Lemma 4.4.1 we have $\mathscr{A} \models_I \phi[t/x]$. Since $\mathscr{A}$ and I were arbitrary, we can conclude that $\Gamma \models \phi[t/x]$.

- ($\exists$-Elimination) *To show that $\Gamma \cup \{(\exists x)\phi(x)\} \vdash \theta$, it suffices to show $\Gamma \cup \{\phi[y/x]\} \vdash \theta$, where y is a new variable that does not appear free in any formula in Γ nor in θ.*

  Suppose that the last line of a proof of length n with premises Γ is given by $\exists$-Elimination. Then $\Gamma \vdash (\exists x)\phi(x)$ in less than n lines and $\Gamma \cup \{\phi[y/x]\} \vdash \theta$ in less than n lines. Let $\mathscr{A} \models_I \gamma$ for every $\gamma \in \Gamma$. Then by the inductive hypothesis, we have $\Gamma \models (\exists x)\phi(x)$, which implies that $\mathscr{A} \models_{I_{b/x}} \phi(x)$ for some $b \in A$. Let $J = I_{b/y}$, so that $J(y) = b$. It follows that $\mathscr{A} \models_{J_{b/x}} \phi(x)$, since $I = J$ except on possibly y and y does not appear free in ϕ. Then by Lemma 4.4.1, $\mathscr{A} \models_J \phi[y/x]$, and hence $\mathscr{A} \models_J \theta$. It follows that $\mathscr{A} \models_I \theta$, since $I = J$ except on possibly y and y does not appear free in θ. Since $\mathscr{A}$ and I were arbitrary, we can conclude that $\Gamma \cup (\exists x)\phi(x) \models \theta$.

- ($\forall$-Introduction) *Suppose that y does not appear free in any formula in Γ, in any temporary assumption, nor in $(\forall x)\phi$. If $\phi[y/x]$ has been derived from Γ, then we may derive $(\forall x)\phi$ from Γ in one additional step.*

  Suppose that the last line of a proof of length n with premises Γ is $(\forall x)\phi(x)$, obtained by $\forall$-Introduction. Thus, we must have derived $\phi[y/x]$ from Γ earlier in the proof (where y satisfies the necessary conditions described above). Let $\mathscr{A} \models_I \gamma$ for every $\gamma \in \Gamma$. Since y does not appear free in Γ, then for any $a \in A$, $\mathscr{A} \models_{I_{a/y}} \Gamma$. For an arbitrary $a \in A$, let $J = I_{a/y}$, so that $J(y) = a$. By the inductive hypothesis, we have $\Gamma \models \phi[y/x]$, which implies that $\mathscr{A} \models_J \phi[y/x]$. Then by Lemma 4.4.1, $\mathscr{A} \models_{J_{a/x}} \phi$. Since $I_{a/x} = J_{a/x}$ except on possibly y, which does not appear free in ϕ, we have $\mathscr{A} \models_{I_{a/x}} \phi$. As a was arbitrary, we have shown $\mathscr{A} \models_{I_{a/x}} \phi$ for every $a \in A$. Hence $\mathscr{A} \models_I (\forall x)\phi(x)$. Since $\mathscr{A}$ and I were arbitrary, we can conclude that $\Gamma \models (\forall x)\phi(x)$.

- ($\exists$-Introduction) *If $\phi[t/x]$ is a legal substitution and $\phi[t/x]$ has been derived from Γ, then we may derive $(\exists x)\phi$ from Γ in one additional step.*

  Suppose that the last line of a proof of length n with premises Γ is $(\exists x)\phi(x)$, obtained by $\exists$-Introduction. Thus, we must have derived $\phi[t/x]$ from Γ earlier in the proof, where t is some term. Let $\mathscr{A} \models_I \gamma$ for every $\gamma \in \Gamma$. Then $I(t) = a$ for some $a \in A$. Since $\Gamma \vdash \phi[t/x]$ in less than n lines, by the inductive hypothesis, it follows that $\mathscr{A} \models_I \phi[t/x]$. Then by Lemma 4.4.1, $\mathscr{A} \models_{I_{a/x}} \phi$, which implies that $\mathscr{A} \models_I (\exists x)\phi(x)$. Since $\mathscr{A}$ and I were arbitrary, we can conclude that $\Gamma \models (\exists x)\phi(x)$.

□

Chapter 5

Models for Predicate Logic

5.1 Models

In this chapter, we will prove the completeness theorem for predicate logic by showing how to build a model for a consistent first-order theory. We will also discuss several consequences of the compactness theorem for first-order logic and consider several relations that hold between various models of a given first-order theory, namely isomorphism and elementary equivalence.

5.2 The Completeness Theorem for Predicate Logic

Fix a first-order theory $\mathscr{L}$. For convenience, we will assume that our $\mathscr{L}$-formulas are built up only using $\neg, \vee$, and $\exists$. We will also make use of the following key facts (the proofs of which we will omit):

1. If A is a tautology in propositional logic, then if we replace each instance of each propositional variable in A with an $\mathscr{L}$-formula, the resulting $\mathscr{L}$-formula is true in all $\mathscr{L}$-structures.
2. For any $\mathscr{L}$-structure $\mathscr{A}$ and any interpretation I into $\mathscr{A}$,

$$\mathscr{A} \models_I (\forall x)\phi \Leftrightarrow \mathscr{A} \models_I \neg(\exists x)(\neg\phi).$$

We will also use the following analogues of results we proved in Chapter 2, the proofs of which are the same:

Lemma 5.2.1. *Let Γ be an $\mathscr{L}$-theory.*

1. *If Γ is not consistent, then $\Gamma \vdash \phi$ for every $\mathscr{L}$-sentence ϕ.*
2. *For an $\mathscr{L}$-sentence ϕ, $\Gamma \vdash \phi$ if and only if $\Gamma \cup \{\neg\phi\}$ is inconsistent.*
3. *If Γ is consistent, then for any $\mathscr{L}$-sentence ϕ, either $\Gamma \cup \{\phi\}$ is consistent or $\Gamma \cup \{\neg\phi\}$ is consistent.*

The following result, known as the Constants Theorem, plays an important role in the proof of the completeness theorem.

Theorem 5.2.2 (Constants Theorem). *Let Γ be an $\mathscr{L}$-theory. If $\Gamma \vdash \phi(c)$ and c does not appear in Γ, then $\Gamma \vdash (\forall x)\phi(x)$.*

Proof. Given a proof of $\phi(c)$ from Γ, let v be a variable not appearing in Γ. If we replace every instance of c with v in the proof of $\phi(c)$, we have a proof of $\phi(v)$ from Γ. Then by $\forall$-Introduction, we have $\Gamma \vdash (\forall x)\phi(x)$. □

Gödel's completeness theorem can be articulated in two ways, which we will prove are equivalent:

Theorem 5.2.3 (Completeness theorem, Version 1)**.** *For any $\mathscr{L}$-theory Γ and any $\mathscr{L}$-sentence ϕ,*

$$\Gamma \models \phi \Rightarrow \Gamma \vdash \phi.$$

Theorem 5.2.4 (Completeness theorem, Version 2)**.** *Every consistent theory has a model.*

We claim that the two versions are equivalent.

Proof of claim. First, suppose that every consistent theory has a model, and suppose further that $\Gamma \models \phi$. If Γ is not consistent, then Γ proves every sentence, and hence $\Gamma \vdash \phi$. If, however, Γ is consistent, we have two cases to consider. If $\Gamma \cup \{\neg\phi\}$ is inconsistent, then by Lemma 5.2.1(2), it follows that $\Gamma \vdash \phi$. In the case that $\Gamma \cup \{\neg\phi\}$ is consistent, by the second version of the completeness theorem, there is some $\mathscr{L}$-structure $\mathscr{A}$ such that $\mathscr{A} \models \Gamma \cup \{\neg\phi\}$, from which it follows that $\mathscr{A} \models \Gamma$ and $\mathscr{A} \models \neg\phi$. But we have assumed that $\Gamma \models \phi$, and hence $\mathscr{A} \models \phi$, which is impossible. Thus, if Γ is consistent, it follows that $\Gamma \cup \{\neg\phi\}$ is inconsistent.

For the other direction, suppose the first version of the completeness theorem holds and let Γ be an arbitrary $\mathscr{L}$-theory. Suppose Γ has no model. Then vacuously, $\Gamma \models \neg(\phi \vee \neg\phi)$, where ϕ is the sentence $(\forall x)x = x$. It follows from the first version of the completeness theorem that $\Gamma \vdash \neg(\phi \vee \neg\phi)$, and hence Γ is inconsistent. □

We now turn to the proof of the second version of the completeness theorem. As in the proof of the completeness theorem for propositional logic, we will use the compactness theorem, which comes in several forms (just as it did in with propositional logic).

Theorem 5.2.5. *Let Γ be an $\mathscr{L}$-theory.*

1. *For an $\mathscr{L}$-sentence ϕ, if $\Gamma \vdash \phi$, there is some finite $\Gamma_0 \subseteq \Gamma$, $\Gamma_0 \vdash \phi$.*
2. *If every finite $\Gamma_0 \subseteq \Gamma$ is consistent, then Γ is consistent.*
3. *If $\Gamma = \cup_n \Gamma_n$ is , $\Gamma_n \subseteq \Gamma_{n+1}$ for every n, and each Γ_n is consistent, then Γ is consistent.*

As in the case of propositional logic, (1) follows by induction on proof length, while (2) follows directly from (1) and (3) follows directly from (2).

Our strategy for proving the completeness theorem is as follows. Given Γ, we want to extend it to a maximally consistent collection of $\mathscr{L}$ -formulas, like the proof of the completeness theorem for propositional logic. The problem that we now encounter (that did not occur in the propositional case) is that it is unclear how to make sentences of the form $(\exists x)\theta$.

The solution to this problem, due to Henkin, is to extend the language $\mathscr{L}$ to a language $\mathscr{L}'$ by adding new constants $c_0, c_1, c_2, \ldots$, which we will use to witness the truth of existential sentences.

Hereafter, let us assume that $\mathscr{L}$ is countably infinite (which is not a necessary restriction), so that we will only need to add countably many new constants to our language. Using these constants, we will build a model of Γ, where the universe of our model consists of certain equivalence classes on the set of all $\mathscr{L}'$ -terms with no variables (the so-called *Herbrand universe* of $\mathscr{L}'$). The model will satisfy a collection $\Delta \supseteq \Gamma$ that is maximally consistent and *Henkin complete*, which means that for each $\mathscr{L}'$ -formula $\theta(v)$ with exactly one free variable v, if $(\exists v)\theta(v)$ is in Δ, then there is some constant c in our language such that $\theta(c)$ is in Δ.

Proof of Theorem 5.2.4. Let $\phi_0, \phi_1, \ldots$ be an enumeration of all $\mathscr{L}'$ -sentences. We define a sequence $\Gamma = \Delta_{-1} \subseteq \Delta_0 \subseteq \Delta_1 \subseteq \ldots$ such that for each $n \in \mathbb{N}$,

$$\Delta_{2n} = \begin{cases} \Delta_{2n-1} \cup \{\phi_n\} & \text{if } \Delta_{2n-1} \cup \{\phi_n\} \text{ is consistent,} \\ \Delta_{2n-1} \cup \{\neg\phi_n\} & \text{otherwise} \end{cases}$$

and

$$\Delta_{2n+1} = \begin{cases} \Delta_{2n} \cup \{\theta(c_m)\} & \text{if } \phi_n \text{ is of the form } (\exists v)\theta(v) \text{ and is in } \Delta_{2n}, \\ \Delta_{2n} & \text{otherwise} \end{cases}$$

where c_m is the first constant in our list of new constants that has not appeared in Δ_{2n}. Then we define $\Delta = \cup_n \Delta_n$.

We now prove a series of claims.

Claim 1: Δ is complete (that is, for every $\mathscr{L}'$ -sentence ϕ, either $\phi \in \Delta$ or $\neg\phi \in \Delta$).

Proof of Claim 1: This follows immediately from the construction.

Claim 2: Each Δ_k is consistent.

Proof of Claim 2: We prove this claim by induction. First, $\Delta_{-1} = \Gamma$ is consistent by assumption. Now suppose that Δ_k is consistent. If $k = 2n$ for some n, then clearly Δ_k is consistent, since if $\Delta_{2n-1} \cup \{\phi_n\}$ is consistent, then we set $\Delta_k = \Delta_{2n-1} \cup \{\phi_n\}$, and if not, then by Lemma 5.2.1(3), $\Delta_{2n-1} \cup \{\neg\phi_n\}$ is consistent, and so we set $\Delta_k = \Delta_{2n-1} \cup \{\neg\phi_n\}$.

If $k = 2n+1$ for some n, then if ϕ_n is not of the form $(\exists v)\theta(v)$ or if it is but it is not in Δ_{2n}, then $\Delta_{2n+1} = \Delta_{2n}$ is consistent by induction. If ϕ_n is of the form $(\exists v)\theta(v)$ and is in Δ_{2n}, then let $c = c_m$ be the first constant not appearing in Δ_{2n}. Suppose that $\Delta_k = \Delta_{2k+1} = \Delta_{2n} \cup \{\theta(c)\}$ is not consistent. Then by Lemma 5.2.1(2), $\Delta_{2n} \vdash \neg\theta(c)$. Then by the Constants Theorem, $\Delta_{2n} \vdash (\forall x)\neg\theta(x)$. But since ϕ_n is the formula $(\exists v)\theta(v)$ and is in Δ_{2n}, it follows that Δ_{2n} is inconsistent, contradicting our inductive hypothesis. Thus $\Delta_k = \Delta_{2n+1}$ is consistent.

Claim 3: $\Delta = \cup_n \Delta_n$ is consistent.

Proof of Claim 3: This follows from the third version of the compactness theorem.

Claim 4: Δ is Henkin complete (that is, for each $\mathscr{L}'$ -formula $\theta(v)$ with exactly one free variable and $(\exists v)\theta(v) \in \Delta$, then $\theta(c) \in \Delta$ for some constant c).

Proof of Claim 4: Suppose that $(\exists v)\theta(v) \in \Delta$. Then there is some n such that $(\exists v)\theta(v)$ is the formula ϕ_n. Since $\Delta_{2n-1} \cup \{\phi_n\} \subseteq \Delta$ is consistent, $(\exists v)\theta(v) \in \Delta_{2n}$. Then by construction, $\theta(c) \in \Delta_{2n+1}$ for some constant c.

Our final task is to build a model $\mathscr{A}$ such that $\mathscr{A} \models \Delta$, from which it will follow that $\mathscr{A} \models \Gamma$ (since $\Gamma \subseteq \Delta$). We define an equivalence relation on the Herbrand universe of $\mathscr{L}'$ (i.e., the set of constant $\mathscr{L}'$ -terms, or equivalently, the $\mathscr{L}'$ -terms that contain no variables). For constant terms s and t, we define

$$s \sim t \Leftrightarrow s = t \in \Delta.$$

Claim 5: $\sim$ is an equivalence relation.

Proof of Claim 5:

- Every sentence of the form $t = t$ must be in Δ since Δ is complete, so $\sim$ is reflexive.
- If $s = t \in \Delta$, then $t = s$ must also be in Δ since Δ is complete, so $\sim$ is symmetric.
- If $r = s, s = t \in \Delta$, then $r = t$ must also be in Δ since Δ is complete, so $\sim$ is transitive.

For a constant term s, let $[s]$ denote the equivalence class of s. Then we define an $\mathscr{L}'$ -structure as follows:

(i) $A = \{[t] : t$ is a constant term of $\mathscr{L}'\}$;

(ii) for each function symbol f of the language $\mathscr{L}$, we define

$$f^{\mathscr{A}}([t_1], \ldots, [t_n]) = [f(t_1, \ldots, t_n)],$$

where n is the arity of f;

(iii) for each predicate symbol P of the language $\mathscr{L}$, we define

$$P^{\mathscr{A}}([t_1],\dots,[t_n]) \text{ if and only if } P(t_1,\dots,t_n) \in \Delta,$$

where n is the arity of P; and

(iv) for each constant symbol c of the language $\mathscr{L}'$, we define

$$c^{\mathscr{A}} = [c].$$

Claim 6: $\mathscr{A} = (A, f, \dots, P, \dots, c, \dots)$ is well-defined.

Proof of Claim 6: We have to show in particular that the interpretation of function symbols and predicate symbols in $\mathscr{A}$ is well-defined. Suppose that $s_1 = t_1, \dots, s_n = t_n \in \Delta$ and

$$f^{\mathscr{A}}([t_1],\dots,[t_n]) = [f(t_1,\dots,t_n)]. \tag{5.1}$$

By our first assumption, it follows that $\Delta \vdash s_i = t_i$ for $i = 1,\dots,n$. Then by term substitution, $\Delta \vdash f(s_1,\dots,s_n) = f(t_1,\dots,t_n)$, and so $f(s_1,\dots,s_n) = f(t_1,\dots,t_n) \in \Delta$. It follows that

$$[f(s_1,\dots,s_n)]. = [f(t_1,\dots,t_n)]. \tag{5.2}$$

Combining (7.1) and (5.2) yields

$$f^{\mathscr{A}}([t_1],\dots,[t_n]) = [f(t_1,\dots,t_n)] = [f(s_1,\dots,s_n)] = f^{\mathscr{A}}([s_1],\dots,[s_n]).$$

A similar argument shows that the interpretation of predicate symbols is well-defined.

Claim 7: Let I be an interpretation into $\mathscr{A}$. Then $I(t) = [t]$ for every constant term t.

Proof of Claim 7: We verify this inductively for constant symbols and then for function symbols applied to constant terms.

- Suppose t is a constant symbol c. Then $I(c) = c^{\mathscr{A}} = [c]$.
- Suppose that t is the term $f(t_1,\dots,t_n)$ for constant symbol f and constant terms $t_1,\dots,t_n$, where $I(t_i) = [t_i]$ for $i = 1,\dots,n$. Then

$$I(f(t_1,\dots,t_n)) = f^{\mathscr{A}}(I(t_1),\dots,I(t_n)) = f^{\mathscr{A}}([t_1],\dots,[t_n]) = [f(t_1,\dots,t_n))].$$

Claim 8: $\mathscr{A} \models \Delta$. We verify this by proving that for every interpretation I into $\mathscr{A}$ and every $\mathscr{L}'$ -sentence ϕ, $\mathscr{A} \models_I \phi$ if and only if $\phi \in \Delta$.

- If ϕ is $s = t$ for some terms s, t, then

$$\begin{aligned} \mathscr{A} \models_I s = t &\Leftrightarrow I(s) = I(t) \\ &\Leftrightarrow [s] = [t] \\ &\Leftrightarrow s \sim t \\ &\Leftrightarrow s = t \in \Delta. \end{aligned}$$

- If ϕ is $P(t_1,\dots,t_n)$ for some predicate symbol P, then

$$\begin{aligned} \mathscr{A} \models_I P(t_1,\dots,t_n) &\Leftrightarrow P^{\mathscr{A}}(I(t_1),\dots,I(t_n)) \\ &\Leftrightarrow P^{\mathscr{A}}([t_1],\dots,[t_n]) \\ &\Leftrightarrow P(t_1,\dots,t_n) \in \Delta. \end{aligned}$$

- If ϕ is $\neg\psi$ for some $\mathscr{L}'$ -sentence ψ, then

$$\begin{aligned}\mathscr{A} \models_I \neg\psi &\Leftrightarrow \mathscr{A} \not\models \psi\\ &\Leftrightarrow \psi \notin \Delta\\ &\Leftrightarrow \neg\psi \in \Delta.\end{aligned}$$

- If ϕ is $\psi \vee \theta$ for some $\mathscr{L}'$ -sentences ψ and θ, then

$$\begin{aligned}\mathscr{A} \models_I \psi \vee \theta &\Leftrightarrow \mathscr{A} \models \psi \text{ or } \mathscr{A} \models \theta\\ &\Leftrightarrow \psi \in \Delta \text{ or } \theta \in \Delta\\ &\Leftrightarrow \psi \vee \theta \in \Delta.\end{aligned}$$

- If ϕ is $(\exists v)\theta(v)$ for some $\mathscr{L}'$ -formula θ with one free variable v, then

$$\begin{aligned}\mathscr{A} \models_I (\exists v)\theta(v) &\Leftrightarrow \mathscr{A} \models_{I_{b/v}} \theta(v) \text{ for some } b \in A\\ &\Leftrightarrow \mathscr{A} \models_I \theta(c) \text{ where } b = [c]\\ &\Leftrightarrow \theta(c) \in \Delta\\ &\Leftrightarrow (\exists v)\theta(v) \in \Delta.\end{aligned}$$

Since $\mathscr{A} \models \Delta$, it follows that $\mathscr{A} \models \Gamma$. Note that $\mathscr{A}$ is an $\mathscr{L}'$ -structure while Γ is only an $\mathscr{L}$ -theory (as it does not contain any expression involving any of the additional constants). Then let $\mathscr{A}^*$ be the $\mathscr{L}$ -structure with the same universe as $\mathscr{A}$ and the same interpretations of the function symbols and predicate symbols, but without interpreting the constants symbols that are in $\mathscr{L}' \setminus \mathscr{L}$ (the so-called *reduct* of $\mathscr{A}$). Then clearly $\mathscr{A}^* \models \Gamma$, and the proof is complete.

□

5.3 Consequences of the completeness theorem

The same consequences we derived from the Soundness and Completeness Theorem for Propositional Logic apply now to Predicate Logic with basically the same proofs.

Theorem 5.3.1. *For any set of sentences Γ, Γ is satisfiable if and only if Γ is consistent.*

Theorem 5.3.2. *If Σ is a consistent theory, then Σ is included in some complete, consistent theory.*

We also have an additional version of the compactness theorem, which is the most common formulation of compactness.

Theorem 5.3.3 (Compactness Theorem for Predicate Logic)**.**
An $\mathscr{L}$ -theory Γ is satisfiable if and only if every finite subset of Γ is satisfiable.

Proof. ($\Rightarrow$) If $\mathscr{A} \models \Gamma$, then it immediately follows that $\mathscr{A} \models \Gamma_0$ for any finite $\Gamma_0 \subseteq \Gamma$.

($\Leftarrow$) Suppose that Γ is not satisfiable. By the completeness theorem, Γ is not consistent. Then $\Gamma \vdash \phi \,\&\, \neg\phi$ for some $\mathscr{L}$ -sentence ϕ. Then by the first formulation of the compactness theorem there is some finite $\Gamma_0 \subseteq \Gamma$ such that $\Gamma_0 \vdash \phi \,\&\, \neg\phi$. It follows that Γ_0 is not satisfiable. □

We now consider two applications of the compactness theorem, the first yielding a model of arithmetic with infinite natural numbers and the second yielding a model of the real numbers with infinitesimals.

Example 5.3.4. Let Ł $= \{+, \times, <, 0, 1\}$ be the language of arithmetic, and let $\Gamma = Th(\mathbb{N})$, the set of $\mathscr{L}$ -sentences true in the standard model of arithmetic. Let us expand $\mathscr{L}$ to $\mathscr{L}'$ by adding a new constant c to our language. We extend Γ to an $\mathscr{L}'$ -theory Γ' by adding all sentences of the form

$$\psi_n : c > \underbrace{1 + \ldots + 1}_{n \text{ times}}$$

We claim that every finite $\Gamma_0' \subseteq \Gamma'$ is satisfiable. Given any finite $\Gamma_0' \subseteq \Gamma'$, Γ_0' consists of at most finitely many sentences from Γ and at most finitely many sentences of the form ψ_i. It follows that

$$\Gamma_0' \subseteq \Gamma \cup \{\psi_{n_1}, \psi_{n_2}, \ldots, \psi_{n_k}\}$$

for some $n_1, n_2, \ldots, n_k \in \mathbb{N}$, where these latter sentences assert that c is larger than each of the values $n_1, n_2, \ldots, n_k$. Let $n = \max\{n_1, \ldots, n_k\}$ then let $\mathscr{A} = (\mathbb{N}, +, \times, <, 0, 1, n)$, so that $c^{\mathscr{A}} = n$ and hence $\mathscr{A} \models \Gamma_0'$. Then by the compactness theorem, there is some $\mathscr{L}'$ -structure $\mathscr{B}$ such that $\mathscr{B} \models \Gamma'$. In the universe of $\mathscr{B}$, we have objects that behave exactly like $0, 1, 2, 3, \ldots$ (in a sense we will make precise shortly), but the interpretation of c in $\mathscr{B}$ satisfies $c^{\mathscr{B}} > n$ for every $n \in \mathbb{N}$ and hence behaves like an infinite natural number. We will write the universe of $\mathscr{B}$ as $\mathbb{N}^*$.

Example 5.3.5. Let $\mathscr{L}$ consist of

- an n-ary function symbol F_f for every $f : \mathbb{R}^n \to \mathbb{R}$;
- an n-ary predicate symbol P_A for every $A \subseteq \mathbb{R}^n$; and
- a constant symbol c_r for every $r \in \mathbb{R}$.

Let $\mathfrak{R}$ be the $\mathscr{L}$ -structure with universe $\mathbb{R}$ satisfying

- $F_f^{\mathfrak{R}} = f$ for every function symbol F_f;
- $P_A^{\mathfrak{R}} = A$ for every predicate symbol P_A; and
- $c_r^{\mathfrak{R}} = r$ for every constant symbol c_r.

Let us expand $\mathscr{L}$ to $\mathscr{L}'$ by adding a new constant d to our language. We extend Γ to an $\mathscr{L}'$ -theory Γ' by adding all sentences of the form

$$\theta_r : c_0 < d < c_r$$

for $r \in \mathbb{R}^{>0}$. As in the previous example, every finite $\Gamma_0' \subseteq \Gamma'$ is satisfiable. Hence by the compactness theorem, Γ' is satisfiable. Let $\mathscr{A} \models \Gamma$. The universe of $\mathscr{A}$ contains a copy of $\mathbb{R}$ (in a sense we will make precise shortly). In addition, $d^{\mathscr{A}}$ is infinitesimal object. For every real number in $\mathscr{A}$, $0 < d^{\mathscr{A}} < r$ holds. We will write the universe of $\mathscr{A}$ as $\mathbb{R}^*$.

Now we consider a question that was not appropriate to consider in the context of propositional logic, namely, what are the sizes of models of a given theory? Our main theorem is a consequence of the proof of the Completeness Theorem. We proved the Completeness Theorem only in the case of a countable language L, and we built a countable model (which was possibly finite). By using a little care (and some set theory), one can modify steps (1) and (2) for an uncountable language to define by transfinite recursion a theory Δ and prove by transfinite induction that Δ has the desired properties. The construction leads to a model whose size is at most the size of the language with which one started. Thus we have:

Theorem 5.3.6 (Löwenheim-Skolem Theorem)**.** *If Γ is an $\mathscr{L}$ -theory with an infinite model, then Γ has a model of size κ for every infinite κ with $|$Ł$| \leq \kappa$.*

Proof Sketch. First we add κ new constant symbols $\langle d_\alpha \mid \alpha < \kappa \rangle$ to our language $\mathscr{L}$. Next we expand Γ to Γ' by adding formulas that say $d_\alpha \neq d_\beta$ for the different constants:

$$\Gamma' = \Gamma \cup \left\{ \neg d_\alpha = d_\beta : \alpha < \beta < \kappa \right\}.$$

Since Γ has an infinite model, each finite $\Gamma'_0 \subseteq \Gamma'$ has a model. Hence by the compactness theorem, Γ' has a model. By the soundness theorem, Γ' is consistent. Then use the proof of the completeness theorem to define a model $\mathscr{B}'$ of Γ' the universe of which has size $|B| \leq \kappa$. Since $\mathscr{B}' \models d_\alpha \neq d_\beta$ for $\alpha \neq \beta$, there are at least κ many elements. Thus $|B| = \kappa$ and so Γ' has a model $\mathscr{B}'$ of the desired cardinality. Let $\mathscr{B}$ be the reduct of $\mathscr{B}'$ obtained by removing the new constant symbols from our expanded language. Then $\mathscr{B}$ is a model of the desired size for Γ. □

5.4 Isomorphism and elementary equivalence

We conclude this chapter with a discussion of isomorphic models and the notion of elementary equivalence.

Definition 5.4.1. Given $\mathscr{L}$ -structures $\mathscr{A}$ and $\mathscr{B}$, a bijection $H : \mathscr{A} \to \mathscr{B}$ is an isomorphism if it satisfies:

1. For every constant $c \in$ Ł, $H(c^{\mathscr{A}}) = c^{\mathscr{B}}$.
2. For every k-ary predicate symbol $P \in$ Ł and every $a_1, \dots, a_k \in A$,

$$P^{\mathscr{A}}(a_1, \dots, a_k) \Leftrightarrow P^{\mathscr{B}}(H(a_1), \dots, H(a_k)).$$

3. For every k-ary function symbol $F \in$ Ł and every $a_1, \dots, a_k \in A$,

$$H(F^{\mathscr{A}}(a_1, \dots, a_k)) = F^{\mathscr{B}}(H(a_1), \dots, H(a_k)).$$

Furthermore, $\mathscr{A}$ and $\mathscr{B}$ are isomorphic, denoted $\mathscr{A} \cong \mathscr{B}$, if there exists an isomorphism between $\mathscr{A}$ and $\mathscr{B}$.

Example 5.4.2. The ordered group $(\mathbb{R}, +, <)$ of real numbers under addition is isomorphic to the ordered group $(\mathbb{R}^{>0}, \cdot, <)$ of positive real numbers under multiplication under the mapping $H(x) = 2^x$. The key observation here is that $H(x + y) = 2^{x+y} = 2^x \cdot 2^y = H(x) \cdot H(y)$.

We compare the relation of isomorphism with the following relation between models.

Definition 5.4.3. $\mathscr{L}$ -structures $\mathscr{A}$ and $\mathscr{B}$ are *elementarily equivalent*, denoted $\mathscr{A} \equiv \mathscr{B}$, if for any $\mathscr{L}$ -sentence ϕ,

$$\mathscr{A} \models \phi \Leftrightarrow \mathscr{B} \models \phi.$$

How do the relations of $\cong$ and $\equiv$ compare? First, we have the following theorem.

Theorem 5.4.4. *If $\mathscr{A}$ and $\mathscr{B}$ are $\mathscr{L}$ -structures satisfying $\mathscr{A} \cong \mathscr{B}$, then $\mathscr{A} \equiv \mathscr{B}$.*

The proof is by induction on the complexity of $\mathscr{L}$ -sentences. The converse of this theorem does not hold, as shown by the following example.

Example 5.4.5. $(\mathbb{Q}, \leq)$ and $(\mathbb{R}, \leq)$, both models of the theory of dense linear orders without endpoints, are elementarily equivalent, which follows from the fact that the theory of dense linear orders without endpoints is complete (which we will prove in Chapter 6). Note, however, that these structures are not isomorphic, since they have different cardinalities.

We conclude this chapter with one last set of definitions and examples.

Definition 5.4.6. Let $\mathscr{A}$ and $\mathscr{B}$ be $\mathscr{L}$ -structures with corresponding domains $A \subseteq B$.

1. $\mathscr{A}$ is a *submodel* of $\mathscr{B}$ ($\mathscr{A} \subseteq \mathscr{B}$) if the following are satisfied:
   (a) for each constant $c \in Ł$, $c^{\mathscr{A}} = c^{\mathscr{B}}$;
   (b) for each n-ary function symbol $f \in Ł$ and each $a_1, \ldots, a_n \in A$,
   $$f^{\mathscr{A}}(a_1, \ldots, a_n) = f^{\mathscr{B}}(a_1, \ldots, a_n);$$
   (c) for each n-ary relation symbol $R \in Ł$ and each $a_1, \ldots, a_n \in A$,
   $$R^{\mathscr{A}}(a_1, \ldots, a_n) \Leftrightarrow R^{\mathscr{B}}(a_1, \ldots, a_n).$$
2. $\mathscr{A}$ is an *elementary submodel* of $\mathscr{B}$ (written $\mathscr{A} \precsim \mathscr{B}$) if
   (a) $\mathscr{A}$ is a submodel of $\mathscr{B}$;
   (b) for each $\mathscr{L}$ -formula $\phi(x_1, \ldots, x_n)$ and each $a_1, \ldots, a_n \in A$,
   $$\mathscr{A} \models \phi(a_1, \ldots, a_n) \Leftrightarrow \mathscr{B} \models \phi(a_1, \ldots, a_n).$$

Example 5.4.7. Consider the rings $(\mathbb{Z}, 0, 1, +, \cdot) \subseteq (\mathbb{Q}, 0, 1, +, \cdot) \subseteq (\mathbb{R}, 0, 1, +, \cdot)$.

- $(\mathbb{Z}, 0, 1, +, \cdot)$ is a submodel of $(\mathbb{Q}, 0, 1, +, \cdot)$ and $(\mathbb{Q}, 0, 1, +, \cdot)$ is a submodel of $(\mathbb{R}, 0, 1, +, \cdot)$.
- $(\mathbb{Z}, 0, 1, +, \cdot)$ is not an elementary submodel of $(\mathbb{Q}, 0, 1, +, \cdot)$, since $\mathbb{Q} \models (\exists x) x + x = 1$ which is false in $\mathbb{Z}$.
- Neither $(\mathbb{Z}, 0, 1, +, \cdot)$ nor $(\mathbb{Q}, 0, 1, +, \cdot)$ is an elementary submodel of $(\mathbb{R}, 0, 1, +, \cdot)$ since $\mathbb{R} \models (\exists x) x \cdot x = 2$, which is false in both $\mathbb{Z}$ and $\mathbb{Q}$.

Example 5.4.8. The following elementary submodel relations hold:

- $(\mathbb{Q}, \leq) \precsim (\mathbb{R}, \leq)$
- $(\mathbb{N}, 0, 1, +, \cdot) \precsim (\mathbb{N}^*, 0, 1, +, \cdot)$.
- $(\mathbb{R}, 0, 1, +, \cdot) \precsim (\mathbb{R}^*, 0, 1, +, \cdot)$.

The latter two items in the previous example justify the claims that the natural numbers are contained in models of non-standard arithmetic and that the real numbers are contained in models of non-standard analysis.

We conclude with one last example.

Example 5.4.9. $\mathbb{Z}_3 = \{0, 1, 2\}$ with addition modulo 3 is not a submodel of $\mathbb{Z}_6 = \{0, 1, 2, 3, 4, 5\}$ with addition modulo 6 because they have different addition functions: In $\mathbb{Z}_3$, $2 + 2 = 1$ whereas in $\mathbb{Z}_6$, $2 + 2 = 4$. However, $\mathbb{Z}_3$ *is* isomorphic to the subgroup of $\mathbb{Z}_6$ consisting of $\{0, 2, 4\}$.

Just as every subset X of a group G generates a subgroup $\langle X \rangle$ of G, every subset X of an arbitrary structure generates a substructure $\langle X \rangle$.

Definition 5.4.10. Let $\mathscr{A}$ be a structure with universe A for some language Ł and let X be a subset of A. Then $\langle X \rangle$ is the smallest substructure of $\mathscr{A}$ which inclludes X.

Example 5.4.11. In $\mathbb{Z}, +)$, $\langle \{20, 30\} \rangle = \langle 10 \rangle = \{10x : x \in \mathbb{Z}\}$.

Example 5.4.12. In the Boolean algebra $\mathscr{B} = (\mathscr{P}(\{1, 2, 3, 4\}, \wedge, \vee, '))$, $\langle \{\{2, 3\}, \{4\}\} \rangle$ contains the sets $\emptyset, \{1\}, \{4\}, \{1, 4\}, \{2, 3\}, \{1, 2, 3\}, \{2, 3, 4\}, \{1, 2, 3, 4\}$.

Example 5.4.13. In $(\mathfrak{R}, 0, 1, +, \cdot)$, $\langle \sqrt{2} \rangle = \{m + n\sqrt{2} : m, n \in \mathbb{Z}\}$.

Proposition 5.4.14. *For any structure $\mathscr{A}$ with universe A and any subset X of A, $\langle X \rangle = \{t(x_1, x_2, \ldots, x_n) :$ t is a term and $x_1, \ldots, x_n \in X\}$.*

Proof. Let $C = \{t(x_1, x_2, \ldots, x_n) : t$ is a term and $x_1, \ldots, x_n \in X\}$. For any substructure $\mathscr{B}$ of $\mathscr{A}$ with universe B which includes X, we must have $C \subseteq B$, since B is closed under all functions and hence all terms. It follows that $C \subseteq \langle X \rangle$. For the other direction, we note that C is closed under all functions and hence is a substructure of $\mathscr{A}$ which inclueldes X. It follows from the definition of $\langle X \rangle$ that $\langle X \rangle \subset C$. □

A submodel $\mathscr{A}$ of a structure $\mathscr{B}$ satisfies the same quantifier-free formulas $\phi(a_1, \ldots, a_n)$ as $\mathscr{B}$. If $\mathscr{A}$ is an elementary submodel, then it satisfies the same first order formulas $\phi(a_1, \ldots, a_n)$ as $\mathscr{B}$. Next we consider some intermediate versions of this notion, where $\mathscr{A}$ agrees with $\mathscr{B}$ on a certain class of formulas.

Definition 5.4.15. A formula ϕ is said to be *universal* if there is a quantifier-free formula θ and variables $y_1, \ldots, y_m$ such that $\phi = (\forall y_1)(\forall y_2)\cdots(\forall y_m)\theta$, and is said to be *logically universal* if it is logically equivalent to a universal formlula. Existential and logically existential formulas are similarly defined using existential quantifiers $(\exists y_i)$. A formula ϕ is said to be Existential-Universal-Existential $(\exists\forall)$ if there is a universal formula θ and variables $y_1, \ldots, y_m$ such that $\phi = (\exists y_1)(\exists y_2)\cdots(\exists y_m)\theta$, and is said to be *logically* $\exists\forall$ if it is logically equivalent to a $\forall\exists$ formlula. Universal-Existential formulas are similarly defined.

Example 5.4.16. The axioms for a group in the language $\{e, *, ^{-1}\}$ are universal.

(a) $(\forall x) x * e = x = e * x$

(b) $(\forall x) x * x^{-1} = e = x * x^{-1}$.

(c) $(forall x)(\forall y)(\forall z) x * (y * z) = (x * y) * z$

If we removed the unary function symbol $^{-1}$ from the language and modified (b) to say that
$(\forall x)(\exists y) x * y = e = y * x$
then this would no longer be a universal sentence.

The importance of universal sentences is in the following notion of persistence.

Definition 5.4.17. A sentence ϕ is said to be *downward persistent* if whenever $\mathscr{A} \subseteq \mathscr{B}$ and $\mathscr{B} \models \phi$, then $\mathscr{A} \models \phi$. This can also apply to a formula $\phi(x_1, \ldots, x_n)$ where $\mathscr{B} \models \phi(a_1, \ldots, a_n)$ should imply that $\mathscr{A} \models \phi(a_1, \ldots, a_n)$ when $a_1, \ldots, a_n \in A$. Similarly, ϕ is *upward persistent* if whenever $\mathscr{A} \subseteq \mathscr{B}$ and $\mathscr{A} \models \phi(a_1, \ldots, a_n)$, then $\mathscr{B} \models \phi(a_1, \ldots, a_n)$.

It is immediate from these definitions that ϕ is upward persistent if and only if $\neg\phi$ is downward persistent.

Proposition 5.4.18. *Any universal formula ϕ is downward persistent.*

Proof. Let $\phi(x_1, \ldots, x_n)$ have the form $(\forall y_1)(forall y_2)\cdots(\forall y_m)\theta(x_1, \ldots, x_n, y_1, \ldots, y_m)$, where θ is quantifier-free. Suppose that $\mathscr{A} \subseteq \mathscr{B}$ and that $\mathscr{B} \models \phi(a_1, \ldots, a_n)$ where $a_1, \ldots, a_n \in A$. Now let $c_1, \ldots, c_n \in A$ be arbitrary. Then $\mathscr{B} \models \theta(a_1, \ldots, a_n, c_1, \ldots, c_m)$ (since $\mathscr{B} \models (\forall y_1)(forall y_2)\cdots(\forall y_m)\theta(a_1, \ldots, a_n, y_1, \ldots, y_m$ Since $\mathscr{A} \subseteq \mathscr{B}$ and θ is quantifier-free, it follows that $\mathscr{A} \models \theta(a_1, \ldots, a_n, c_1, \ldots, c_m)$, Since $c_1, \ldots, c_m$ are arbitrary elements of A, we may now conclude that $\mathscr{A} \models (\forall y_1)(forall y_2)\cdots(\forall y_m)\theta(a_1, \ldots, a_n, y_1, \ldots, y_m))$, as desired. □

It follows that existential formulas are upward persistent.

The converse of this proposition, that any downward persistent formula is logically universal, is also true but beyond the scope of this book. We know that the family of universal formulas is closed under conjunction and disjunction, so it must be the case that the family of downward persistent formulas is also closed under conjunction and disjunction. To see this for conjunction, let us suppose that ϕ and ψ are downward persistent and show that $\phi \vee \psi$ is also downward persistent. That is, suppose that $\mathscr{A} \subseteq \mathscr{B}$ and $\mathscr{B} \models \phi \vee \psi$. Then without loss of generality, $\mathscr{B} \models \phi$. Since ϕ is downward persistent, $\mathscr{A} \models \phi$, and it follows that $\mathscr{A} \models \phi \vee \psi$.

Other interesting notions of persistence include *product persistence* and persistence under unions of chains.

If $\mathscr{A}$ and $\mathscr{B}$ are two structures over the same language, with universes A and B, respectively, then $\mathscr{A} \times \mathscr{B}$ is the structure $\mathscr{C}$ with universe $A \times B$ defined as follows. For each constant symbol c, $c^C = (c^A, c^B)$. For each n-afry function symbol f, $f^C((a_1,b_1),\dots,(a_n,b_n)) = (f^A(a_1,\dots,a_n), f^B(b_1,\dots,b_n))$. For each n-ary relation symbol R $R^C(((a_1,b_1),\dots,(a_n,b_n)) \iff R^A(a_1,\dots,a_n)) \wedge R^B(b_1,\dots,b_n))$. We say that a formula ϕ is *product persistent* if whenever $\mathscr{A} \models \phi$ and $\mathscr{B} \models \phi$, then $\mathscr{A} \times \mathscr{B} \models \phi$.

For example, the group axioms given above are all product persistent, as the product of two groups is a group. The axioms for a commutative ring with unity in the language $(+,\cdot,0,1)$ are also product persistent. However, consider the additional axiom for being an integral domain, that there are no zero-divisors.

$$(\forall x)(\forall y)[x \cdot y = 0 \implies (x = 0 \ \vee y = 0)]$$

The natural example here is that $\mathbb{Z}_2$ and $\mathbb{Z}_3$ are both integral domains, but the product $\mathbb{Z}_2 \times \mathbb{Z}_3$ has zero divisors $(0,1)\cdot(1,0) = (0,0)$.

Definition 5.4.19. The family of *Horn* formulas is the smallest family of formulas generated as follows. For all atomic formulas $\rho_1,\dots,\rho_n,\theta$, the formulas $\neg(\rho_1\&\rho_2\&\cdots\&\rho_n)$ and $(\rho_1\&\rho_2\&\cdots\&\rho_n) \to \theta$ are Horn formulas. If ϕ and ψ are Horn formulas, then the conjunction $\phi\&\psi$ is a Horn formula and both $(\exists x)\phi$ and $(\forall x)\phi$ are Horn formulas.

We note that the statement about zero-divisors above is NOT a Horn formula.

Proposition 5.4.20. *Any Horn formula ϕ is product persistent.*

Definition 5.4.21. A sequence $\mathscr{A}_0 \subseteq \mathscr{A}_1 \subseteq \cdots$ is said to be a *chain*. The union $\mathscr{C}$ of such a chain has universe $\bigcup_i A_i$, where each structure $\mathscr{A}_i$ has universe A_i, and is defined as follows. For each constant symbol c, $c^C = c^{A_0}$. For each n-ary function symbol and for elements $a_1,\dots,a_n$, let k be the least such that each $a_1,\dots,a_n \in A_k$ and let $f^C(a_1,\dots,a_k) = f^{A_k}(a_1,\dots,a_n)$. For each n-ary relation symbol and for elements $a_1,\dots,a_n$, let k be the least such that each $a_1,\dots,a_n \in A_k$ and let $R^C(a_1,\dots,a_k) \iff R^{A_k}(a_1,\dots,a_n)$.

Proposition 5.4.22. *If $\mathscr{C}$ is the union of a chain $\mathscr{A}_0 \subseteq \mathscr{A}_1 \subseteq \cdots$, then for each k, $\mathscr{A}_k \subseteq \mathscr{C}$.*

Unions of chains preserve formulas with two levels of quantification.

Definition 5.4.23. A formula ϕ is said to be *existential-universal* ($\exists\forall$) if there is a universal formula θ such that $\phi = (\forall y_1)\cdots(\forall y_m)\theta$. Similarly, ϕ is *universal-existential* ($\forall\exists$) if there is an existential formula θ such that $\phi = (\exists y_1)\cdots(\exists y_m)\theta$.

As for universal and existential formulas, we can define the notion of *logically* $\exists\forall$ as being logically equivalen to a $\exists\forall$ formulas and similarly for the $\forall\exists$ formulas.

As an example, the statement about an ordering that it has no greatest element can be written

$$(\forall y)(\exists x)(y < x)$$

and is therefore seen to be universal-existential.

It turnes out that $\forall\exists$ formulas are preserve under unions of chains.

Theorem 5.4.24. *For any $\forall\exists$ formula ϕ and any chain $\mathscr{A}_0 \subseteq \mathscr{A}_1 \subseteq \cdots$, if $\mathscr{A}_k \models \phi$ for all k, then the union $\mathscr{C}$ of the chain also satisfies ϕ.*

Proof. Let ϕ have the form $(\forall y_1)\cdots(\forall y_m)\theta(y_1,\dots,y_m)$, where θ is existential and suppose that $\mathscr{C}$ is the union of a chain $\mathscr{A}_0 \subseteq \mathscr{A}_1 \cdots$ such that $\mathscr{A}_k \models \phi$ for each k. Let $c_1,\dots,c_m \in C$ be given and take k large enough so that each $c_i \in \mathscr{A}_k$. Since $\mathscr{A}_k \models \phi$, it follows that $\mathscr{A}_k \models \theta(c_1,\dots,c_m)$. Now θ is an existential formula and $\mathscr{A}_k \subseteq \mathscr{C}$, hence it follows from Proposition 5.4.18 that $\mathscr{C} \models \theta(c_1,\dots,c_m)$. Since $c_1,\dots,c_m$ are arbitrary, it follows that $\mathscr{C} \models (\forall y_1)\cdots(\forall y_m)\theta$, as desired. □

Now we can see that the negation of the sentence above, which states that there *is* a greatest element in a linear ordering, cannot be logically $\forall\exists$, because the union of the chain $\mathscr{A}_k = [0,k]$ (under the usual ordering on the real numbers) would have no greatest element, whereas each $[0,k]$ does have a greatest element.

5.5 Axioms and Theories

Given a set Γ of sentences in a fixed language $\mathscr{L}$, we let $Mod(\Gamma)$ be the set of structures $\mathscr{A}$ such that $\mathscr{A}$ mod γ for all $\gamma \in \Gamma$. For a single sentence γ, we let $Mod(\gamma) = Mod(\{gamma\})$. Given a class K of models in a fixed language $\mathscr{L}$, we let $Th(K)$ be the set of sentences γ such that $\mathscr{A}$ mod γ for all $\mathscr{A} \in K$. For a single structure $\mathscr{A}$, we let $Th(\mathscr{A}) = Th(\{\mathscr{A}\})$.

For example, in the language $\mathscr{L} = \{<\}$, consider the axioms of a partial ordering:

(a) $(\forall x)\neg x < x$

(b) $(\forall x)(\forall y)\neg x < y \& y < x$

(c) $(\forall x)(\forall y)(\forall z)(x < y \& y < z) \to x < z)$

These axioms state that $<$ is irreflexive, antisymmetric, and transitive. If Γ_0 is the set of these three sentences, then $Mod(\Gamma)$ is the class of partial orderings. If we add the axiom of trichotomy

(d) $(\forall x)(\forall y)(x < y \vee x = y \vee y < x)$

then the resulting set of four axioms define the class of linear orderings. Note that all of these axioms are universal, so that any substructure of a linear ordering is also a linear ordering.

It is easy to see that $K \subseteq Mod(Th(K))$ for any class K of models and that $\Gamma \subseteq Th(Mod(\Gamma)$ for any theory Γ. However, $Th(K)$ is always closed under implication, so Γ may be a proper subset. For example, if $\Gamma = \emptyset$, then $Mod(\Gamma)$ would consist of all possible structures for the given language and $Th(Mod(\Gamma)$ would consist of all logically valid sentences.

A class K of models is said to be *axiomatizable* if $K = Mod(\Gamma)$ for some set Γ of sentences. K is said to be *finitely axiomatizable* if $K = Mod(\Gamma)$ for a finite set Γ; equivalently $K = Mod(\gamma)$ for some single sentence γ. If K is not axiomatizible, then clearly $K \neq Mod(Th(K)$.

For a finite language $\mathscr{L}$, any finite structure $\mathscr{A}$ is finitely axiomatizable. For example, let $\mathscr{A} = (\{1,2\},<)$ where $1 < 2$. Then the axiom for $\mathscr{A}$ is the following:

$$(\exists x)(\exists y)[(x \neq y \ \& \ (\forall z)(x = z \ lor \ y = z) \ \& \ x < y]$$

In general, the axiom for a finite structure $\mathscr{A}$ just states the existence of the distinct elements and defines the constants, relations and functions on $\mathscr{A}$ for those elements.

The language of pure equality provides a very interesting illustration of these notions, as well as the use of compactness.

Let γ_n be the sentence stating that there are at least n distinct elements, that is

$$(\exists x_1)(\exists x_2)\cdots(\exists x_n)[x_1 \neq x_2 \& x_1 \neq x_3 \& x_2 \neq x_3 \& \cdots \& x_{n-1} \neq x_n]$$

The conjunct inside the quantifiers can be abbreviated as $\bigwedge_{i \neq j} x_i \neq x_j$. Note that $\gamma_{n+1} \to \gamma_n$ for each n.

It is clear that a structure $\mathscr{A}$ is infinite if and only if $\mathscr{A} \models \gamma_n$ for all n. Let $\Gamma_{INF} = \{\gamma_1, \gamma_2, \dots\}$. Then the class INF of infinite structures equals $Mod(\Gamma_{INF})$. Hence INF is axiomatizable.

Proposition 5.5.1. *INF is not finitely axiomatizable.*

Proof. Suppose by way of contradiction that $INF = Mod(\gamma)$ for some sentence γ. This means that a structure $\mathscr{A}$ is infinite if and only if $\Gamma \models \gamma$. Then $\Gamma_{INF} \vdash \gamma$. By Compactness, there must be a finite subset $\Gamma_n = \{\gamma_1, \dots, \gamma_n\}$ of Γ such that $\Gamma_n \vdash \gamma$. Now consider the structure $\{1, 2, \dots, n\}$, which satisfies Γ_n but is not infinite, and hence does not satisfy γ. This contradicts the statment above that $\Gamma_n \vdash \gamma$. ☐

Now consider the complementary set FIN of finite models.

Proposition 5.5.2. *FIN is not axiomatizable.*

Proof. Suppose by way of contradiction that there were a set Δ of sentences such that $FIN = Mod(\Delta)$. Then $\Delta \cup \Gamma_{INF}$ is inconsistent. So by Compactness, there is a finite subset $\Gamma_n = \{\gamma_1, \dots, \gamma_n\}$ such that $\Delta \cup \Gamma_n$ is inconsistent. But the structure $\{1, 2, \dots, n\}$ satisfies Γ_n and it is finite, so it must satisfy Δ. This contradiction proves the result. □

We finish this chapter with a brief introduction to the notion of quantifier-elimination. We assume that the relevant language is finite.

Definition 5.5.3. A theory Γ is said to have quantifier elimination if for any formula $\phi(x_1, \dots, x_n)$, there is a quantifier-free formula θ such that $\Gamma \vdash \phi \iff \theta$.

Note that in a relational language (with no functions or constants) a quantifier-free sentence can only be either the trivial true statement or the trivial false statement. Thus a theory with quantifier elimination is complete. If there is an algorithm to produce from any sentence ϕ a quantifier-free θ such that $\Gamma \vdash \phi \iff \theta$, then this algororithm also decides whether $\Gamma \vdash \phi$ or $\Gamma \vdash \neg\phi$. Structures with functions and constants are a bit more complicated.

The theory of infinity, where $\Gamma = \{\gamma_1, \gamma_2, \dots\}$ as above, is a natural example of a theory which satisfies quantifier elimination.

Theorem 5.5.4. *The theory of infinity satisfies quantifier-elimination, and there is an algorithm which produces the equivalent quantifier-free formula from any given formula.*

Proof. In the language of equality, the atomic formulas have the form "$u = v$" where u and v are two variables, possibly identical. Now by disjunctive normal form, any quantifier-free formula $\theta(x_1, \dots, x_n)$ is a disjunction $C_1 \vee \cdots \vee C_k$ of conjuncts $C_1, \dots, C_k$ where each C_i is a conjunct of literals, either of the form $u = v$ or the form $u \neq v$.

The result is proved by induction on the set of formulas. For the base case, any atomic formula is quantifier-free already. For the connectives $\neg, \vee, \&$, if two formulas ϕ_1 and ϕ_2 are logically equivalent to quantifier-free formulas θ_1 and θ_2 (under Γ), then $\neg\phi_1$ is logically equivalent to $\neg\theta_1$, $\phi_1 \vee \phi_2$ is logically equivalent to $\theta_1 \vee \theta_2$ and $\phi_1 \& \phi_2$ is logically equivalent to $\theta_1 \& \theta_2$. But these formula ($\neg\theta_1$, $\theta_1 \vee \theta_2$, $\theta_1 \& \theta_1$) are all quantifier-free.

Since $(\forall x)\theta$ is logically equivalent to $\neg(\exists x)\neg\theta$, it suffices to show that if $\theta(x, x_1, \dots, x_n)$ is quantifier-free, then there is a quantifier-free formula psi such that

$$\Gamma \vdash (\exists x)\theta(x, x_1, \dots, x_n) \iff \psi(x_1, \dots, x_n)$$

Now for any disjunction $C_1 \vee \cdots \vee C_k$), $(\exists x)[C_1 \vee \cdots \vee C_k]$ is logically equivalent to $(\exists x)C_1 \vee (\exists x)C_2 \vee \cdots \vee (\exists x)C_k$, thus we may assume without loss of generality that θ is a conjunct of literals.

There are three cases.

Case 0: One of the literals has the form $x \neq x$. In this case, $(\exists x)\theta$ is simply false.

Case 1: One of the literals has the form $x = x_i$. In this case we can modify the formula θ by replacing every occurrence of x with an occurrence of x_i to obtain the desired quantifier-free formula $\psi = \theta(x_i, x_1, \dots, x_n)$ which is equivalent to $(\exists x)\theta$.

Case 2: Each of the literals has the form $x \neq x_i$. Then the sentence γ_{n+1} from Γ implies the existence of an element x which is different from each of $x_1, \dots, x_k$. Thus we can modify θ by eliminating each literal in which x occurs to obtain the desired quantifier-free formula ψ. □

5.6 Exercises

1. Show that $Mod(Th(K)) = K$ for any axiomatizable class K of models.

2. Show that if a theory Γ is closed under implication, then $Th(Mod(\Gamma) = \Gamma$.

3. Write the sentence γ such that $\mathscr{A} = Mod(\gamma)$ where $\mathscr{A} = (\{0,1\},+)$ with addition mod 2.
4. Show that the theory of dense linear orderings without endpoints satisfies quantifier elimination.
5. Show that the class of equivalence structures with exactly three elements in each equivalence class is not downward persistent and therefore cannot have a universal axiomatization.
6. Show that the familly of product persistent formulas is closed under conjunctions.
7. Show that the class of well-orderings is not closed under unions of chains and therefore cannot have a $\forall\exists$ axiomatization.
8. Show that the class of graphs of infinite degree is not finitely axiomatizable and that the class of graphs of finite degree is not axiomatizable. An unordered graph $G = (V, E)$ consists of a set V of elements (called vertices) and a binary edge relation E. The *degree* of a vertex v is the cardinality of $\{u : uEv\}$. G has finite degree k if k is the maximum of the degrees of the vertices in V. If no such finite maximum exists, then G is said to have infinite degree.
9. Show by induction on Horn formulas that every Horn formula is product persistent.
10. If $\mathscr{C}$ is the union of a chain $\mathscr{A}_0 \subseteq \mathscr{A}_1 \subseteq \cdots$, then for each k, $\mathscr{A}_k \subseteq \mathscr{C}$.

Chapter 6

Computability Theory

6.1 Introduction and Examples

There are many different approaches to defining the collection of computable number-theoretic functions. The intuitive idea is that a function $F : \mathbb{N} \to \mathbb{N}$ (or more generally $F : \mathbb{N}^k \to \mathbb{N}$) is computable if there is an algorithm or effective procedure for determining the output $F(m)$ from the input m. To demonstrate that a particular function F is computable, it suffices to give the corresponding algorithm.

Example 6.1.1. Basic computable functions include

(i) the successor function $S(x) = x + 1$,

(ii) the addition function $+(x, y) = x + y$, and

(iii) the multiplication function $\cdot(x, y) = x \cdot y$.

Example 6.1.2. Some slightly more complicated examples of computable functions:

(i) The Division Algorithm demonstrates that the two functions that compute, for inputs a and b, the unique quotient $q = q(a, b)$ and remainder $r = r(a, b)$, with $0 \le r < a$, such that $b = qa + r$, are both computable.

(ii) The Euclidean Algorithm demonstrates that the function $gcd(a, b)$ which computes the greatest common divisor of a and b is computable. It follows that least common multiple function $lcm(a, b) = (a \cdot b)/gcd(a, b)$ is also computable

The notion of computability for functions can be extended to subsets of $\mathbb{N}$ or relations on $\mathbb{N}^k$ for some k as follows. First, a set $A \subseteq \mathbb{N}$ is said to be computable if the characteristic function of A, defined by

$$\chi_A(n) = \begin{cases} 1 & \text{if } n \in A \\ 0 & \text{if } n \notin A, \end{cases}$$

is a computable function. Similarly, a relation $R \subseteq \mathbb{N}^k$ is said to be computable if its characteristic function

$$\chi_A(n_1, \dots, n_k) = \begin{cases} 1 & \text{if } (n_1, \dots, n_k) \in R \\ 0 & \text{if } (n_1, \dots, n_k) \notin R \end{cases}$$

is computable. These definitions are equivalent to saying that there is an algorithm for testing whether a given number is in A or whether a given finite sequence $(n_1, \dots, n_k)$ is in R.

Example 6.1.3. The following are computable:

(i) The set of perfect squares is computable, since given a number n, we can test whether it is a square by computing m^2 for all $m \le n$ and checking whether $m^2 = n$.

(ii) The relation $x \mid y$ ("x divides y") is computable, since by the Division Algorithm, $x \mid y$ if and only if the remainder $r(x, y) = 0$.

(iii) The set of even numbers is computable, since n is even if and only if $2 \mid n$.

(iv) The set of prime numbers is computable, since p is prime if and only if

$$(\forall m < p)[(m \neq 0 \,\&\, m \neq 1) \rightarrow m \nmid p].$$

Two formalizations of the class of computable functions that will we consider are the collection of Turing machines and the collection of partial recursive functions.

The first general model of computation that we will consider is the Turing machine, developed by Alan Turing in the 1930's. The machine consists of one or more infinite *tapes* with cells on which symbols from a finite alphabet may be written, together with *heads* which can read the contents of a given cell, write a new symbol on the cell, and move to an adjacent cell. A program for such a machine is given by a finite set of *states* and a transition function which describes the action taken in a given state when a certain symbol is scanned. Possible actions are (1) writing a new symbol in the cell; (2) moving to an adjacent cell; (3) switching to a new state.

6.2 Finite State Automata

As a warm-up, we will first consider a simplified version of a Turing machine, known as a finite state automaton. A finite state automaton over a finite alphabet Σ (usually $\{0,1\}$) is given by a finite set of states $Q = \{q_0, q_1, \ldots, q_k\}$ and a transition function $\delta : Q \times \Sigma \rightarrow Q$. There may also be an output function $F : Q \times \Sigma \rightarrow \Sigma$. The state q_0 is designated as the *initial* state and there may be a set $A \subseteq Q$ of *accepting* states.

The action of a finite automaton M on input $w = a_0 a_1 \ldots a_k$ occurs in *stages*.

- At stage 0, the machine begins in state q_0, scans the input a_0, and then and transitions to state $s_1 = \delta(q_0, a_0)$, possibly writing $b_0 = F(q_0, a_0)$.

- At stage n, the machine (in state s_n) scans a_n, transitions to state $s_{n+1} = F(s_n, a_n)$, and possibly writes $b_n = F(s_n, a_n)$.

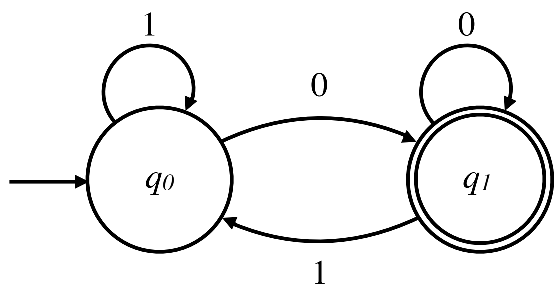
- After reading a_k during stage k, the machine halts in state s_{k+1}. The input word w is *accepted* by M if $s_{k+1} \in A$. If there is an ouput function, then the output word will be written as $M(w) = b_0 b_1 \ldots b_k$.

The language $L(M)$ is the set of words accepted by M. We will sometimes refer to such a collection as a *regular language*.

Example 6.2.1. Let M_1 be the machine, depicted by the state diagram below, with transition function

$$\delta(q_i, j) = \begin{cases} q_{1-i} & \text{if } i = j \\ q_i & \text{if } i \neq j, \end{cases}$$

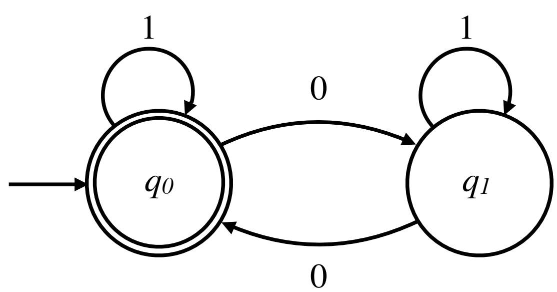
for $i = 0, 1$ and $j = 0, 1$. Thus $L(M_1)$ is the set of words which end in a 0.



Example 6.2.2. Let M_2 be the machine, depicted by the state diagram below, with transition function

$$\delta(q_i, j) = \begin{cases} q_{1-i} & \text{if } j = 0 \\ q_i & \text{if } j = 1, \end{cases}$$

for $i = 0, 1$. Thus $L(M_2)$ is the set of words which contain an even number of 0's.

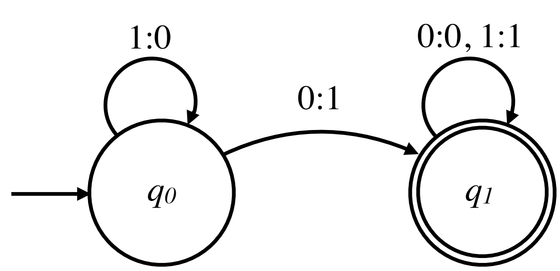


Next we consider some finite state automata which compute functions. These are called finiet state *transducers*. We want to express natural numbers in reverse binary notation, so that the word $a_0a_1 \dots a_k$ represents $a_0 + 2a_1 + \cdots + 2^k a_k$.

Example 6.2.3. The successor machine M_3 computes $S(x) = x + 1$. State q_0 is the *carry* state and state q_1 is the *no-carry* state. The edges in the state diagram below are labelled with symbols of the form $i : j$, which means that i is an input bit and j is an output bit.

For reasons that will be clear shortly, we require that any number of the form $2^n - 1$ (normally represented by a string of the form 1^n) to be represented by 1^n0. For any other number, adding additional zeros to the end of its representation will make no difference.

Starting in state q_0, the machine outputs $(i + 1) \mod 2$ on input i and transitions to state q_{1-i}. From state q_1 on input i, the machine outputs i and remains in state q_1. We take the liberty of adding an extra 0 at the end of the input in order to accommodate the carry.



More precisely, the transition function and output function of M_3 are defined by

$$\delta(q_i, j) = \begin{cases} q_0 & \text{if } i = 0 \ \& \ j = 1 \\ q_1 & \text{if } (i = 0 \ \& \ j = 0) \vee i = 1 \end{cases}$$

and

$$F(q_i, j) = \begin{cases} 1 - j & \text{if } i = 0 \\ j & \text{if } i = 1 \end{cases}.$$

We see that this computes $S(x)$ by the following reasoning. Assume that x ends with 0 (if x represents the number $2^n - 1$ for some n, then this terminal 0 is necessary to end up in an accepting state; if x does not, then the terminal 0 is inconsequential).

We consider two cases:

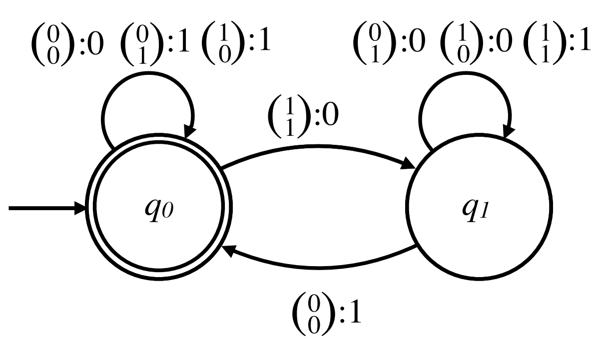
- Case 1: x contains a 0. Let i be least such that $a_i = 0$. Then M will write $i-1$ 1's and remain in state q_0 until arriving at the i-th bit, which is a 0. Then it will output $b_i = 1$ and transition to state q_1. After that it will simply write $b_j = a_j$ for all $j > i$.
- Case 2: If x consists of all 1's, then M will write n 0's and remain in state q_0 until reaching the extra 0, when M will output a final 1 and transition to q_1.

In each case, $b_1 \ldots b_n = M(x)$ is the reverse binary representation of $S(x)$ and we will end up in an accepting state.

Using the carry and no-carry states, we can also perform addition with a finite state automaton.

Example 6.2.4. The addition machine M_4 computes $S(x, y) = x + y$. Unlike the previous example, state q_0 is the *no-carry* state and state q_1 is the *carry* state. Moreover, we work with a different input alphabet: the input alphabet consists of pairs $\binom{i}{j} \in \{0,1\} \times \{0,1\}$. To add two numbers n_1 and n_2 represented by σ_1 and σ_2, if σ_1 is shorter than σ_2, we append 0s to σ_1 so that the resulting string has the same length as σ_2. As in the previous example, we will also append an additional 0 to the end of σ_1 and σ_2, which is necessary in case that the string representing $n_1 + n_2$ is strictly longer than the strings representing n_1 and n_2.

The state diagram is given by:



The transition function and output function of M_3 are defined in the following tables:

δ	$\binom{0}{0}$	$\binom{0}{1}$	$\binom{1}{0}$	$\binom{1}{1}$
q_0	q_0	q_0	q_0	q_1
q_1	q_0	q_1	q_1	q_1

F	$\binom{0}{0}$	$\binom{0}{1}$	$\binom{1}{0}$	$\binom{1}{1}$
q_0	0	1	1	0
q_1	1	0	0	1

For any finite state automaton M, the set $L(M)$ of words accepted by M is a computable set. Moreover, if M computes a function f, then f is also computable. However, many computable sets and functions cannot be computed by a finite automaton.

Proposition 6.2.5. *$L = \{a^n b^n : n \in \omega\}$ is not a regular set.*

Proof. Suppose by way of contradiction that $L = L(M)$ for some FSA M and let k be the number of states of M. Consider the sequence of states $s_0, s_1, \dots, s_k$ resulting when M reads the input a^k. It follows from the pigeonhole principle that $s_i = s_j$ for some $i < j \leq k$. Thus M ends up in the same state s_i after reading a^i and after reading a^j. But this means that M ends up in the same state q after reading $a^i b^i$ that it ends up in after reading $a^j b^i$. By assumption, M accepts $a^i b^i$, so that q is an accepting. However, M does not accept $a^i b^j$, so that q is not accepting. This contradiction shows that $L \neq L(M)$. □

Example 6.2.6. The function $f(x) = x^2$ is not computable by a finite state transducer. Suppose that M computes x^2 on input x, where we have appended a sufficient number of 0's to the end of the input so that we can output x^2 (recall that each input bit yields at most one output bit). In particular, for any n, M will output $0^{2n}1$ on input 0^n1, since $f(2^n) = (2^n)^2 = 2^{2n}$. Thus, on input 0^n1, after reading the first $n+1$ bits, M needs to examine at least n additional 0's and write at least n additional 0's before it finishes the output with a final 1. Now suppose that M has k states and let $n > k+1$. Let s_j be the state of the machine after reading 0^n10^j. Then there must be some $i, j \leq k+1 < n$ such that $s_i = s_j$. Furthermore, every time M transitions from state s_i upon reading a 0, M must output a 0. But the machine is essentially stuck in a *loop* and hence can only print another 0 after reading 0^n10^n when it needs to print the final 1.

6.3 Exercises

1. Define a finite automaton M such that $L(M)$ is the set of words from $\{0,1\}^*$ such that all blocks of zeroes have length a multiple of three.

2. Define a finite automaton M such that $L(M)$ is the set of words from $\{0,1\}^*$ such that every occurence of 11 is followed by 0.

3. Show that there is no finite automaton M such that $L(M)$ is the set of words with an equal number of 1's and 0's.

4. Define a finite automaton M on the alphabet $\{0,1,2\}$ such that $M(w)$ is the result of erasing all 0's from w.

5. Define a finite automaton M such that $M(w) = 3 \cdot w$ where w is a natural number expressed in reverse binary form.

6. Show that if L_1 and L_2 are regular languages, then $L_1 \cup L_2$ is a regular language.

7. Show that if L is a regular language, then L^* is a regular language.

   Hint: Use a non-deterministic FSA.

8. Show that the set $\{1^n : n \text{ is a square}\}$ is not a regular language.

6.4 Turing Machines

A Turing machine is a simple model of a computer that is capable of computing *any* function that can be computed. It consists of these items:

1. a finite state control component with a finite number of read/write heads; and
2. a finite number of unbounded memory tapes (one or more for the input(s), one for the output, and the rest for scratchwork), each of which is divided into infinitely many consecutive squares in which symbols can be written.

Furthermore, it must satisfy these conditions:

1. there is a specified initial state q_0 and a specified final q_H, or *halting*, state; and
2. each read/write head reads one cell of each tape and either moves one cell to the left (L), moves one cell to the right (R), or stays stationary (S) at each stage of a computation.

The notion of Turing machine is formalized in the following definition.

Definition 6.4.1. A k-tape *Turing machine* M for an alphabet Σ consists of

(i) a finite set $Q = \{q_0, \dots, q_n\}$ of states;

(ii) an alphabet Σ;

(iii) a transition function $\delta : Q \times \Sigma^k \to \Sigma^k \times \{\text{L,R,S}\}^k$;

(iv) $q_0 \in Q$ is the start state; and

(v) $q_H \in Q$ is the halting or final state.

A *move* of M in a given state q_i, scanning the symbols $a_1, \dots, a_k$ on the k tapes, where $\delta(q_i, a_1, \dots, a_k) = (q_j, b_1, \dots, b_k, D_1, \dots, D_k)$, consists of the following actions:

1. switching from state q_i to state q_j;
2. writing b_i (and thus erasing a_i) on tape i; and
3. moving the head on tape i in the direction D_i.

A *computation* always begins with

(i) the machine in state q_0;

(ii) some finite input on each of the input tapes; and

(iii) each of the input heads scanning the first symbol on each of the input tapes.

The *configuration* of a machine at a given stage of a computation consists of

(i) the current state of the machine;

(ii) the contents of each tape; and

(iii) the location of each of the heads on each tape.

M machine *halts* after n moves if it transitions to the halting state in the n-th stage of the computation. A machine M *accepts* a word w, denoted $M(w)\downarrow$, if the machine halts (i.e. ends up in the halting state) when given the input w. In this case, we say M *halts* on the input w. Otherwise, we say that M *diverges* on input w, denoted $M(w)\uparrow$.

It is an essential feature of Turing machines that they may fail to halt on some inputs. This gives rise to the *partial computable function* f_M which has domain $\{w : M(w)\downarrow\}$. That is, $f_M(w) = y$ if and only if M halts on input w and y appears on the output tape when M halts, meaning that the output tape contains y surrounded by blanks on both sides. (For the sake of elegance, we may insist that the first symbol of y is scanned at the moment of halting.) Sometimes we will write the value $f_M(w)$ as $M(w)$.

Example 6.4.2. We define a Turing machine M_1 that computes $x+y$. There are two input tapes and one output tape. The numbers x and y are written on separate input tapes in reverse binary form. M has the states: the initial state q_0, a *carry* state q_1 and the halting state q_H. The two input tapes are read simultaneously in the form a/b. We need to consider blanks squares as symbols # in case one input is longer and/or there is a carry at the end. The behavior of M_1 is summed up in the following table.

State	Read	Write	Move	New State
q_0	0/0	0	R	q_0
q_0	0/#	0	R	q_0
q_0	#/0	0	R	q_0
q_0	0/1	1	R	q_0
q_0	1/0	1	R	q_0
q_0	1/#	1	R	q_0
q_0	#/1	1	R	q_0
q_0	1/1	0	R	q_1
q_0	#/#	#	S	q_H
q_1	0/0	1	R	q_0
q_1	0/#	1	R	q_0
q_1	#/0	1	R	q_0
q_1	0/1	0	R	q_1
q_1	1/0	0	R	q_1
q_1	1/#	0	R	q_1
q_1	#/1	0	R	q_1
q_1	1/1	1	R	q_1
q_1	#/#	1	S	q_H

Example 6.4.3. We roughly describe a Turing machine M_2 that computes $x \cdot y$. Again there are two input tapes and one output tape. The idea is that if $y = \sum_{i \in I} 2^i$ for some finite $I \subseteq \mathbb{N}$, then $x \cdot y = \sum_{i \in I} 2^i \cdot x$. For each i, $2^i \cdot x$ has the form of i 0's followed by x. For example, if $x = 1011$ (thirteen), then $4 \cdot x = 001011$ (fifty-two).

To multiply 1011 by 100101 we add

$$2^0 \cdot 13 + 2^3 \cdot 13 + 2^5 \cdot 13 = 1011 + 0001011 + 000001011.$$

We begin in state q_0 with x and y each written on one of the input tapes (tapes 1 and 2) in reverse binary notation, and all three reader heads lined up. We first add a terminal 0 to the end of y, which is necessary to ensure that our computation will halt.

Suppose there are k initial 0's in y. For each such 0, we replace it with a #, write a 0 on tape 3 in the corresponding cell, move the heads above tapes 2 and 3 one cell to the right, and stay in state q_0.

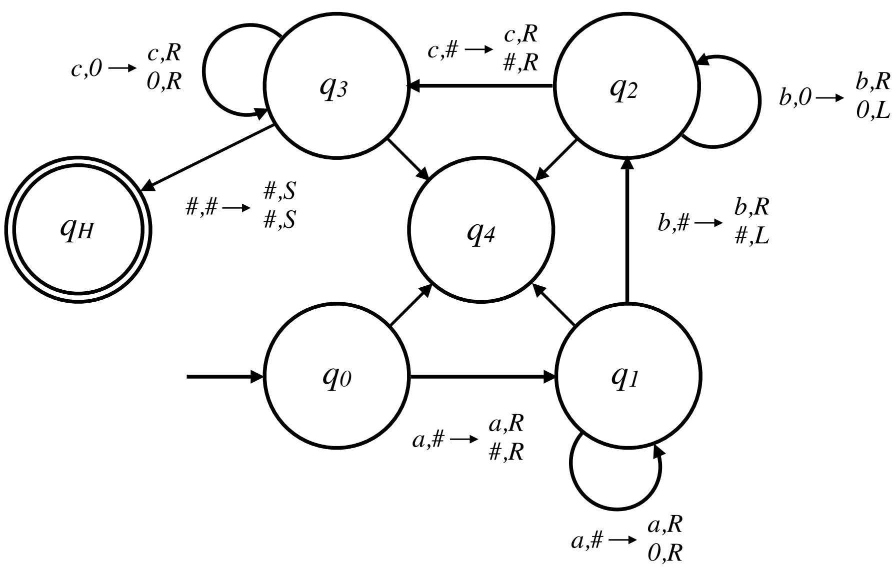
When we encounter the first 1 of y, we replace this 1 with a # and transition to a state q_C in which we copy x to tape 3 (the output tape), beginning in the i-th cell of this tape. As the contents of tape 1 are being copied onto tape 3, we also require that head above tape 2 moves in the same directions as the head above tapes one and three until we encounter the first # on tape one. (One can verify that the heads above tapes 2 and 3 will always be lined up). We then transition to a state in which we reset the position of the tape 1 head to scanning the first bit of x, and then the tape 2 head is scanning the leftmost bit in y that has not been erased (i.e. replaced with a #). As above, we require that the tape 3 head moves in the same directions as the tape 2 head during this phase (which will ensure that we begin copying x in the correct place on the tape if the next bit of y happens to be a 1. We then transition back to state q_0 and continue as before.

Suppose that the next bit of y that is equal to 1 is the i-th bit of y. Then we start the addition by adding the first bit of x to the i-th bit of the output tape. We note that we may need to utilize a carry state during the addition. When we reach the # at the end of y, the computation is complete, and we transition to the halting state. For an example of the product $7 \ldots 8 = 7 \cdot 8 = 56$, see the supplementary document

Like finite state automata, Turing machines can be represented a state diagram.

Example 6.4.4. There is a Turing machine that accepts every string in the set $\{a^n b^n c^n \mid n > 0\}$ over the alphabet $\Sigma = \{a, b, c, 0, \#\}$.

Here we will use the "0" as a special marking symbol, although we can do without it (and use "a" instead). The following state diagram gives a Turing machine which accepts this every string in the above set.



Observe that q_4 functions as a reject state. Any input not listed on any of the edges will cause a transition to state q_4. To see that this Turing machine accepts the desired set, see the supplementary document for an example with input $aaaabbbbcccc$.

Example 6.4.5. We define a Turing machine M that computes the function $f(x) = |x|$ (the length of a string x). There are two tapes, the input tape and the output tape. The input tape is read-only but we allow writing on the output tape. Let the input alphabet be $\Sigma = \{a\}$. Let α/β indicate that the machine is currently reading α on the input tape and β on the output tape. Similarly D_1/D_2 indicates that the head on the input tape moves in direction D_1 while the head on the output tape moves in direction D_2. The idea is to add one to the output tape after reading each symbol of the input tape. State q_1 arises when we need to add one to the output by carrying. State q_2 simply brings the output tape back to the first bit. Certain transitions are omitted from the table since they lead to divergent computation (we will assume that incorrect inputs will immediately cause a transition to a reject state). For example, we only get to state q_1 when we have read a on the first tape and we continute to read that a, so that the input $\#/0$ is not a legal input when we have reached state q_1.

The following table describes the behavior of the three main states.

state	read	write	move	new state
q_0	$a/\#$	$a/1$	R/S	q_0
q_0	$a/0$	$a/1$	R/S	q_0
q_0	$a/1$	$a/0$	S/R	q_1
q_0	$\#/0$	$\#/0$	S/S	q_H
q_0	$\#/1$	$\#/0$	S/S	q_H
q_0	$\#/\#$	$\#/0$	S/S	q_H
q_1	$a/\#$	$a/1$	S/L	q_2
q_1	$a/0$	$a/1$	S/L	q_2
q_1	$a/1$	$a/0$	S/R	q_1
q_2	$a/\#$	$a/\#$	R/R	q_0
q_2	$a/0$	$a/0$	S/L	q_2
q_2	$a/1$	$a/1$	S/L	q_2

We now prove some general facts about certain kinds of languages that are central to the study of Turing machines.

For a fixed alphabet Σ, a *language* is simply a set $L \subseteq \Sigma^*$ (recall that Σ^* is the collection of all finite sequences of elements of Σ).

Definition 6.4.6. A language L is said to be *Turing semicomputable* if there is a Turing machine M such that $L = \{w : M(w)\downarrow\}$. L is a *Turing computable* language if the characteristic function of L is Turing computable.

Example 6.4.7. Here is a simple Turing machine M such that $M(w)\downarrow$ if and only if w contains a 0. In state q_0, M moves right and remains in state q_0 upon reading a 1 or a blank. M immediately halts upon reading a 0.

Proposition 6.4.8. *Every Turing computable language is also Turing semicomputable.*

Proof. Let M be a Turing machine that computes the characteristic function of L. We modify M to define a machine M' as follows. First we introduce new states q_A and q_B. Replace any transition that goes to the halting state q_H with a transition that goes the the state q_A. For the q_A state, add two transitions. If the output tape reads 1, then transition to the halting state q_H. If the output tape reads 0, then move the output tape head one cell to the right and transition to state q_B. In state q_B, move the output tape head one cell to the left and return to state q_A. Then $M'(w)$ will halt if and only if $M(w) = 1$ and will endlessly alternate between states q_A and q_B if and only if $M(w) = 0$. □

Proposition 6.4.9. *L is Turing computable if and only if both L and its complement are Turing semicomputable.*

Proof. First observe that if $L \subseteq \Sigma^*$ is Turing computable, then $\Sigma^* \setminus L$ is Turing computable. Indeed, if M computes the characteristic function of L, then define M' to be the machine that behaves exactly like M except that for $i = 0, 1$ whenever M writes i on its output tape, M' writes $1 - i$ on its output tape. It follows from Proposition 6.4.8 that if L is Turing computable, then both L and its complement are semicomputable.

Now suppose that $L = \{w : M_0(w)\downarrow\}$ and that $\Sigma^* \setminus L = \{w : M_1(w)\downarrow\}$ for two Turing machines M_0 and M_1. We define a Turing machine M such that the function f_M computed by M is the characteristic function of L. Suppose for the sake of simplicity that M_0 and M_1 each have one input tape and have no output tape. Then M will have one input tape, two scratch tapes, and one output tape. The states of M will include pairs (q, q') where q is a state of M_0 and q' is a state of M_1. Given w on the input tape of M, M will begin by copying w onto each of the scratch tapes and transitioning to the pair (q_0, q_0') of initial states of M_0 and M_1. On the first scratch tape, M will simulate M_0, while on the second scratch tape, M will simulate M_1. Eventually M will enter a state of one of the two forms: (q_H, r), where q_H is the halting state of M_0 and r is a non-halting state of M_1, or (q, q_H'), where q_H' is the halting state of M_1 and q is a non-halting state of M_0.

- If M enters a state of the form (q_H, r), this means that the machine M_0 has halted on w, and hence $w \in L$. M will thus write a 1 on its output tape and transition to its halting state.
- If M enters a state of the form (q, q'_H), this means that the machine M_1 has halted on w, and hence $w \notin L$. M will thus write a 0 on its output tape and transition to its halting state.

Note that M will never enter the state (q_H, q'_H), since M_0 and M_1 can never accept the same word. It thus follows that M computes the characteristic function of L. □

Given two Turing machines M_0 and M_1, we write $M_0(x) \simeq M_1(x)$ to mean that (i) $M_0(x)\downarrow$ if and only if $M_1(x)\downarrow$ and (ii) $M_0(x)\downarrow$ implies that $M_0(x) = M_1(x)$.

Theorem 6.4.10. *Fix a finite alphabet $\Sigma = \{0, 1, q, \#, *, L, R, S, H, \downarrow\}$. There is a* universal *Turing machine $U : \Sigma^* \times \Sigma^* \to \Sigma^*$ such that, for any Turing machine $M : \{0,1\}^* \to \{0,1\}^*$, there exists $w_M \in \Sigma^*$ such that, for all inputs $x \in \{0,1\}^*$, $U(w_M, x) \simeq M(x)$.*

We will only sketch the main ideas in the proof of Theorem 6.4.10. To simplify matters, we will use the following lemma.

Lemma 6.4.11. *Let M be a k-tape Turing machine for some $k > 1$. Then there is a single-tape Turing machine M' such that $M(x) \simeq M'(x)$ for all $x \in \Sigma^*$.*

Before beginning the proof, we want to consider further the notion of a Turing machine configuration. Suppose that M is a single-tape Turing machine working on the alphabet $\Sigma = \{0, 1\}$ with states $q_0, q_1, \dots, q_k$. The computation of $M(w)$ on an input w is accomplished in a series of steps which may be given by a *configuration*. The configuration $ua * q_i \downarrow bv$ indicates that M is in state q_i, that the word $uabv$ is written on the tape and that the head is located at the symbol b; here u and v are words in the language $\{0,1\}^*$. The *next* configuration of M in the computation may be determined by looking at the transition $\delta(q_i, b)$. This configuration indicates the current state of M, the symbols written on the tape (for any squares which have been used), and the position of the pointer.

Proof of Theorem 6.4.10. (Sketch) We define a universal Turing machine with two input tapes, a scratch tape, and an output tape. For each machine M, we would like to define a word $w_M \in \Sigma^*$ that encodes all of the information of M. We will let w_M be the entire transition table of M written as one long string in the alphabet Σ^*, with a $*$ separating each entry on a given row of a table and $**$ separating each row of the table. The string $q0$ will stand for the initial state, qH will stand for the halting state, and all other states will be coded by a q followed by a finite string of 1s ($q1$, $q11$, $q111$, etc.)

Now, given the code w_M for a machine M, to compute $U(w_M, x)$ (i.e. $M(x)$), U proceeds by writing the initial configuration of M with input x on its scratch tape. Suppose, for example that $x = 000$. Then U will write

$$q0 \downarrow 000$$

where the $\downarrow$ specifies that the reader head is above the first 0 on the input tape of M. To proceed, U simply consults the transition table w_M written on its first input tape and writes the resulting configuration of M after one move on its scratch tape. Continuing the example from above, if the first move of M is to change to state q_1, replace the first 0 of x with a 1 and move the head one cell to the right, the resulting configuration will be

$$1 * q1 \downarrow 00$$

Continuing in this way, if $M(x)\downarrow$, then U will eventually come to a stage in which a halting configuration is written on its scratch tape. In this halting configuration, the value $y = M(x)$ will be written, and so U can simply copy this value y to its output tape and transition to its own halting state. Lastly, if $M(x)\uparrow$, then $U(w_M, x)\uparrow$.

□

It is important to note that our use of the alphabet Σ in the definition of a universal Turing machine is not strictly necessary. For instance, we can also define a universal Turing machine $U : \{0,1\}^* \times \{0,1\}^* \to \{0,1\}^*$ by representing each of the symbols in Σ as a unique binary string. Some caution is necessary to make sure the coding is unambiguous. We will not discuss the details of such a coding here, but we will assume that for each Turing machine M, there is some unique $w_M \in \{0,1\}^*$ that codes M. Moreover, we will assume that every $x \in \{0,1\}^*$ codes some Turing machine, which we will write as M_x. (We will justify these assertions in Chapter 7 when we discuss Gödel numbering.)

Programs nearly always have bugs, so they may not do want we want them to do. The problem of determining whether a given Turing machine M halts on input string w is the *Halting Problem*. Let us define the halting set to be $H = \{(x, y) \in \{0,1\} * \times \{0,1\}^* : M_x(y)\downarrow\}$. Observe that H is semicomputable: $(x, y) \in H$ if and only if $U(x, y)\downarrow$. By contrast, we have the following.

Theorem 6.4.12. *The Halting Problem is not computable.*

Proof. We will show that the complement of H is not semicomputable, so that by Proposition 6.4.9, H is not computable. Suppose by way of contradiction that there is a Turing machine M such that, for all $x, y \in \{0,1\}^*$,

$$M(x, y)\downarrow \iff M_x(y)\uparrow.$$

We can define a Turing machine N so that $N(w) \simeq M(w, w)$. Then

$$N(w)\downarrow \iff M_w(w)\uparrow.$$

But this Turing machine N must have some code e. So for all w,

$$M_e(w)\downarrow \iff M_w(w)\uparrow.$$

The contradiction arises when $w = e$. □

Thus there exists a set which is semicomputable but not computable.

6.5 Recursive Functions

In this section, we define the primitive recursive and the (partial) recursive functions and show that they are all Turing computable. Each function f maps from $\mathbb{N}^k$ to $\mathbb{N}$ for some fixed k (the *arity* of f).

Definition 6.5.1. The collection of *primitive recursive functions* is the smallest collection $\mathscr{F}$ of functions from $\mathbb{N}^k$ to $\mathbb{N}$ for each $k > 0$ that includes the following *initial functions*

1. the constant function $c(x) = 0$,
2. the successor function $s(x) = x + 1$,
3. the projection functions $p_i^k(x_1, \dots, x_k) = x_k$ for each $k \in \mathbb{N}$ and $i = 1, \dots, k$,

and are closed under the following schemes for defining new functions:

(4) (composition) if $f : \mathbb{N}^k \to \mathbb{N}$ and $g_i : \mathbb{N}^j \to \mathbb{N}$ for $i = 1, \dots, k$ (where $j, k > 0$) are in $\mathscr{F}$, then the function $h : \mathbb{N}^j \to \mathbb{N}$ defined by

$$h(x_1, \dots, x_j) = f(g_1(x_1, \dots, x_j), \dots, g_k(x_1, \dots, x_j))$$

is in $\mathscr{F}$, and

(5) (primitive recursion) if $g : \mathbb{N}^k \to \mathbb{N}$ and $h : \mathbb{N}^{k+2} \to \mathbb{N}$ are in $\mathscr{F}$, then the function $f : \mathbb{N}^{k+1} \to \mathbb{N}$ defined by

$$f(0, x_1, \ldots, x_k) = g(x_1, \ldots, x_k)$$
$$f(n+1, x_1, \ldots, x_k) = h(n, x_1, \ldots, x_k, f(n, x_1, \ldots, x_k))$$

is in $\mathscr{F}$.

Example 6.5.2.

1. For any constant $c \in \mathbb{N}$, the function $h(x) = c$ is primitive recursive. The proof is by induction on c. For $c = 0$, this follows from the fact that the initial function $c(x) = 0$ is primitive recursive. Supposing that $g(x) = c$ is primitive recursive and using the fact that $s(x) = x+1$ is primitive recursive, we can use composition to conclude that $h(x) = s(g(x)) = g(x)+1 = c+1$ is primitive recursive.

2. For any k and any c, the constant function $f : \mathbb{N}^k \to \mathbb{N}$ with $f(x_1, \ldots, x_k) = c$ is primitive recursive. We have $h(x) = c$ by (1) and we have $p_1^k(x_1, \ldots, x_k) = x_1$ as a basic function, so that $f(x_1, \ldots, x_k) = h(p_1^k(x_1, \ldots, x_k)) = h(x_1) = c$ is also primitive recursive.

3. The addition function $f(x, y) = x + y$ is primitive recursive. Let $g(y) = p_1^1(y)$ and $h(x, y, z) = c(p_3^3(x, y, z))$. Then f is given by

$$f(0, y) = g(y) = y$$
$$f(n+1, y) = h(n, y, f(n, y)) = f(n, y) + 1.$$

4. The *predecessor* function $f(x) = x \dot{-} 1 = \begin{cases} x-1, & \text{if } x > 0 \\ 0, & \text{if } x = 0 \end{cases}$ is primitive recursive. Let $g(x) = c(x)$ and $h(x, y) = p_1^2(x, y)$. Then f is given by

$$f(0) = g(y) = 0$$
$$f(n+1) = h(n, f(n)) = n.$$

5. The truncated subtraction function $f(x, y) = \begin{cases} y \dot{-} x, & \text{if } x \leq y \\ 0, & \text{otherwise} \end{cases}$ is primitive recursive. Let $g(y) = p_1^1(y)$ and $h(x, y, z) = z \dot{-} 1$. Then f is given by

$$f(0, y) = g(y) = y$$
$$f(n+1, y) = h(n, y, f(n, y)) = f(n, y) \dot{-} 1.$$

6. The function $sg = \begin{cases} 0, & \text{if } x = 0 \\ 1, & \text{if } x > 0 \end{cases}$ is clearly primitive recursive.

7. The multiplication function $f(x, y) = x \cdot y$ is primitive recursive. Let $g(x) = c(x) = 0$ and let $h(x, y, z) = p_3^3(x, y, z) + p_2^3(x, y, z)$. Then f is given by

$$f(0, y) = g(y) = 0$$
$$f(n+1, y) = h(n, y, f(n, y)) = f(n, y) + y.$$

We can extend the collection of primitive recursive functions to the collection of partial recursive functions by adding one additional scheme for defining new functions from previously defined ones. It is this scheme that allows for the possibility that a function be undefined on a given input. Given a function $f : \mathbb{N}^k \to \mathbb{N}$, if f is defined on input $(x_1, \ldots, x_k)$, we write $f(x_1, \ldots, x_k)\downarrow$; if f is undefined on $(x_1, \ldots, x_k)$, we write $f(x_1, \ldots, x_k)\uparrow$.

Definition 6.5.3. The collection of *partial recursive functions* is the smallest collection $\mathscr{F}$ of functions from $\mathbb{N}^k$ to $\mathbb{N}$ for each $k > 0$ that includes the primitive recursive functions and is closed under the following scheme:

(6) (unbounded search) if $g : \mathbb{N}^{k+1} \to \mathbb{N}$ is in $\mathscr{F}$, then the function $f : \mathbb{N}^k \to \mathbb{N}$ defined by

> $f(x_1, \ldots, x_k) =$ the least n such that $g(n, x_1, \ldots, x_k) = 0$ and $g(i, x_1, \ldots, x_k)\downarrow$ for $i = 0, \ldots n$ (and $f(x_1, \ldots, x_k)\uparrow$ otherwise)

is in $\mathscr{F}$.

We will refer to total recursive functions simply as recursive functions.

We say that f is defined from the total function g by *bounded search* if $f(n, x)$ equals the least $i < n$ such that $g(i, x) = 0$ and otherwise $f(n, x) = n$. Certainly if g is a recursive function, then f will be recursive function. We note that if we add a scheme of *bounded search* to the collection of primitive recursive functions, we do not add any new functions.

Lemma 6.5.4. *If g is primitive recursive and f is defined from g by bounded search, then f is primitive recursive.*

Proof. We have $f(0, x) = 0$ and for each n,

$$f(n+1, x) = \begin{cases} f(n, x) & \text{if } g(f(n, x), x) = 0 \\ n+1 & \text{otherwise} \end{cases}.$$

□

The collection of partial recursive functions is equivalent to the collection of Turing computable functions, in the sense that every partial recursive function can be computed by a Turing machine, and every Turing computable function is partial recursive. We will prove one direction of this equivalence.

Theorem 6.5.5. *Every partial recursive function can be computed by a Turing machine.*

Proof. The proof is by induction on the family of partial recursive functions. For the base case, it is clear that the initial functions are Turing computable. We now verify that the schemes of composition, primitive recursion, and unbounded search yield Turing computable functions when applied to Turing computable functions.

Composition. For simplicity let $f(x) = h(g(x))$ where h is computable by machine M_h and g is computable by machine M_g. Assume without loss of generality that each machine uses one input tape and one output tape and that the sets of states of the two machines are disjoint. We define a machine M_f that computes f with six tapes:

1. the input tape;
2. a scratch tape to serve as the input tape of g;
3. a scratch tape to serve as the output tape of g;
4. a scratch tape to serve as the input tape for h;
5. a scratch tape to serve as the output tape of h; and

6. an output tape.

M will have the states of g and h together plus a few new states to handle the transfer from M_g to M_h. The transition function for M_g will be changed so that instead of halting when the output $g(x)$ is ready, M will go into a subroutine which copies from the g-output tape to the h-input tape and then hands over the controls to M_h. The halting states of M will be the halting states of M_h.

Primitive Recursion. For simplicity let $f(0,x) = g(x)$ and $f(n+1,x) = h(n,x,f(n,x))$. Let M_g compute g and let M_h compute h. Assume without loss of generality that each machine uses one input tape and one output tape and that the sets of states of the two machines are disjoint. We define a machine M that computes $f(n,x)$ with nine tapes:

1. the input tape for n;
2. the input tape for x;
3. a scratch tape to serve as the input tape for g;
4. a scratch tape to serve as the output tape of g;
5. a tape to keep track of the ongoing value of $m < n$;
6. a tape to keep track of the ongoing value of $f(m,x)$;
7. a scratch tape to serve as the input tape for h;
8. a scratch tape to serve as the output tape of $h(m,x,f(m,x))$; and
9. an output tape.

M will have the states of g and h together plus a few new states to handle the transfer from M_g to M_h and the ongoing recursion. The transition function for M_g will be changed so that instead of halting when the output $g(x)$ is ready, M will copy the value $g(x)$ onto tape (6), write $m = 0$ onto tape (5), and then hand over control to M_h. The inputs for M_h are found on tapes (2), (5) and (6). M_h uses these to compute $h(m,x,f(m,x)) = f(m+1,x)$. When M_h is ready to halt and give its output, M does the following:

(i) M compares m from tape (5) with n from tape (1); if $n = m+1$, then the value on tape (8) equals the desired $f(n,x)$, so M copies this to tape (9) and halts.

(ii) Otherwise, M erases tape (6) and then copies the value from tape (8) onto tape (6).

(iii) Then M adds one to the value of m on tape (5), erases tapes (7) and (8) and hands control back to M_h again.

Unbounded Search. For simplicity let $f(x) =$ the least n such that $g(n,x)\downarrow = 0$ and for all $i \leq n$ $g(i,x)\downarrow$. Let M_g compute g using one input tape and one output tape.

We define a machine M that computes $f(x)$ with five tapes:

1. the input tape for x;
2. a tape for the ongoing value of n;
3. a scratch tape to serve as the input tape for g;
4. a tape to keep track of the ongoing value of $g(n,x)$; and
5. an output tape.

M will have the states of g plus a few new states to handle the the ongoing computations of $g(n,x)$. M begins by writing $n = 0$ on tape (2) and handing control to M_g. The transition function for M_g will be changed so that instead of halting when the output $g(x)$ is ready, M will do the following:

(i) Compare the value $g(n,x)$ from tape (4) with 0. If $g(n,x) = 0$, then the value n on tape (2) equals the desired $f(x)$, so M copies this to tape (5) and halts.

(ii) Otherwise, M increments the value of n on tape (2), erases tapes (3) and (4) and hands control back to M_g again.

□

For the other direction, we need to associate to each Turing machine a natural number. We will consider now the problem of coding words on a finite alphabet $\Sigma = \{a_1, \dots, a_k\}$ into words over $\{0,1\}^*$ and then into natural numbers.

For the first part, we can code the word $w = a_{i_1} a_{i_2} \cdots a_{i_n}$ into the string $0^{i}_{1}10^{i_2}1\cdots 0^{i_n}1$. For the second part, we can code the word $v = i_0 \cdots i_n \in \{0,1\}^*$ by the reverse binary natural number $i_0 \cdots i_n 1$. Note that this will work for an arbitrary finite alphabet and even for the potentially infinite alphabet $\mathbb{N}$. Let $< w >$ be the natural number code for the word w.

It is clear that we can use a Turing machine to compute the code $< w >$ for a string $w \in \{1, 2, \dots, k\}^*$ by going into state q_i when reading each symbol (i) and then writing i 0's followed by a 1 and returning to state q_0.

For the other direction, given a binary number $0^{r_1}0^{r_2}1\cdots 0^{r_k}1$, we can compute a sequence $n_0 = n, r_0, n_1, r_1, \dots, n_k, r_k$ as follows. Recall the primitive recursive functions $Q(a,b)$, the quotient when a is divided by b and the corresponding remainder $R(a,b)$, and note that b divides a if the remainder is 0. Now let r_0 be the least r such that 2^{r+1} does not divide n and then let $n_1 = Q(n, 2^{r_0}) - 1$. After this, let $r_{i+1} = (least r) R(n_i, 2^{r+1}) \neq 0$ and let $n_{i+1} = Q(n_i, 2^{r_i}) - 1$. Thus the function which computes r_i from n and i is computable.

For example, if $n = 0^3 10^5 101$, then $r_0 = 3, n_1 = 0^5 101, r_1 = 5, n_2 = 01, r_2 = 1, n_3 = 0$.

Then we may use the universal Turing machine U to provide an enumeration $\{M_e : e \in \omega\}$ of all Turing computable functions by letting M_e be the machine whose program may be coded by the natural number e. Then the proof of Theorem 6.4.10 also proves the following.

Proposition 6.5.6. *For natural numbers s, e, w, let $F(s,e,w)$ be the natural number which codes the sth configuration in the computation of M_e on input word coded by w. Let $M^s_e(w)$ be the output given by M_e if the computation halts by stage s. Then*

1. *The function F is primitive recursive.*
2. *The set $\{(e,s,w) : M^s_e(w)\downarrow\}$ is primitive recursive.*
3. *The set $\{e,s,w,v) : M^s_e(w)$ is the string coded by $v\}$ is primitive recursive.*

If we consider Turing computable functions on natural numbers, then we have primitive recursive functions coding the string 1^n into $< 1^n >= (01)^n 1$ and back again, so the result above holds for natural numbers v and w.

Now given any Turing computable function M_e, $M_e(w)$ may be computed by searching for the least s such that $M^s_e(w)\downarrow$ and then the least $v \leq s$ such that $M^s_e(w) = v$.

Since we can code finite sequences as strings and hence as natural numbers, the equivalence of Turing computable functions from $\mathbb{N}^k \to \mathbb{N}$ easily follows. This completes the proof that any Turing computable function is recursive, and hence verifies this case of Church's Thesis, as described below.

All of the formalizations of the intuitive notion of a computable number-theoretic function has given rise to the same collection of functions. For this and other reasons, the community of mathematical logicians have come to accept the *Church-Turing thesis*, which is the claim that the collection of Turing computable functions is the same as the collection of intuitively computable functions. In practice, the Church-Turing thesis has two main consequences:

(1) if we want to show that a given problem cannot be solved by any algorithmic procedure, it suffices to show that solutions to the problem cannot be computed by any Turing computable functions;

(2) to show that a given function is computable, it suffices to give an informal description of an effective procedure for computing the values of the function.

Given the equivalence of the various notions of computable function, we will hereafter refer to the formal notions as *computable functions* and *partial computable functions* (as opposed to *recursive functions* and *partial recursive functions*).

We now recast the earlier definitions of computability and semi-computability in terms of natural numbers and introduce a new notion, namely, computable enumerability.

Definition 6.5.7. Let $A \subseteq \mathbb{N}$.

1. A is said to be *primitive recursive* if the characteristic function χ_A is primitive recursive.

2. A is said to be *computable* if the characteristic function χ_A is computable.

3. A is said to be *semi-computable* if there is a partial computable function ϕ such that $A = dom(\Phi)$.

4. A is said to be *computably enumerable* if there is some computable function f such that $A = ran(f)$. That is, A can be enumerated as $f(0), f(1), f(2), \dots$.

Example 6.5.8. The set of even numbers is primitive recursive since its characteristic function may be defined by $f(0) = 1$ and $f(n+1) = 1 - f(n)$.

Example 6.5.9. Define the functions Q and R as follows. Let $Q(a, b)$ be the quotient when b is divided by $a+1$ and let $R(a, b)$ be the remainder, so that $b = Q(a, b) \cdot (a+1) + R(a, b)$. Then both Q and R are primitive recursive. That is, $Q(a, b)$ is the least $i \leq b$ such that $i \cdot (a+2) \geq b$ and $R(a, b) = b - Q(a, b) \cdot (a+1)$.

Example 6.5.10. The relation $x \mid y$ (x divides y) is primitive recursive, since $x \mid y$ if and only if $(\exists q < y+1)\, x \cdot q = y$.

Example 6.5.11. The set of prime numbers is primitive recursive and the function P which enumerates the prime numbers in increasing order is also primitive recursive. To see this, note that $p > 1$ is prime if and only if $(\forall x < p)[x \mid p \to p = 1]$. Now we know that for any prime p, there is another prime $q > p$ with $q < p! + 1$. By one of the exercises, the factorial function is primitive recursive. Then we can recursively define P by $P(0) = 2$ and, for all i,

$$P(i+1) = (\text{least } x < P(i+1)! + 1)\ x \text{ is prime.}$$

We conclude this chapter with the following result.

Theorem 6.5.12. *A is computably enumerable if and only if A is semicomputable.*

Proof. Suppose first that A is computably enumerable. If $A = \emptyset$, then certainly A is semicomputable, so we may assume that $A = rng(f)$ for some computable function f. Now define the partial computable function ϕ by $\phi(x) = (\text{least } n) f(n) = x$ for $x \in \mathbb{N}$. Then $A = dom(\phi)$, so that A is semicomputable.

Next suppose that A is semicomputable and let ϕ be a partial computable function so that $A = dom(\phi)$. If A is empty, then it is computably enumerable. If not, select $a \in A$ and define the computable function f by

$$f(2^s \cdot (2m+1)) = \begin{cases} m, & \text{if } \phi(m)\downarrow \text{ in } < s \text{ steps,} \\ a, & \text{otherwise.} \end{cases}$$

Then $A = rng(f)$, so that A is computably enumerable. ☐

6.6 Exercises

1. Define a Turing machine M such that $M(w) = ww$ for any word $w \in \{a, b\}^*$. (For example, $M(aab) = aabaab$.)

2. Define a Turing machine M such that $M(u, v)$ halts exactly when $u = v$ for $u, v \in \{a, b\}^*$. (M begins with u on one input tape and with v on a second input tape. There might be a scratchwork tape.)

3. Define a Turing machine M which halts on input (u, v) if and only if $u \leq v$ (in the lexicographic order on alphabet $\{a, b, c\}$.

4. Define a Turing machine M such that $M(1^n) = n$, that is, M converts a string of n 1's into the reverse binary form of n.

5. Show that function $F(a, b) = LCM(a, b)$ is primitive recursive.

6. Show that the factorial function is primitive recursive.

7. Show that the general exponentiation function $f(x, y) = y^x$ is primitive recursive.

8. $S \subseteq \mathbb{N}^k$ is said to be Σ_1^0 if there is a computable relation R such that $S(x_1, \ldots, x_k) \iff (\exists y)R(y, x_1, \ldots, x_k)$. Prove that S is Σ_1^0 if and only if S is semi-computable.

9. Suppose that f is a (total) recursive function and let $g(x, y) = \prod_{i<y} f(x, i)$. Show carefully that g is also recursive. Use this to show that the function $f(n) = n!$ is recursive.

10. 13 Suppose that $f : \mathbb{N} \to \mathbb{N}$is a (total) recursive one-to-one function. Show that f^{-1} is also recursive.

Chapter 7

Decidable and Undecidable Theories

7.1 Introduction

In this chapter we bring together the two major strands that we have considered thus far: logical systems (and, in particular, their syntax and semantics) and computability theory. We will briefly discuss decidable and undecidable logical systems, but our primary focus will be decidable and undecidable first-order theories.

7.1.1 Gödel numbering

We began to consider coding of strings from a finite alphabet in Chapter 6 in order to define a universal Turing machine. In order to apply the tools of computability theory to the study of various logical systems and first-order theories, we need to represent the objects such as formulas and proofs as objects that we can give as input to computable functions (such as strings over some fixed alphabet or natural numbers). Here we will code formulas as natural numbers in the binary notation.To do so, we first need to code each symbol in our logical language as a natural number. Below are two such coding schemes for the symbols used in propositional and predicate logic.

A coding of the symbols of proposition logic:

symbol	code	symbol	code
(	1	$\vee$	5
)	2	$\rightarrow$	6
&	3	$\leftrightarrow$	7
$\neg$	4	P_i	$8+i$

A coding of the symbols of predicate logic:

symbol	code	symbol	code
(	1	$=$	8
)	2	$\exists$	9
&	3	$\forall$	10
$\neg$	4	v_i	$11+4i$
$\vee$	5	c_i	$12+4i$
$\rightarrow$	6	P_i	$13+4i$
$\leftrightarrow$	7	F_i	$14+4i$

Then, for instance, the propositional formula P_0 & P_1 will be represented by the number $0^8 10^3 10^9 1$.

7.2 Decidable vs. Undecidable Logical Systems

Let us consider some examples of decidable and undecidable logical systems.

Definition 7.2.1. A logical system is decidable if there is an effective procedure that, given the Gödel number of a sentence ϕ, outputs 1 if ϕ is logically valid and outputs 0 if ϕ is not logically valid.

For a given logical system such as propositional logic or predicate logic, we say that the *decision problem* for this system is the problem of determining whether a given formula is logically valid. Moreover, for a decidable logical system, we say that its decision problem is *solvable*; similarly, for an undecidable logical system, we say that its decision problem is *unsolvable*.

We now turn to our examples.

Example 7.2.2. *The decision problem for propositional logic is solvable.*

The method of truth-tables provides an algorithm for determining whether a given propositional formula ϕ is logically valid.

Example 7.2.3. *The decision problem for monadic predicate logic is solvable.*

Monadic predicate logic is first-order logic with only 1-place predicates such as $R(x), B(y)$, etc. A key result needed to show the decidability of monadic predicate logic is the following:

Theorem 7.2.4. *If ϕ is a sentence of monadic predicate logic consisting of k distinct monadic predicates and r distinct variables, then if ϕ is satisfiable, it is satisfiable in a model of size at most $2^k \cdot r$.*

As a corollary, we have:

Corollary 7.2.5. *If ϕ is a sentence of monadic predicate logic consisting of k distinct monadic predicates and r distinct variables and is not logically valid, then there is a model of size at most $2^k \cdot r$ in which $\neg\phi$ is satisfied.*

Thus to determine if a sentence of monadic predicate logic is logically valid, we must check to see whether ϕ is true in all models of cardinality less than some finite bound, which can be done mechanically.

We now turn to some examples of undecidable logical systems.

Example 7.2.6. *The decision problem for predicate logic is unsolvable.*

To show this, one can prove that, given the transition table of a Turing machine M, there is a finite set Γ_M of $\mathscr{L}$-sentences in some first-order language $\mathscr{L}$ and an additional collection of Ł-sentences $\{\phi_n\}_{n\in\mathbb{N}}$ such that for every n,

$$\Gamma_M \models \phi_n \Leftrightarrow M(n)\downarrow.$$

By the Completeness Theorem, it follows that

$$\Gamma_M \vdash \phi_n \Leftrightarrow M(n)\downarrow,$$

which is equivalent to

$$\vdash \Gamma_M \rightarrow \phi_n \Leftrightarrow M(n)\downarrow.$$

Now suppose there is an effective procedure that, given the Gödel number of any first-order sentence ϕ, will output a 1 if $\vdash \phi$ and outputs a 0 otherwise. Then for all sentences of the form $\Gamma_M \rightarrow \phi_n$, this procedure will output a 1 if and only if $\vdash \Gamma_M \rightarrow \phi_n$, which holds if and only if $M(n)\downarrow$. But this contradicts the unsolvability of the halting problem, and thus there can be no such effective procedure.

Example 7.2.7. *The decision problem for dyadic first-order logic is unsolvable.*

Dyadic predicate logic is first-order logic with only 2-place predicates such as $R(x, y), B(y, z)$, etc. One can in fact show that if our language includes just *one* 2-place predicate, this is sufficient to create a collection of sentences for which the decision problem is unsolvable.

7.3 Decidable Theories

In this section we identify a sufficient condition for a theory T to have an algorithm that enables us to determine the consequences of T. Then we will provide a specific example of a theory satisfying this condition, namely, the theory of dense linear orders without endpoints.

Hereafter, let us fix a first-order language $\mathscr{L}$. Given a set of $\mathscr{L}$-sentences S, we will often identify it with the set of Gödel numbers of the sentences in S. Thus, when we say that a set S of formulas is, say, computably enumerable, we really mean that the set of Gödel numbers of the formulas in S is computably enumerable.

Definition 7.3.1. An $\mathscr{L}$-theory Γ is *decidable* if there is an algorithm to determine for any $\mathscr{L}$-sentence φ whether $\Gamma \vdash \varphi$ or $\Gamma \nvdash \varphi$.

Recall that for an $\mathscr{L}$-theory Γ, $Th(\Gamma) = \{\phi : \Gamma \vdash \phi\}$.

Definition 7.3.2. Let Γ be an $\mathscr{L}$-theory.

(i) Γ is *finitely axiomatizable* if there is a finite set Σ of $\mathscr{L}$-sentences such that $Th(\Sigma) = Th(\Gamma)$.

(ii) Γ is *computably axiomatizable* if there is a computable set Σ of $\mathscr{L}$-sentences such that $Th(\Sigma) = Th(\Gamma)$.

Lemma 7.3.3. *If Γ is a computably axiomatizable $\mathscr{L}$-theory, then $Th(\Gamma)$ is computably enumerable.*

Proof Sketch. Let Σ be a computable collection of $\mathscr{L}$-sentences such that $Th(\Sigma) = Th(\Gamma)$. Since the collection of all possible proofs with premises from Σ is computably enumerable, the collection of $\mathscr{L}$-sentences that are the conclusion of some proof with premises from Σ is also computably enumerable. This collection of $\mathscr{L}$-sentences is precisely the collection of consequences of Σ and hence of Γ. It follows that $Th(\Gamma)$ is computably enumerable. □

Theorem 7.3.4. *If Γ is a computably axiomatizable, complete $\mathscr{L}$-theory, then Γ is decidable.*

Sketch of Proof. By the previous lemma, $Th(\Gamma)$ is computably enumerable. Let f be a total computable function whose range is $Th(\Gamma)$. Since Γ is complete, for every sentence φ, either the sentence or its negation is in $Th(\Gamma)$. Thus the characteristic function χ of $Th(\Gamma)$ can be defined in terms of f as follows. First, suppose that the Gödel numbers of the collection of $\mathscr{L}$-sentences is precisely $\mathbb{N}$. Then for an $\mathscr{L}$-sentence ϕ with Gödel number n, we let

$$\chi(n) = \begin{cases} 1, & \text{if } \phi \text{ is in the range of } f \\ 0, & \text{if } \neg\phi \text{ is in the range of } f. \end{cases}$$

□

We now give an example of a computably axiomatizable, complete theory, namely the theory of dense linear orders without endpoints.

The theory of *linear orders*, denoted **LO**, in the language $\{\leq\}$ has four sentences in it, which state that the relation $\leq$ is reflexive, antisymmetric, transitive and that the order is total. The last statement for *total order* is the following:

$$(\forall x)(\forall y)(x \leq y \vee y \leq x).$$

Not that this theory does not rule out the possibility that two points in the relation $x \leq y$ are actually equal. Since we include equality in every language of predicate logic, we can also define the strict order $x < y$ to be $x \leq y \;\&\; x \neq y$.

Next, the theory of *dense linear orders*, denoted **DLO**, is obtained from the theory of linear orders by the addition of the *density property*:

$$(\forall x)(\forall y)(x < y \rightarrow (\exists z)(x < z < y))$$

Consider two more sentences that state the existence of endpoints of our linear order, REnd for right endpoint and LEnd for left endpoint:

$$(\exists x)(\forall y)(x \leq y)$$
$$(\exists x)(\forall y)(y \leq x)$$

The theory of *dense linear orders without first and last element*, denoted **DLOWE**, is the theory **DLO** with the addition of the negations of REnd and LEnd. It is this theory that we will show is decidable.

One model of the theory is quite familiar, since the rationals with the usual order is a model:

$$\langle \mathbb{Q}, \leq \rangle \models \textbf{DLOWE}$$

To show that **DLOWE** is decidable, we only need to show that it is complete, since it is clearly finitely axiomatizable (and hence computably axiomatizable). We will establish the completeness of **DLOWE** by proving a series of results.

Theorem 7.3.5. *Any non-empty model of* **DLOWE** *is infinite.*

Proof. Left to the reader. □

Theorem 7.3.6. *Any two non-trivial countable models of* **DLOWE** *are isomorphic.*

Proof. Suppose that $\mathscr{A} = \langle A, \leq_A \rangle$ and $\mathscr{B} = \langle B, \leq_B \rangle$ are two non-empty countable models of **DLOWE**. Suppose that $\langle a_i \mid i < \omega \rangle$ and $\langle b_i \mid i < \omega \rangle$ are enumerations of A and B, respectively. We define an isomorphism $h : A \rightarrow B$ by defining h in a sequence of stages.

At stage 0 of our construction, set $h(a_0) = b_0$. Suppose at stage $m > 0$, h has been defined on $\{a_{i_1}, \dots, a_{i_k}\}$ where the elements of A are listed in increasing order under the relation $\leq_A$. Further suppose that we denote by b_{r_j} the value of $h(a_{i_j})$. Then since by hypothesis h is an isomorphism on the points on which it is defined, the elements $b_{r_1}, \ldots, b_{r_k}$ are listed in increasing order under the relation $\leq_B$.

At stage $m = 2n$ of our construction, we ensure that a_n is in the domain of h. If $h(a_n)$ has already been defined at an earlier stage, then there is nothing to do at stage $m = 2n$. Otherwise, either (i) $a_n <_A a_{i_1}$, (ii) $a_{i_k} <_A a_n$, (iii) or for some ℓ, $a_{i_\ell} <_A a_n <_A a_{i_{\ell+1}}$. Choose b_r as the element of B of least index in the enumeration $\langle b_i \mid i < \omega \rangle$ that has the same relationship to $b_{r_1}, \ldots, b_{r_k}$ that a_n has to $a_{i_1}, \ldots, a_{i_k}$. It is possible to choose such a b_r in (i) the first case because B has no left endpoint, (ii) the second case because B has no right endpoint, and (iii) the last case because the order is dense. Extend h to a_n by setting $h(a_n) = b_r$.

At stage $m = 2n + 1$ of our construction, we ensure that the point b_n is in the range of h. As above, if b_n is in the range of h, then there is nothing to do at stage $m = 2n + 1$. Otherwise, it has a unique position relative to $b_{r_1}, \ldots, b_{r_k}$. As above, either it is a left endpoint, a right endpoint, or it lies strictly between b_{r_ℓ} and $b_{r_{\ell+1}}$ for some ℓ. As in the previous case, choose a_r as the element of A of least index in the enumeration of $\langle a_i \mid i < \omega \rangle$ which has the same relationship to $a_{i_1}, \ldots, a_{i_k}$ as b_n has to $b_{r_1}, \ldots, b_{r_k}$, and extend h to a_r by setting $h(a_r) = b_n$.

This completes the recursive definition of h. One can readily prove by induction that the domain of h is A, the range of h is B, and that h is an isomorphism. □

This property of **DLOWE** is an example of a more general property.

Definition 7.3.7. A theory Γ is *κ-categorical* for some infinite cardinal κ if and only if every two models of Γ of cardinality κ are isomorphic.

Corollary 7.3.8. *The theory* **DLOWE** *is $\aleph_0$-categorical.*

The key result here is that theories that are categorical in some power and have only infinite models are also complete.

Theorem 7.3.9 (Los-Vaught Test)**.** *Suppose that Γ is a theory of cardinality κ with no finite models. If Γ is λ-categorical for some (infinite) $\lambda \geq \kappa$, then Γ is complete.*

Proof. Suppose by way of contradiction that Γ is not complete, but is λ-categorical for some $\lambda \geq \kappa$. Then there is a sentence φ in the language of Γ such that neither $\Gamma \vdash \varphi$ nor $\Gamma \vdash \neg\varphi$. Let $\Gamma_1 = \Gamma \cup \{\varphi\}$ and $\Gamma_2 = \Gamma \cup \{\neg\varphi\}$. Since Γ is consistent, so are Γ_1 and Γ_2. By the Completeness Theorem, each of these theories has a model. Since both of these models are also models of Γ, by hypothesis, they must be infinite. Therefore by the Löwenheim-Skolem Theorem, they each have models of cardinality λ. Let $\mathcal{N}_1$ and $\mathcal{N}_2$ be models of Γ_1 and Γ_2 of cardinality λ. Both $\mathcal{N}_1$ and $\mathcal{N}_2$ are models of Γ of power λ, so they are isomorphic. By recursion on the definition of $\models$, one can prove that they model the same set of sentences. However $\mathcal{N}_1$ is a model of φ while $\mathcal{N}_2$ is a model of $\neg\varphi$. This contradiction proves that Γ is complete. □

Corollary 7.3.10. *The theory* **DLOWE** *is complete and hence decidable.*

Other examples of decidable theories are:

1. Presburger arithmetic (arithmetic without multiplication);

2. arithmetic with only multiplication and no addition;

3. the theory of real closed fields (fields F in which every polynomial of odd degree has a root in F);

4. the first-order theory of algebraically closed fields of a fixed characteristic; and

5. the first-order theory of Euclidean geometry.

7.4 Gödel's Incompleteness Theorems

We now turn to undecidable theories. We will prove Gödel's first incompleteness theorem (G1), which states that any computably axiomatizable theory that contains elementary arithmetic (such as Peano Arithmetic and ZFC) is incomplete and hence undecidable. Gödel's second incompleteness theorem (G2) states that any computably axiomatizable theory that contains a sufficient amount of arithmetic cannot prove its own consistency. We will show this holds for Peano Arithmetic.

Just how much arithmetic is sufficient to prove the incompleteness theorems? For G1, it suffices to satisfy the theory of arithmetic in the language $\{+, \times, S, 0\}$ known as Robinson's Q, given by the following axioms:

(Q_1) $\neg(\exists x)S(x) = 0$

(Q_2) $(\forall x)(\forall y)(S(x) = S(y) \rightarrow x = y)$

(Q_3) $(\forall x)(x \neq 0 \rightarrow (\exists y)x = S(y))$

(Q_4) $(\forall x)(x + 0 = x)$

(Q_5) $(\forall x)(\forall y)(x + S(y) = S(x + y))$

(Q_6) $(\forall x)(x \times 0 = 0)$

(Q_7) $(\forall x)(\forall y)(x \times S(y) = (x \times y) + x)$

Note that Q is finitely axiomatizable and hence is computably axiomatizable.

To carry out the proof of $G2$, a stronger theory of arithmetic is necessary (while there is a version of G2 for Q, it is generally accepted that Q lacks the resources to recognize that the statement asserting its consistency really is its own consistency statement). Such a theory can be obtained by expanding our language to include symbols for every primitive function, adding axioms to Q that define every primitive recursive function, and adding mathematical induction for quantifier-free formulas (that is, for every quantifier-free formula ϕ in our language, we include as an axiom the sentence $(\phi(0) \,\&\, (\forall n)(\phi(n) \rightarrow \phi(S(n))) \rightarrow (\forall n)\phi(n))$. The resulting theory is known as *primitive recursive arithmetic*, which one can verify is computably axiomatizable.

We will restrict our attention to Peano Arithmetic (hereafter, *PA*), which is obtained by adding to Q the schema of mathematical induction for all formulas in the language of arithmetic. *PA* is also computably axiomatizable.

In order to prove Gödel's theorems, we have to represent the notion of provability *within PA*. To do so requires us to code much of the meta-theory of *PA* inside of *PA*. We begin by coding the natural numbers in the language of arithmetic as follows:

number	name	abbreviation
0	0	$\underline{0}$
1	S(0)	$\underline{1}$
2	S(S(0))	$\underline{2}$
⋮	⋮	⋮

Hereafter, the name of each $n \in \mathbb{N}$ in *PA* will be written $\underline{n}$.

Next, we represent functions in *PA*.

Definition 7.4.1. A function $f : \mathbb{N}^k \rightarrow \mathbb{N}$ is *definable in PA* by a formula ϕ if the following equivalence holds: For every $u_1, \dots, u_k, v \in \mathbb{N}$

$$PA \vdash \phi(\underline{u_1}, \dots, \underline{u_k}, \underline{v}) \text{ if and only if } f(u_1, \dots, u_k) = v.$$

A relation is *definable* if its characteristic function is definable.

We now code sequences of natural numbers in a computable and definable fashion. Although we could use a coding like the one using prime numbers as in Section 7.1, we will use an alternative coding.

Lemma 7.4.2. *The following relations are both computable and definable in PA.*

1. $x \leq y$;
2. $rem(x, y) = z$ *where* z *is the remainder* x *is divided by* y;
3. $Code(x, y, z) = rem(x, 1 + (z + 1)y)$.

Proof. 1. $(x \leq y)$: The formula that defines this relation in *PA* is

$$x \leq y \text{ if and only if } (\exists z)(x + z = y).$$

To see that the relation is computable, notice that since $x \leq y$ if and only if $x \dot{-} y = 0$, the characteristic function of the relation $x \leq y$ is

$$\chi_{\leq}(x, y) = 1 \dot{-} (x \dot{-} y)$$

2. $(rem(x, y))$: The formula that defines this relation in *PA* is

$$rem(x, y) = z \text{ if and only if } (\exists q)((x = yq + z)\&(0 \leq z < y)).$$

To see that the relation is computable, notice that the desired value of q is the least q so that $y(q+1) > x$. That is, this is the least $q \leq x$ such that it is not the case that $y(q+1) \leq x$, or equivalently, the least $q \leq x$ such that $\chi_{\leq}(y(q+1), x) = 0$, which can be found by bounded search. If such a q exists, then we check to see if $z = x - yq$, which can be done computably.

3. $(Code(x, y, z))$: The definition of *Code* in *PA* is given above. Since addition, multiplication and *rem* are all computable, so is *Code*.

□

Theorem 7.4.3. *For any sequence* $k_1, k_2, \ldots, k_n$ *of natural numbers, there exist natural numbers* a, b *such that* $Code(a, b, 0) = n$ *and* $\text{Code}(a, b, i) = k_i$ *for* $i = 1, 2, \ldots, n$.

Proof. We prove this using the Chinese Remainder Theorem, which is as follows: Let $m_1, \ldots, m_n$ be pairwise relatively prime (i.e., $gcd(m_i, m_j) = 1$ for $i \neq j$). Then for any $a_1, \ldots, a_n \in \mathbb{N}$, the system of equations

$$\begin{gathered} x \equiv a_1 \mod m_1 \\ \vdots \\ x \equiv a_n \mod m_n \end{gathered}$$

has a unique solution modulo $M = m_1 \cdot \ldots \cdot m_n$. To use the Chinese Remainder Theorem to prove the theorem, let $s = \max\{n, k_1, \ldots, k_n\}$ and set $b = s!$. We claim that the numbers

$$s! + 1, 2s! + 1, \ldots, (n+1)s! + 1$$

are pairwise relatively prime. Suppose not. Then there are $c, d \in \mathbb{N}$ such that $0 \leq c < d \leq n + 1$ and $p \in \mathbb{N}$ such that $p \mid (cs! + 1)$ and $p \mid (ds! + 1)$. Note that $p > s$, since $p \leq s$ implies that $rem(cs! + 1, p) = 1$. $p \mid (cs! + 1)$ and $p \mid (ds! + 1)$ together imply that

$$p \mid ((ds! + 1) - (cs! + 1)) = (d - c)s!.$$

However, $p \nmid s!$, since $p \mid s!$ implies $p \nmid (cs! + 1)$. It follows that $p \mid (d - c)$. But $d - c < n \leq s$, so $p \leq s$, which contradicts our earlier statement. By the Chinese Remainder Theorem, there is a unique solution x to

$$\begin{aligned} x &\equiv n \mod s!+1 \\ x &\equiv k_1 \mod 2s!+1 \\ &\vdots \\ x &\equiv k_n \mod (n+1)s!+1 \end{aligned}$$

modulo $M = \prod_{i=1}^{n+1} is!+1$. Let $a = x$.

We now check that a and b are the desired values. First,

$$\begin{aligned} Code(a,b,0) &= rem(a,b+1) \\ &= rem(a,s!+1) = n. \end{aligned}$$

For $i = 1,\ldots,n$,

$$\begin{aligned} Code(a,b,i) &= rem(a,(i+1)b+1) \\ &= rem(a,(i+1)s!+1) = k_i. \end{aligned}$$

□

Hereafter, let us fix a Gödel number of the symbols in the language of arithmetic:

symbol	code	symbol	code
(	1	$=$	8
)	2	$\exists$	9
&	3	$\forall$	10
$\neg$	4	$+$	11
$\vee$	5	$\times$	12
$\rightarrow$	6	S	13
$\leftrightarrow$	7	v_i	$14+i$

Using the function *Code*, we now code as natural numbers the following objects: $\mathscr{L}$-formulas, $\mathscr{L}$-sentences, sequences of $\mathscr{L}$-sentences, and proofs. First, we note that a pair of natural numbers (a,b) can be coded as a single natural number, denoted $\langle a,b\rangle$, by means of the following function:

$$\langle x,y\rangle = \frac{1}{2}(x+y)(x+y+1)+y.$$

Now, given an $\mathscr{L}$-formula ϕ, if a,b are natural numbers such that $Code(a,b,z)$ outputs the Gödel numbers of the symbols that make up ϕ, then we will set $\ulcorner\phi\urcorner = \langle a,b\rangle$. That is, $\ulcorner\phi\urcorner$ will denote the Gödel number of ϕ.

Next, if $\phi_1,\ldots,\phi_k$ is a sequence of $\mathscr{L}$-formulas, then n is the Gödel number of this sequence if $n = \langle c,d\rangle$ and

$$\begin{aligned} Code(c,d,0) &= k \\ Code(c,d,i) &= \ulcorner\phi_i\urcorner \end{aligned}$$

for $i = 1,\ldots,k$.

Recall that a proof $\Sigma \vdash \phi$ is a finite sequence of $\mathscr{L}$-formulas $\psi_1,\ldots,\psi_n,\phi$, where each $\psi_i \in \Sigma$ or follows from some subcollection of $\{\psi_1,\ldots,\psi_{i-1}\}$ by one of the rules of inference. We will make use of the following, which can be proved by showing that each of the rules of inference in the predicate calculus can effectively verified to hold.

Lemma 7.4.4. *There is a computable, definable predicate $Proof(n,m) \subseteq \mathbb{N}^2$ (represented by an $\mathscr{L}$-formula that we will also denote $Proof(n,m)$ such that $PA \vdash Proof(\underline{n},\underline{m})$ if and only if $\underline{m}$ is the code of a sequence of $\mathscr{L}$-sentences $\psi_1,\ldots,\psi_n,\phi$ such that $\{\psi_1,\ldots,\psi_n\} \vdash \phi$ and $\ulcorner\phi\urcorner = n$.*

We will make use of the following in the proof of Gödel's first incompleteness theorem.

Theorem 7.4.5. *The partial computable functions are exactly the functions definable in PA.*

We prove two auxiliary lemmas.

Lemma 7.4.6. $f : \mathbb{N}^k \to \mathbb{N}$ *is partial computable if and only if*

$$Graph(f) = \{(x_1, \dots, x_k, y) : f(x_1, \dots, x_k) = y\}$$

is a computably enumerable set.

Proof. ($\Rightarrow$) If f is partial computable, then we can define an effective procedure that enumerates a tuple $(n_1, \dots, n_k, m)$ into $Graph(f)$ whenever we see $f(n_1, \dots, n_k)\downarrow = m$.

($\Leftarrow$) To compute $f(n_1, \dots, n_k)$, enumerate $Graph(f)$ until we see some tuple $(n_1, \dots, n_k, m)$ that belongs to $Graph(f)$. If no such tuple exists, then we will wait forever and hence $f(n_1, \dots, n_k)\uparrow$. □

Lemma 7.4.7. $S \subseteq \mathbb{N}^k$ *is computably enumerable if and only if there is a computable relation* $R \subseteq \mathbb{N}^{k+1}$ *such that*

$$S = \{(n_1, \dots, n_k) : \exists z\, R(n_1, \dots, n_k, z)\}.$$

We will omit the proof this lemma.

Proof of Theorem 7.4. ($\Leftarrow$): Suppose that $f : \mathbb{N}^k \to \mathbb{N}$ is definable in *PA*. Then there is some $\mathscr{L}$-formula ϕ_f such that for all $u_1, \dots, u_k, v \in \mathbb{N}$,

$$\begin{aligned} PA \vdash \phi(\underline{u_1}, \dots, \underline{u_k}, \underline{v}) &\Leftrightarrow f(u_1, \dots, u_k) = v \\ &\Leftrightarrow (u_1, \dots, u_k, v) \in Graph(f). \end{aligned}$$

However, $PA \vdash \phi(\underline{u_1}, \dots, \underline{u_k}, \underline{v})$ if and only if there is some finite $X \subseteq PA$ such that $X \vdash \phi(\underline{u_1}, \dots, \underline{u_k}, \underline{v})$. Equivalently, there is some m such that $Proof(m, \ulcorner \phi_f(\underline{u_1}, \dots, \underline{u_k}, \underline{v}) \urcorner)$ holds (where m codes a sequence $\psi_1, \dots, \psi_j, \phi_f(\underline{u_1}, \dots, \underline{u_k}, \underline{v})$ where $\{\psi_1, \dots, \psi_j\} \vdash \phi_f(\underline{u_1}, \dots, \underline{u_k}, \underline{v})$). It follows that

$$Graph(f) = \{(u_1, \dots, u_k, v) : \exists m\, R(u_1, \dots, u_k, v, m)\}$$

where $R(u_1, \dots, u_k, v, m)$ is the computable relation $Proof(m, \ulcorner \phi_f(\underline{u_1}, \dots, \underline{u_k}, \underline{v}) \urcorner)$. By Lemma 7.4.7, $Graph(f)$ is computably enumerable, and so by Lemma 7.4.6, f is partial computable.

($\Rightarrow$) We show all partial computable functions are definable in *PA* by induction. It is not hard to show that the initial functions are definable.

1. (Constant function): The defining formula $\phi_c(x, y)$ for the constant function $c(x) = 0$ is

$$(y = \underline{0}) \,\&\, (x = x).$$

2. (Projective functions): The defining formula $\phi_{p_j^k}(x_1, \dots, x_k, y)$ for the projection function $p_j^k(x_1, \dots, x_k) = x_j$ is

$$(y = x_j) \,\&\, (x_1 = x_1) \,\&\, \dots \,\&\, (x_k = x_k).$$

3. (Successor function): The defining formula $\phi_S(x, y)$ for the successor function $S(x) = x + 1$ is $y = S(x)$.

For the induction step of the proof, we must show that the set of *PA*-definable functions is closed under the production rules for the set of partial computable functions.

(4) (Composition): Suppose that f and $g_1, g_2, \ldots, g_k$ are definable in *PA* by the formulas ϕ_f and $\phi_{g_1}, \ldots, \phi_{g_k}$, respectively and that h is the function $h(\vec{x}) = f(g_1(\vec{x}), \ldots, g_k(\vec{x}))$. Then the defining formula $\phi_h(\vec{x}, y, z)$ for h is

$$(\exists y_1)(\exists y_2)\ldots(\exists y_k)(\phi_{g_1}(\vec{x}, y_1) \mathbin{\&} \ldots \mathbin{\&} \phi_{g_k}(\vec{x}, y_k) \mathbin{\&} \phi_f(y_1, \ldots, y_k, z)).$$

(5) (Primitive Recursion): Suppose that f and g are definable in *PA* by ϕ_f and ϕ_g, respectively, and that h is the function defined by recursion with

$$h(\vec{x}, 0) = f(\vec{x})$$
$$h(\vec{x}, y+1) = g(\vec{x}, y, h(\vec{x}, y)).$$

To define h, we will use a pair of numbers a and b that code up the sequence

$$h(\vec{x}, 0), h(\vec{x}, 1), \ldots, h(\vec{x}, y),$$

via the function *Code*, where $h(\vec{x}, 0) = f(\vec{x})$ and $h(\vec{x}, n+1) = g(\vec{x}, n, h(\vec{x}, n))$ for every $n < y$. Thus the defining formula $\phi_h(\vec{x}, y, z)$ for h is

$$\begin{aligned}(\exists a)(\exists b)(&\phi_f(\vec{x}, \mathit{Code}(a, b, 1))\\ &\mathbin{\&} (\forall i < y)(\phi_g(\vec{x}, i, \mathit{Code}(a, b, i+1), \mathit{Code}(a, b, i+2))\\ &\mathbin{\&} (z = \mathit{Code}(a, b, y+1))).\end{aligned}$$

(6) (Unbounded search): Left to the reader.

Thus the class of functions definable in *PA* includes all the initial functions and is closed under the production rules. Therefore every partial computable function is definable in *PA*. □

The following is a key ingredient of the proofs of the incompleteness theorems.

Lemma 7.4.8 (Diagonal lemma). *Let $\phi(x)$ be an $\mathscr{L}$-formula with one free variable. Then there is an $\mathscr{L}$-sentence ψ such that*

$$PA \vdash \psi \leftrightarrow \phi(\ulcorner \psi \urcorner).$$

Proof. First, observe that the operation that, given inputs $n \in \mathbb{N}$ and a formula $A(x)$ with one free variable, outputs $A(n)$ is purely mechanical. Similarly, the operation that, given inputs $n \in \mathbb{N}$ and the Gödel number of such a formula $A(x)$, outputs $\ulcorner A(\underline{n}) \urcorner$, the Gödel number of $A(\underline{n})$, is also purely mechanical, and hence is intuitively computable.

Let $\mathit{subst}(\ulcorner A(x) \urcorner, n) = \ulcorner A(\underline{n}) \urcorner$ be this function, which is partial computable (since it may receive as input some number that is not the Gödel number of any formula, or the Gödel number of a formula with more than one free variable, etc.). By Theorem 7.4, there is a formula $S(x, y, z)$ that defines *subst* in *PA*.

The key observation to make here is that for any $\mathscr{L}$-formula $A(x)$ with one free variable, we can consider the value $\mathit{subst}(\ulcorner A(x) \urcorner, \ulcorner A(x) \urcorner)$. Given $\phi(x)$ as above, let $\theta(x)$ be the formula

$$(\exists y)(\phi(y) \mathbin{\&} S(x, x, y)).$$

That is, $\theta(x)$ says "There is some y satisfying ϕ that is obtained by substituting x into the formula with Gödel number x." Let $k = \ulcorner \theta(x) \urcorner$ and consider the sentence $\theta(\underline{k})$, which has Gödel number $\ulcorner \theta(\underline{k}) \urcorner$. Call this sentence ψ.

Unpacking the sentence $\theta(\underline{k})$, we have

$$(\exists y)(\phi(y) \mathbin{\&} S(\underline{k}, \underline{k}, y))$$

Moreover, since $subst(k,k) = \ulcorner\theta(\underline{k})\urcorner$, one can show that

$$PA \vdash (\forall y)(S(\underline{k},\underline{k},y) \leftrightarrow y = \ulcorner\theta(\underline{k})\urcorner). \tag{7.1}$$

Then by definition of ψ

$$PA \vdash \psi \leftrightarrow (\exists y)(\phi(y) \,\&\, S(\underline{k},\underline{k},y)).$$

However, by Equation 7.1,

$$PA \vdash \psi \leftrightarrow (\exists y)(\phi(y) \,\&\, y = y = \ulcorner\theta(\underline{k})\urcorner).$$

Equivalently,

$$PA \vdash \psi \leftrightarrow \phi(\ulcorner\theta(\underline{k})\urcorner),$$

which can be rewritten as

$$PA \vdash \psi \leftrightarrow \phi(\ulcorner\psi\urcorner).$$

□

One last necessary result that we will state without proof is the following.

Lemma 7.4.9. *Given an $\mathscr{L}$-sentence ψ of the form $(\exists x)\phi(x)$, if ψ is true in the standard model of arithmetic, then ψ is provable in PA. That is,*

$$\mathbb{N} \models (\exists x)\phi(x) \Leftrightarrow PA \vdash (\exists x)\phi(x).$$

Theorem 7.4.10 (Gödel's First Incompleteness Theorem)**.** *If PA is consistent, then PA is not complete. That is, there is a sentence ψ_G true in the standard model of arithmetic such that $PA \nvdash \psi_G$ and $PA \nvdash \neg\psi_G$.*

Proof. We apply the Diagonal Lemma to the formula $\neg(\exists x)Proof(x,y)$, which we will abbreviate as $\neg Prov(y)$ (Informally, this sentence asserts that there is no proof of the sentence with Gödel number y). Thus, there is some $\mathscr{L}$-sentence ψ_G (the "Gödel sentence") such that

$$PA \vdash \psi_G \leftrightarrow \neg Prov(\ulcorner\psi_G\urcorner). \tag{7.2}$$

That is, ψ_G is equivalent to the $\mathscr{L}$-sentence that asserts that ψ_G is not provable.

We now show that if *PA* is consistent, then $PA \nvdash \psi_G$ and $PA \nvdash \neg\psi_G$. In fact, we will show that $PA \vdash \psi_G$ if and only if $PA \vdash \neg\psi_G$, which is clearly impossible if *PA* is consistent and from which the desired conclusion follows.

Observe that $PA \vdash \psi_G$ if and only if $(\exists n)Proof(n, \ulcorner\psi_G\urcorner)$ holds, which is equivalent to $PA \vdash (\exists n)Proof(n, \ulcorner\psi_G\urcorner)$ by Lemma 7.4.9. This is equivalent to $PA \vdash Prov(\ulcorner\psi_G\urcorner)$, which is equivalent to $PA \vdash \neg\psi_G$ by (7.2).

Lastly, if *PA* is consistent, then $\mathbb{N} \models \psi_G$. Suppose instead that $\mathbb{N} \models \neg\psi_G$. Since $\neg\psi_G$ is equivalent to $(\exists n)Proof(n, \ulcorner\psi_G\urcorner)$, it follows that $\mathbb{N} \models (\exists n)Proof(n, \ulcorner\psi_G\urcorner)$. But then by Lemma 7.4.9, it follows that $PA \vdash (\exists n)Proof(n, \ulcorner\psi_G\urcorner)$, i.e., $PA \vdash Prov(\ulcorner\psi_G\urcorner)$. But this implies that $PA \vdash \neg\psi_G$, which we have shown is impossible.

□

We now turn to Gödel's second incompleteness theorem. Informally, Gödel's second theorem states that if *PA* is consistent, then it cannot prove that it is consistent. To express this formally, we need to formalize that statement that *PA* is consistent *within PA*. The standard way to do this by the formula $\neg Prov(\ulcorner\underline{0} = \underline{1}\urcorner)$.

Theorem 7.4.11 (Godel's Second Incompleteness Theorem)**.** *If PA is consistent, then*

$$PA \nvdash \neg Prov(\ulcorner\underline{0} = \underline{1}\urcorner).$$

To prove G2, one can show (with a considerable amount of work) that the proof of G1 can be carried out entirely *within PA*. We proved above that if *PA* is consistent, then $PA \nvdash \psi_G$; within *PA*, this yields

$$PA \vdash \neg Prov(\ulcorner \underline{0} = \underline{1} \urcorner) \to \neg Prov(\ulcorner \psi_G \urcorner).$$

By G1, we have

$$PA \vdash \psi_G \leftrightarrow \neg Prov(\ulcorner \psi_G \urcorner).$$

It follows from the previous two statements that

$$PA \vdash \neg Prov(\ulcorner \underline{0} = \underline{1} \urcorner) \to \psi_G.$$

Thus if $PA \vdash \neg Prov(\ulcorner \underline{0} = \underline{1} \urcorner)$, it would follow that $PA \vdash \psi_G$, which is impossible by G1. Hence $PA \nvdash \neg Prov(\ulcorner \underline{0} = \underline{1} \urcorner)$.

Let us take a more general approach to proving G2. Let $B(x)$ be an $\mathscr{L}$-formula with one free variable. The following three conditions are referred to as *derivability conditions*:

(D1) If $PA \vdash \phi$, then $PA \vdash B(\ulcorner \phi \urcorner)$.

(D2) $PA \vdash B(\ulcorner \phi \to \psi \urcorner) \to (B(\ulcorner \phi \urcorner) \to B(\ulcorner \psi \urcorner))$.

(D3) $PA \vdash B(\ulcorner \phi \urcorner) \to B(\ulcorner B(\ulcorner \phi \urcorner) \urcorner)$.

With some effort, one can prove that if we let $B(x)$ be the formula $Prov(x)$, then (D1)-(D3) hold.

We can now formulate an abstract version of G2:

Theorem 7.4.12 (Abstract G2). *If $B(x)$ satisfies (D1)-(D3), then $PA \nvdash \neg B(\ulcorner \underline{0} = \underline{1} \urcorner)$.*

To prove this theorem, we will first prove the following, which tells us that *PA* can only prove that soundness for sentences that it can prove to hold.

Theorem 7.4.13 (Löb's Theorem). *If $B(x)$ satisfies (D1)-(D3), then for any $\mathscr{L}$-sentence ϕ, if $PA \vdash B(\ulcorner \phi \urcorner) \to \phi$, then $PA \vdash \phi$.*

Proof. Suppose that $B(x)$ satisfies (D1)-(D3) and that

$$PA \vdash B(\ulcorner \phi \urcorner) \to \phi. \tag{7.3}$$

Let $\theta(y)$ be the formula $B(y) \to \phi$. By the Diagonal Lemma there is an $\mathscr{L}$-sentence ψ such that

$$PA \vdash \psi \leftrightarrow (B(\ulcorner \psi \urcorner) \to \phi) \tag{7.4}$$

and hence

$$PA \vdash \psi \to (B(\ulcorner \psi \urcorner) \to \phi). \tag{7.5}$$

From (7.5) and (D1) it follows that

$$PA \vdash B(\ulcorner \psi \to (B(\ulcorner \psi \urcorner) \to \phi) \urcorner). \tag{7.6}$$

By (D2)

$$PA \vdash B(\ulcorner \psi \to (B(\ulcorner \psi \urcorner) \to \phi) \urcorner) \to (B(\ulcorner \psi \urcorner) \to B(\ulcorner B(\ulcorner \psi \urcorner) \to \phi \urcorner)) \tag{7.7}$$

Then by applying modus ponens to (7.6) and (7.7), we have

$$PA \vdash B(\ulcorner \psi \urcorner) \to B(\ulcorner B(\ulcorner \psi \urcorner) \to \phi \urcorner). \tag{7.8}$$

Again by (D2),

$$PA \vdash B(\ulcorner B(\ulcorner \psi \urcorner) \to \phi \urcorner) \to (B(\ulcorner B(\ulcorner \psi \urcorner) \urcorner) \to B(\ulcorner \phi \urcorner)). \tag{7.9}$$

From (7.8) and (7.9) we have

$$PA \vdash B(\ulcorner \psi \urcorner) \to (B(\ulcorner B(\ulcorner \psi \urcorner) \urcorner) \to B(\ulcorner \phi \urcorner)). \tag{7.10}$$

By (D3),

$$PA \vdash B(\ulcorner\psi\urcorner) \to B(\ulcorner B(\ulcorner\psi\urcorner)\urcorner), \tag{7.11}$$

and so from (7.10) and (7.11) we can conclude

$$PA \vdash B(\ulcorner\psi\urcorner) \to B(\ulcorner\phi\urcorner). \tag{7.12}$$

By (7.3) and (7.12) we have

$$PA \vdash B(\ulcorner\psi\urcorner) \to \phi. \tag{7.13}$$

From (7.4) and (7.13) it follows that

$$PA \vdash \psi. \tag{7.14}$$

From (D1) we can infer from (7.14) that

$$PA \vdash B(\ulcorner\psi\urcorner). \tag{7.15}$$

Applying modus ponens to (7.13) and (7.15) gives the desired conclusion

$$PA \vdash \phi.$$

□

Proof of Abstract G2. Suppose $B(x)$ satisfies (D1)-(D3) and that $PA \vdash \neg B(\ulcorner \underline{0} = \underline{1}\urcorner)$. Then

$$PA \vdash B(\ulcorner \underline{0} = \underline{1}\urcorner) \to \phi$$

for any $\mathscr{L}$-sentence ϕ. In particular,

$$PA \vdash B(\ulcorner \underline{0} = \underline{1}\urcorner) \to \underline{0} = \underline{1}.$$

Then by Löb's Theorem, it follows that $PA \vdash \underline{0} = \underline{1}$, which contradicts our assumption that PA is consistent. Thus $PA \nvdash \neg B(\ulcorner \underline{0} = \underline{1}\urcorner)$. □

7.5 Exercises

1. Find the Gödel number for the predicate logic formula

$$(\exists v_1)(\forall v_2)F_0(v_1, v_2) = v_2$$

Chapter 8

Computable Mathematics

In this chapter, we will examine computability for several problems in mathematics. We are looking for families of problems which may have a general solution, where the methods have an algorithmic flavor. Our goal is to determine whether there is really an algorithm to solve the problem. This will include the problems from combinatorics such as finding a coloring of a graph, and finding a homogeneous set for a partition. Problems from analysis include finding the zeros and the extreme values of a continuous function on an interval, and solving certain standard differential equations. We will also examine the problem of finding a maximal ideal of a Boolean algebra, and the related problem from logic of finding a complete consistent extension of a theory.

8.1 Computable Combinatorics

In this section, we consider computable aspects of the coloring problem for graphs, and the partition problem from Ramsey theory.

A graph $G=(V,E)$ is given by a set V of *vertices* and a binary relation E; pairs (u,v) such that $E(u,v)$ (sometimes written uEv) are said to be *edges*. We assume that E is symmetric, that is the edges are unordered, and irreflexive, that is, there are no edges from a vertex v to itself. For any vertex v, let $N(v)$ denote $\{u : uEv\}$. The *degree* $\delta(v)$ of a vertex v is the number of edges from v, that is, the cardinality of $N(v)$; more generally $N(A)=\bigcup\{N(v) : v\in A\}$ for $A\subseteq V$. Then we can inductively define $N^i(A)$ by letting $N^0(A)=A$ and letting $N^{i+1}(A)=N(N^i(A))$. Note that $A\subseteq N^i(A)$ for $i\geq 2$ as long as $\delta(v)\neq 0$.

A *coloring* of G is a map F from V to a set K of *colors*. G is said to be k-colorable if there is a coloring mapping to the set $\{1,2,\dots,k\}$. One of the most famous results of recent years is the solution of the 4-coloring problem, showing that every planar graph can be colored with 4 colors. The problem we want to examine is this: Given that G has a k-coloring, how hard is it to find such a coloring. In particular, suppose that $G=(\omega,E)$ is an infinite computable graph which is k-colorable. Does G have a computable k-coloring?

We say that the graph $G=(\omega,E)$ is computable if E is a computable relation. We will consider here only *highly computable* graphs, where the degree function δ is also computable.

Proposition 8.1.1 (Bean 1976). *If the highly computable graph G is k-colorable for some fixed finite k, then G has a computable $2k$-coloring.*

Proof. We will give the argument when G is connected. Let $v_0=0$, let $G_1=\{v_0\}\cup N(v_0)$ and let $G_{n+1}=N(G_n)$ for each $n\geq 1$. Then we have $G_n\subseteq G_{n+1}$ for all n, and $\bigcup_n G_n=V$. Furthermore, is $v\in G_n$, then $N(v)\subseteq G_{n+1}$. Now let $X_1=G_1$ and let $X_{n+1}=G_{n+1}\setminus G_n$ for all $n\geq 1$. Then $\bigcup_n X_n = G_1\cup[G_2\setminus G_1]\cup[G_3\setminus G_2]\cup\cdots[G_n\setminus G_{n-1}]=\bigcup_n G_n = V$. Now for $v\in X_i$, $v\in G_i$ and hence $N(v)\subseteq G_{i+1}$, so that $N(v)\cap X_n=\emptyset$ for all $n>i+1$. It follows by symmetry that $N(X_i)\subseteq X_{i-1}\cup X_i\cup X_{i+1}$. Now let $K=\{1,2,\dots,k,k+1,\dots,2k\}$. We compute the coloring $F: V\to K$ on each set X_i as follows. If i is odd, then we compute, by trial and error, a coloring

F_i of X_i using the colors $\{1,\ldots,k\}$ and if i is even, then we compute F_i using the colors from $\{k+1,\ldots,2k\}$. Then $F(v)$ is defined to be $F_i(v)$ when $v \in X_i$. We now check that F is a legal coloring of V. Suppose that uEv and that $v \in X_i$. There are two cases by the above. Either $u \in X_i$, so that $F(u) = F_i(u) \neq F_i(v) = F(v)$, or $u \in X_{i-1} \cup X_{i+1}$, in which case $F(u)$ and $F(v)$ are chosen from disjoint sets of colors, so that again $F(u) \neq F(v)$.

If G is not connected, we just let $G_{n+1} = N(G_n) \cup \{v_n\}$ and proceed as before. □

Theorem 8.1.2. *There is a highly computable,* 3*-colorable graph G with no computable 3-coloring.*

Proof. Let $\phi_0, \phi_1, \ldots$ enumerate all (partial) computable functions from $V = \omega$ to $\{1,2,3\}$. We build the computable graph G consisting of, for each e, a pair of "towers" to show that ϕ_e cannot possibly be a 3-coloring of G. So there will be vertices u_e and v_e at the base of the e-towers. The first step of the construction for each tower adds 3 vertices w_1, w_2, w_3, so that (for the u_e tower) u_e, w_1, w_2 are joined by edges and also w_1, w_2, w_3 are joined by edges. Thus in any 3-coloring F of G, $F(u_e), F(w_1), F(w_2)$ are all different and $F(w_3) = F(u_e)$, and similarly for any v_e column. For each e, we continue this construction so that at stage s, we have vertices u_e^s and v_e^s at the top of the u_e and v_e columns which must receive the same color as u_e and v_e (respectively) in any 3-coloring of G. Now we use these two e-towers to ensure that ϕ_e is not a legal 3-coloring of G. We keep building the towers until $\phi_e(u_e)$ and $\phi_e(v_e)$ are both defined at some stage s. At that point there are two cases.

Case I: If $\phi_e(u_e) = \phi_e(v_e)$, then we create an edge between u_e^{s+1} and v_e^{s+1}. Since the towers are constructed to force $\phi_e(u_e^{s+1}) = \phi_e(v_e^{s+1})$, but the edge forces them to have different colors, it follows that ϕ_e is not a legal 3-coloring of G.

Case 2: If $\phi_e(u_e) \neq \phi_e(v_e)$, then we connect u_e^{s+1} and v_e^{s+1} with two new vertices x and y such that xEy as well. This link will force them to have the same color, so that again ϕ_e is not a legal 3-coloring of G. □

Next we consider a special case of the (Ramsey) partition problem. Let $[\mathbb{N}]^i$ be the family of subsets of $\mathbb{N}$ of cardinality i. In particular $[\mathbb{N}]^2$ is the set of unordered pairs of (distinct) natural numbers. If $C : [\mathbb{N}]^2 \to \{1,2,\ldots,k\}$ is a k-coloring of $[\mathbb{N}]^2$, then a subset A of $\mathbb{N}$ is said to be homogeneous for C if there is a single color i such that $C(\{a,b\}) = i$ for all pairs a, b of elements from A. Note that here a k-coloring is *any* map from $[\mathbb{N}]^i$ to $\{1,2,\ldots,k\}$.

The infinite version of Ramsey's Theorem for pairs shows that for any k-coloring of $[N]^2$, there exists an infinite homogeneous set. More generally we have.

Theorem 8.1.3. *For natural numbers r,k, for every function $f : [\omega]^r \to k$, there is an infinite set $a \subset \omega$ such that f is constant on $[a]^r$.*

Proof. The proof is by induction on r. The base case $r = 1$ is immediate since for $f : \omega \to k$, f induces a partition of ω into k sets, $A_i = \{a : f(a) = i\}$, one of which must be infinite.

xxx □

Theorem 8.1.4. *There exists a computable partition $f : [\mathbb{N}]^2 \to \{1,2\}$ such that no homogeneous set for f is computably enumerable.*

Proof. Let W_e be the e^{th} computable enumerable set. We define our computable coloring C to satisfy, for each e, the requirement $\boldsymbol{R_e}$ that W_e is not both infinite and homogeneous for C.

For an individual requirement $\boldsymbol{R_e}$, we try to identify a pair $a_e < b_e$ in W_e which will have a certain color, say $C(\{a_e, b_e\}) = k$, and ensure that, for almost all n, $C(\{a_e, n\}) \neq C(\{b_e, n\})$. Thus if W_e is infinite, then it cannot be homogeneous for C. It is important to note that another requirement $\boldsymbol{R_i}$ might want to use one or both of a_e, b_e. So we need to organize and prioritize the requirements. That is, we give the ith requirement *higher priority* than the eth requirement if $i < e$.

The construction is in stages s so that we have colored all pairs $\{a, b\}$ with $a, b \leq s$ by stage s. This will make C a computable coloring. So at stage 1, we let $C(\{0,1\}) = 1$.

After stage s, we will have defined $C(\{a,b\})$ for all $a, b \leq s$ and, for certain $e \leq s$, we will have designated $a_e^s < b_e^s < s$.

Now at stage $s+1$, we check to see whether there is some new $e \leq s+1$ so that W_e^{s+1} now has two elements $a < b \leq s$ which have not been used by any higher priority requirement, and that a_e^s and b_e^s are not defined. If so, then we take the least such e and the least such pair $a, b \in W_e^{s+1}$ and set $a_e^{s+1} = a$ and $b_e^{s+1} = b$. If there is $i > e$ such that a_i^s and b_i^s were defined but one or both of them is now needed by requirement $\boldsymbol{R_e}$, then we cancel the designation for the i^{th} requirement, so that a_i^{s+} and b_i^{s+1} are undefined. For other $i \leq s+1$, we let $a_i^{s+1} = a_i^s$ and $b_i^{s+1} = b_i^s$. Then we define $C(\{x, s+1\})$ for all x to be 1, *unless* $x = b_e^{s+1}$ for some i, in which case we let $C(x, s+1) = 2$.

Let us say that a_e^s and b_e^s are *permanently defined at stage s* if for all $t > s$, $a_e^t = a_e^s$ and $b_e^t = b_e^s$, and let $a_e = lim_s a_e^s$ and $b_e = b_e^s$ if this exists. a_e^s and b_e^s are *permanently undefined at stage s* if for all $t \geq s$, a_e^t b_e^t are undefined.

Lemma 8.1.5. *For any e, there is a stage s such that a_e^s, b_e^s are permanently defined or are permanently undefined at stage s. Furthermore, if W_e is infinite, then there is a stage s such that a_e^s, b_e^s are permanently defined at stage s.*

Proof. The proof is by induction on e. For $e = 0$, there are two cases. First, it may be that W_0 has ≤ 1 elements, and in that case we never act for requirement $\boldsymbol{R_0}$so that a_0 and b_0 are permanently undefined at stage 0. Otherwise, consider the least s such that W_e^{s+1} has two distinct elements $a < b$ and let $\{a, b\}$ be the least such pair. Then since there is no higher priority requirement, the construction will make $a = a_e^{s+1}$ and $b = b_e^{s+1}$ and these designations will never be canceled. Now suppose the lemma holds for all $i < e$, and let s be large enough so that, for all $i < e$, a_e^s and b_e^s are either permanently defined at stage s or permanently undefined at stage s. It follows that if a_e^s and b_e^s are defined at stage s or at any later stage t, then they can never be undefined at a later stage. This proves the first part of the lemma. Now suppose that W_e is infinite. Then there must exist $a < b$ in W_e different from all permanently defined a_i^s and b_i^s with $i < e$. Let $a < b$ be the least such pair and let $a, b \in W^{t+1}$ for some $t > s$. There are two cases. If a_e^t and b_e^t are already defined, then as above they are permanently defined at stage t. If not, then the construction will make $a_e^{t+1} = a$ and $b_e^{t+1} = b$ and they will be permanently defined at stage $t+1$. □

Lemma 8.1.6. *For any e such that W_e is infinite, W_e is not a homogeneous set for C.*

Proof. By Lemma 8.1.5, there is stage s such that a_e^s and b_e^s are permanently defined after stage s. Then for all $x \geq s+1$, we have $C(\{x, a_e\}) = 1$ and $C(\{x, b_e\}) = 2$. Since W_e is infinite, it must contain an element $x \geq s+1$. It follows that W_e is not homogeneous for the coloring C. □

This complete the proof of Theorem **??**. □

8.2 Computable Analysis

In this section, we discuss computablity of real numbers and of continuous functions, as well as computable aspects of such standard results as the Intermediate Value Theorem.

8.2.1 Computable Real Numbers

Definition 8.2.1. A real number r is said to be *computable* if there is a computable sequence $\{q_n : n \in \omega\}$ of rationals such that $|r - q_n| \leq 2^{-n}$ for all n.

Thus any rational number q is itself computable as the limit of a constant sequence. For a more interesting example, it is easy to see that the transcendental number e is computable, by examining the sequence of partial sums $1 + 1/2 + 1/6 + \cdots + 1/n$. To see that $\sqrt{2}$ is computable, just compute decimals $q_n < r_n$ to n places such that $q_n^2 < 2 < r_n^2$. Then certainly $lim_n q_n = lim_n r_n = \sqrt{2}$. Since $r_n - q_n = 10^{-n}$, it follows that $\sqrt{2} - q_n < 10^{-n}$.

Given an infinite sequence $x \in \{0,1\}^\omega$, we may form the real number $r_x = \sum_{n=1}^\infty x(n) 2^{-n-1}$.

Weaker notions of definability are also of interest.

Definition 8.2.2. 1. A real number r is *lower semi-computable* if there is a computable, non-decreasing, sequence $\{q_n : n \in \omega\}$ of rationals such that $\lim_n q_n| = r$.

2. A real number r is *upper semi-computable* if there is a computable, non-increasing, sequence $\{q_n : n \in \omega\}$ of rationals such that $\lim_n q_n| = r$.

3. A real number r is *approximable* if there is a computable sequence $\{q_n : n \in \omega\}$ of rationals such that $\lim_n q_n| = r$.

Proposition 8.2.3. *For any real r, r is computable if and only if r is both lower semi-computable and upper semi-computable.*

Proof. Suppose first that r is computable and let $\{q_n : n \in \omega\}$ be a computable sequence of rationals such that $|q_n - r| \leq 2^{-n}$. This means that $r - 2^{-n} \leq q_n \leq r + 2^{-n}$ for each n. Now let $s_n = max\{q_n - 2^{-n} : i \leq n\}$ for each n; this is clearly a non-decreasing sequence. For each n, we have $q_n - 2^{-n} \leq s_n \leq q_n$, so that $lim_n s_n = lim_n q_n = r$. It follows that r is lower semi-computable. The argument for upper semi-computability is similar.

For the other direction, let $\{p_n : n \in \omega\}$ be a non-decreasing sequence of rationals and $\{r_n : n \in \omega\}$ be a non-increasing sequence of rationals such that $lim_n p_n = r = lim_n r_n$. Then we have $p_n \leq r \leq r_n$ for each n and $lim_n r_n - p_n = 0$. So we can choose a sequence n_i such that $r_{n_i} - p_{n_i} < 2^{-i}$ and let $q_i = p_{n_i}$. Then $lim_i q_i = r$ and $r - q_i < r_{n_i} - p_{n_i} < 2^{-i}$. □

Definition 8.2.4. A set D of rationals is a *Dedekind cut* if is closed downward, bounded above, and contains no greatest element. For any real r, let $D_r = \{q \in \mathbb{Q} : q < r\}$. This will always be a Dedekind cut.

One version of the Completeness Principle for the real numbers is that every Dedekind D cut has a supremum, which will be a real r such that $D = D_r$.

Proposition 8.2.5. *For any real r,*

1. *r is lower semi-computable if and only if D_r is computably enumerable.*

2. *r is upper semi-computable if and only if the complement of D_r is computably enumerable.*

3. *r is computable if and only if D_r is computable.*

Proof. Suppose that r is lower semi-computable and let $\{q_n : n \in \omega\}$ be a computable, non-decreasing sequence such that $r = lim_n q_n$. Then we have, for any rational q,

$$q < r \iff (\exists n) q < q_n$$

For the other direction, suppose that D_r is computably enumerable and that r is not a rational itself. Then let $p_0, p_1, \dots$ be a computable enumeration of D_r and define the increasing sequence $q_n = p_{i_n}$ as follows: Let$i_0 = 0$, so that $q_0 = p_0$. For each n, let i_{n+1} be the least $i > i_n$ such that $q_n < p_i$. This is an increasing sequence by construction, so the limit s must exist and $s \leq r$ since each $q_n < r$. Furthermore, for each n and each $i < i_n$, we can see by induction that $p_i < p_{i_n}$. (This is trivial for $i = 0$, so suppose that $i < i_{n+1}$. There are two cases. If $i \leq i_n$, then by induction $p_i \leq p_{i_n} < p_{i_{n+1}}$. If $i_n < i < i_{n+1}$, then by the choice of i_{n+1}, we must have $p_i \leq p_{i_n}$.) Since the sequence i_n is strictly increasing, we have $n \leq i_n$ and therefore $p_n \leq p_{i_n}$. Now suppose by way of contradiction that $s = lim_n q_n < r$. Then by the density of $\mathbb{Q}$, there is a rational p with $s < p < r$. Then $p = p_n$ for some n, so that $p \leq p_{i_n} \leq s$. It follows that r is lower semi-computable.

A similar argument works for upper semi-computable in each direction.

For the third clause, r is computable if and only if r is both lower and upper semi-computable, by Proposition 8.2.3, which is if and only if D_r is both lower and upper semi-computable, which is if and only if D_r is computable, by Proposition 6.4.9 □

Proposition 8.2.6. *For any real $r \in [0, 1]$, r is computable if and only if $r = r_x$ for some computable $x \in 2^\omega$.*

Proof. For the first direction, suppose that r is computable and therefore $\{q \in \mathbb{Q} : q < r\}$ is computable. If r is a dyadic rational, that is, of the form $i2^{-n}$ for some nonnegative integers i and n, then $r = r_x$ where x is eventually 0. Otherwise, for all dyadic rationals q, $q < r$ if and only if $q \leq r$. We define the function $x \in 2^\omega$ such that $r = r_X$ by recursion as follows. Let $x(0) = 1$ if $\frac{1}{2} < r$ and $x(0) = 0$ otherwise (in which case $r < \frac{1}{2}$. Given $x(0), \ldots, x(n)$, let $q_n = \sum_{i \leq n} x(i)2^{-i-1}$ and let $x(n+1) = 1$ if $q_n + 2^{-n-2} < r$, and $x(n+1) = 0$, otherwise. It follows from this construction that $q_n < r, q_n + 2^{-n-1}$, so that $r = lim_n q_n$. But by definition of x, $r_x = \sum_{i=0}^{\infty} x(i)2^{-i-1} = lim_n q_n$.

For the other direction, suppose that we are given a computable x and $r = r_x$ and let $q_n = \sum_{i=0}^{n} x(i)2^{-i-1}$. Then for each n,

$$r_x - q_n = \sum_{i=n+1} x(n)2^{-i-1} \leq \sum_{i=n+1} 2^{-i-1} = 2^{-n}.$$

Then $r = lim_n q_n$ and it follows that $r = r_x$ is computable. □

8.2.2 Computable Real Functions

There are several equivalent notions of computability for real functions. For simplicity, we will consider functions $F : [a, b] \to [a, b]$ for some computable reals $a < b$. A computable real function may be viewed as an effectively continuous function.

Definition 8.2.7. Fix computable reals $a < b$. A real function $F : [a, b] \to [a, b]$ is computable if there is a computable sequence f_n of functions $f_n : \mathbb{Q}\times \to \mathbb{Q}$ and a computable function $\phi : \mathbb{N} \to \mathbb{N}$ such that, for every real $r \in [a, b]$, every rational $q \in [a, b]$, and every $n \in \mathbb{N}$, if $|q - r| < 2^{-phi(n)}$, then $F(r) - f_n(q)| < 2^{-n}$.

Definition 8.2.8. A function $\mu : \mathbb{N} \to \mathbb{N}$ is a *modulus function* for a continuous function $F : [a, b] \to [a, b]$ if for every n and every $r, s \in [a, b]$, $|r - s| \leq 2^{-\mu(n)}$ implies that $F(r) - F(s)| \leq 2^{-n}$.

It is easy to see that F is continuous if and only if it has a modulus function.

Proposition 8.2.9. *If $F : [a, b] \to [a, b]$ is computable, then F has a computable modulus function and F is continuous.*

Proof. Let F, $\{f_n : n \in \omega\}$, and ϕ be given as above and suppose that $|r - s| \leq 2^{-\phi(n+1)}$ for reals $r < s \in [a, b]$. Then there exists a rational q between r and s such that $q - r$ and $s - q$ are both $\leq 2^{-\phi(n+1}$. It follows that $|F(r) - f_{n+1}(q)|$ and $F(s) - f_{n+1}(q)|$ are both $\leq 2^{-n-1}$ and therefore $|F(r) - F(s)| \leq 2^{-n}$. Thus $\mu(n) = \phi(n + 1)$ is a computable modulus function for F. □

Another approach to computable real functions is via approximations of continuous functions by simple functions such as polynomials or piecewise linear functions. In particular, polynomials with rational coefficients, as well as piecewise linear functions with rational critical points, will map rational points to rational points. Thus a computable sequence of such functions will provide a sequence of rational functions $\{f_n : n \in \omega\}$ as above. The following is equivalent to our definition above.

Definition 8.2.10 (Pour-El)**.** Fix computable reals $a < b$. A real function $F : [a, b] \to [a, b]$ is computable if there is a computable sequence P_n of rational polynomials and a function $\phi : \mathbb{N} \to \mathbb{N}$ such that, for all $r \in [a, b]$, $|F(r) - P_k(r)| \leq 2^{-n}$ whenever $k \geq \phi(n)$.

It is not hard to approximate piecewise linear functions by polynomials and vice versa, hence we have another equivalent alternative definition.

Definition 8.2.11. Fix computable reals $a < b$. A real function $F : [a, b] \to [a, b]$ is computable if there is a computable sequence P_n of rational piecewise linear functions and a function $\phi : \mathbb{N} \to \mathbb{N}$ such that, for all $r \in [a, b]$, $|F(r) - P_k(r)| \leq 2^{-n}$ whenever $k \geq \phi(n)$.

Theorem 8.2.12 (Computable Intermediate Value Theorem)**.** *For any computable real function $F : [a,b] \to [a,b] \to [a,b]$ and any computable real c such that $F(a) < c < F(b)$, there exists a computable real r such that $F(r) = c$.*

Proof. For simplicity, we give the argument when $a = 0$, $b = 1$ and $c = 0$. There are two distinct cases. First suppose that $F(r) = 0$ for some dyadic rational $i/2^k$. Then clearly such a zero r is computable. Thus we may assume that $F(r) \neq 0$ for any dyadic rational r. It follows that $F(1/2) \neq 0$ and without loss of generality $F(1/2) > 2^{-n}$ for some n. Now for all $m \geq \mu(n+1)$, $|f_m(1/2) - F(1/2)| \leq 2^{-n-1}$, so that $f_m(1/2) \geq 2^{-n-1}$. So we just compute $f_m(1/2)$ for $m = 0, 1, \ldots$ until we find m and n such that $m \geq \mu(n+1)$ and $f_m(1/2) \geq 2^{-n-1}$. This will tell us that $F(1/2) > 0$. Similarly, if $F(1/2) < 0$, then we will eventually learn this. If we see that $F(1/2) > 0$, then there must be a zero in the smaller interval $[0, 1/2]$, so we next examine the value $F(1/4)$; if $F(1/2) < 0$, then we look at $[1/2, 1]$ and examine $F(3/4)$. Continuing in this fashion, after n steps we will have an interval of diameter 2^{-n} which must contain a zero of F; let q_n be the midpoint of this interval. Then the sequence q_n is computable and the intersection of these intervals willl be a singleton $\{r\}$ such that $r = lim_n q_n$, $|q_n - r| \leq 2^{-n}$, and $F(r) = 0$. □

It can be shown that there is no uniform algorithm which computes the zero $r \in [a,b]$ from the values a, b.

On the other hand, if say $F(r) \geq 0$ for all $r \in [a,b]$, then it is possible that $F(r) = 0$ for some r but not for any computable r.

Theorem 8.2.13. *There is a computable real function F on $[-1,1]$ such that $F(r) = 0$ for infinitely many reals in $[-1,1]$ but there is no lower semi-computable r real in $(-1,1)$ such that $F(r) = 0$.*

Proof. Let ϕ_e be the e^{th} partial computable function mapping $\mathbb{Q} \cap [0,1] \to \{0,1\}$, so that $r \in [0,1]$ is a computable real if and only if ϕ_e is the characteristic function of $\{q \in \mathbb{Q} : q < r\}$. We want to satisfy for each e, the following requirement:

$\boldsymbol{R_e}$: If ϕ_e is total and is the characteristic function of a Dedekind cut D_r for some $r = r_e$, then $F(r_e) > 0$.

At the same time, we need to have $F(x) = 0$ for *some* $x \in [-1,1]$.

For each individual requirement $\boldsymbol{R_e}$, we look for rationals $p < q$ with $q - p < 2^{-e-1}$, and either $q > -1$ or $p < 1$, and a stage s such that $\phi_e^s(p) = 1$ and $\phi_e^s(q) = 0$, so that we know that $p < r < q$.Then we can ensure that $F(r) > 0$ by adding a triangular bump F_e^s to F, where $F_e^s(x) = 0$ if either $x \leq p$ or $x \geq q$, $F(\frac{p+q}{2}) = 2^{-s-1}$, and the function is linear from $(p, 0)$ to $(\frac{p+q}{2}, 2^{-s-1}$ and from there to $(q, 0)$.

The construction of $F : [-1,1] \to [-1,1]$ is now given in stages. Initially we have $F(x) = 0$ for all x. At stage $s+1$, we look for the least e such that p and q exist as above, and such that requirement R_e has not yet been satisfied. Then we add the function $F_s = F_e^s$, described above, to F. Once this has been done for e, we say that the eth requirement is satisfied, and we no longer act on it at future stages. Then we define $F(x) = \sum_{s \leq n} F_s(x)$. and let the piecewise linear function $P_n(x) = \sum_{s \leq n} F_s(x)$. Then we have $F_s(x) \leq 2^{-s-1}$ for all reals x, so that

$$F(x) - P_n(x) = \sum_{s>n} F_s(x) \leq \sum_{s>n} 2^{-s-1} = 2^{-n-1}.$$

Thus $F(x)$ is a computable function, according to Definition 8.2.10.

We need to check that $F(x)$ does equal zero for some $x \in [-1,1]$. The explanation is that for each e, the domain of the bump F_e^s corresponding to action on Requirement e has diameter $< 2^{-e-1}$. Thus the total measure of the union of these domains is $< \sum_e 2^{-e-1} = 1$, and it follows that the measure of $\{x \in [-,1,1] : F(x) = 0\}$ is at least 1.

Now suppose that $r \in [-,1,1]$ is a computable real. Then for some e, ϕ_e is the characteristic function of $\{q : q < r\}$. Since the rationals are dense, there must exist rationals p, q with $p < r < q$ and $q - p < 2^{-e}$. Then at some stage s, we will see $\phi_e^s(p) = 1$ and $\phi_e^s(q) = 0$. Since there are only finitely many natural numbers $d < e$, eventually there will be a stage s where e will be the least not yet acted on, and we will add the function F_e^s at that stage. Therefore $F(r) > 0$. □

8.3 Exercises

1. Find a computable sequence q_n of rationals to show that $sin\ 2$ is computable.
2. Show that a real r is lower semi-computable if and only if $\{q \in \mathbb{Q} : q < r\}$ is computably enumerable.

Chapter 9

Boolean Algebras

Chapter 10

Real Numbers

Chapter 11

Nonstandard Analysis

Chapter 12

Algorithmic Randomness

Bibliography