Equivalence Structures and Isomorphisms in the Difference Hierarchy

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Abstract

We examine the notion of structures and functions in the Ershov difference hierarchy, and of equivalence structures in particular. A equivalence structure \( A = (A, E) \) has universe \( A = \omega \) and an equivalence relation \( E \). The equivalence class \([a]\) of \( a \in A \) is \( \{b \in A : aEb\} \) and the character \( K \) of \( A \) is \( \{(k, n) \in (\omega - \{0\})^2 : A \text{ has at least } n \text{ classes of size } k\} \). It is known that for any character \( K \), there exists an equivalence structure with character \( K \) if and only if \( K \) is \( \Sigma^0_2 \) but that there exists a \( \Delta^0_2 \) character such that any equivalence structure with character \( K \) must have infinite equivalence classes. We show: (1) for any \( n \)-c.e. character \( K \), there is an equivalence structure with character \( K \) and no infinite equivalence classes; (2) there is a \( \omega \)-c.e. character \( K \) such that any equivalence structure with character \( K \) must have infinite equivalence classes; (3) For any \( \Delta^0_2 \) character \( K \), there exists a d.c.e equivalence structure with no infinite equivalence classes and character \( K \). We define the notions of \( \alpha \)-c.e. functions and graph-\( \alpha \)-c.e. functions and show: (1) Any nonempty \( \Sigma^0_2 \) set is the range of 2-c.e. function; (2) for every \( n \), there is an \( (n + 1) \)-c.e. function which is not graph-\( n \)-c.e.; (3) there is a graph-2-c.e function that is not \( \omega \)-c.e.; (4) there is a 2-c.e. bijection such that \( f^{-1} \) is not \( \omega \)-c.e. We define the notions of (weakly) \( \alpha \)-c.e. and of graph-\( \alpha \)-c.e. isomorphisms and show: (1) For each \( n \), there exist computable equivalence structures which are \( n+1 \)-c.e. isomorphic but not weakly \( n \)-c.e. isomorphic; (2) there are computable equivalence structures which are graph-2-c.e isomorphic but not weakly \( \omega \)-c.e. isomorphic. We show that a computable equivalence structure is computably categorical if and only if it is weakly \( \omega \)-c.e. categorical, by examining all cases. We show that any computable equivalence structure with bounded character \( K \) (and any number of infinite equivalence classes) is relatively graph-2-c.e. categorical and we show that any computable equivalence structure with a finite number of infinite equivalence classes is relatively graph-\( \omega \)-c.e. categorical. It follows that a computable equivalence structure is \( \Delta^0_2 \) categorical if and only if it is graph-\( \omega \)-c.e. categorical.