Σ_1^0 and Π_1^0 equivalence structures

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Abstract

We study computability theoretic properties of Σ_1^0 and Π_1^0 equivalence structures and how they differ from computable equivalence structures or equivalence structures that belong to the Ershov difference hierarchy. This includes the complexity of isomorphisms between Σ_1^0 equivalence structures and between Π_1^0 equivalence structures. Any Σ_1^0 equivalence structure $\mathcal{A}$ with infinitely many infinite equivalence classes is isomorphic to a computable structure. On the other hand, there are Σ_1^0 equivalence structures with finitely many infinite equivalence classes, which are *not* isomorphic to any computable structure. If Σ_1^0 equivalence structures $\mathcal{A}_1$ and $\mathcal{A}_2$ are isomorphic to a computable structure $\mathcal{A}$ that is computably categorical or relatively Δ_2^0 categorical, then $\mathcal{A}_1$ and $\mathcal{A}_2$ are Δ_2^0 isomorphic.

If $\mathcal{B}$ is a computably categorical computable equivalence structure and $\mathcal{A}$ is a Π_1^0 structure which is isomorphic to $\mathcal{B}$, then $\mathcal{A}$ and $\mathcal{B}$ are Δ_2^0 isomorphic. If $\mathcal{B}$ is a computable equivalence structure which is *not* computably categorical, then in several cases we construct a Π_1^0 structure $\mathcal{A}$ which is isomorphic to $\mathcal{B}$ but is *not* Δ_2^0 isomorphic to $\mathcal{B}$. The simplest case is when $\mathcal{B}$ consists of infinitely many classes of sizes 1 or 2, and no other classes; if $\mathcal{B}$ is Δ_2^0 categorical, then we show that the Π_1^0 structure $\mathcal{A}$ is moreover not Δ_2^0 isomorphic to any Σ_1^0 structure.

The *spectrum question* is to determine the possible sets (or degrees of sets) that can compose the elements in equivalence classes of size k, for some fixed k, in a computable equivalence structure. For example, for any infinite c.e. set B, there is a computable equivalence structure with infinitely many equivalence classes of size 1, infinitely many classes of size 2, and no other equivalence classes, such that $B = \{x : card([x]) = 2\}$.

We explore the connection between the complexity of the character $\chi(\mathcal{A})$ and the theory $Th(\mathcal{A})$. If $Th(\mathcal{A})$ is decidable, then the character $\chi(\mathcal{A})$ is computable. If an equivalence structure $\mathcal{B}$ has a computable character, then there is a decidable structure $\mathcal{A}$ isomorphic to $\mathcal{B}$.