\[ \Sigma^0_1 \text{ and } \Pi^0_1 \text{ equivalence structures} \]

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Abstract

We study computability theoretic properties of \( \Sigma^0_1 \) and \( \Pi^0_1 \) equivalence structures and how they differ from computable equivalence structures or equivalence structures that belong to the Ershov difference hierarchy. This includes the complexity of isomorphisms between \( \Sigma^0_1 \) equivalence structures and between \( \Pi^0_1 \) equivalence structures. Any \( \Sigma^0_1 \) equivalence structure \( A \) with infinitely many infinite equivalence classes is isomorphic to a computable structure. On the other hand, there are \( \Sigma^0_1 \) equivalence structures with finitely many infinite equivalence classes, which are not isomorphic to any computable structure. If \( \Sigma^0_1 \) equivalence structures \( A_1 \) and \( A_2 \) are isomorphic to a computable structure \( A \) that is computably categorical or relatively \( \Delta^0_2 \) categorical, then \( A_1 \) and \( A_2 \) are \( \Delta^0_2 \) isomorphic.

If \( B \) is a computably categorical computable equivalence structure and \( A \) is a \( \Pi^0_1 \) structure which is isomorphic to \( B \), then \( A \) and \( B \) are \( \Delta^0_2 \) isomorphic. If \( B \) is a computable equivalence structure which is not computably categorical, then in several cases we construct a \( \Pi^0_1 \) structure \( A \) which is isomorphic to \( B \) but is not \( \Delta^0_2 \) isomorphic to \( B \). The simplest case is when \( B \) consists of infinitely many classes of sizes 1 or 2, and no other classes; if \( B \) is \( \Delta^0_2 \) categorical, then we show that the \( \Pi^0_1 \) structure \( A \) is moreover not \( \Delta^0_2 \) isomorphic to any \( \Sigma^0_1 \) structure.

The spectrum question is to determine the possible sets (or degrees of sets) that can compose the elements in equivalence classes of size \( k \), for some fixed \( k \), in a computable equivalence structure. For example, for any infinite c.e. set \( B \), there is a computable equivalence structure with infinitely many equivalence classes of size 1, infinitely many classes of size 2, and no other equivalence classes, such that \( B = \{ x : \text{card}(\{x\}) = 2 \} \).

We explore the connection between the complexity of the character \( \chi(A) \) and the theory \( Th(A) \). If \( Th(A) \) is decidable, then the character \( \chi(A) \) is computable. If an equivalence structure \( B \) has a computable character, then there is a decidable structure \( A \) isomorphic to \( B \).