Equivalence Structures and Isomorphisms in the Difference Hierarchy

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August 23, 2007

Abstract

We examine the notion of structures and functions in the Ershov difference hierarchy, and of equivalence structures in particular. A equivalence structure $\mathcal{A} = (A, E)$ has universe $A = \omega$ and an equivalence relation E. The equivalence class $[a]$ of $a \in A$ is $\{b \in A : aEb\}$ and the *character* K of A is $\{\langle k, n\rangle \in (\omega - \{0\})^2 : \mathcal{A}$ has at least n classes of size $k\}$. It is known that for any character K, there exists an equivalence structure with character K if and only if K is Σ^0_2 but that there exists a Δ^0_2 character such that any equivalence structure with character K must have infinite equivalence classes. We show: (1) for any n-c.e. character K, there is an equivalence structure with character K and no infinite equivalence classes; (2) there is an ω-c.e. character K such that any equivalence structure with character K must have infinite equivalence classes; (3) For any Δ^0_2 character K, there exists a d.c.e equivalence structure with no infinite equivalence classes and character K. We define the notions of α-c.e. functions and graph-α-c.e. functions and show: (1) Any nonempty Σ^0_2 set is the range of 2-c.e. function; (2) for every n, there is an $(n + 1)$-c.e. function which is not graph-n-c.e.; (3) there is a graph-2-c.e function that is not ω-c.e.; (4) there is a 2-c.e. bijection such that f^{-1} is not ω-c.e. We define the notions of (weakly) α-c.e. and of graph-α-c.e. isomorphisms and show: (1) For each n, there exist computable equivalence structures which are $n + 1$-c.e. isomorphic but not weakly n-c.e. isomorphic; (2) there are computable equivalence structures which are graph-2-c.e isomorphic but not weakly ω-c.e. isomorphic. We show that a computable equivalence structure is computably categorical if and only if it is weakly ω-c.e. categorical, by examining all cases. We show that any computable equivalence structure with bounded character K (and any number of infinite equivalence classes) is relatively graph-2-c.e. categorical and we show that any computable equivalence structure with a finite number of infinite equivalence classes is relatively graph-ω-c.e. categorical. It follows that a computable equivalence structure is Δ^0_2 categorical if and only if it is graph-ω-c.e. categorical.