Effectively Closed Sets
Π^0_1 Classes
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Preface

Effectively closed sets have been a central theme in computability theory, algorithmic randomness and applications to computability and effectiveness in mathematics. This book is intended to be a self-contained introduction to the theory and applications of effectively closed sets, or Π^0_1 classes. It may be used for a graduate-level course and also as reference for researchers in computability theory and related areas.

Part A begins with some basic facts from computability theory which will be needed. The members of a Π^0_1 class are real numbers, often represented by infinite strings of natural numbers, or by sets of natural numbers. Background is taken from the classic book of Soare [198] on computably enumerable (c.e.) sets and degrees. The fundamental problem, going back to work of Kleene [108] in the period 1940-1960, is to determine the complexity of the members of a Π^0_1 class, as measured by the Turing degree, or by the definition in the hyperarithmetic hierarchy, or by the amount of resources in time and space required. The Kleene basis theorem showed that every Π^0_1 class contains a member which is recursive in some Σ^1_1 set and the Kreisel-Shoenfield basis theorem [188], which showed that every c. b. Π^0_1 class contains a member of degree $< 0'$. Two fundamental papers in this area are [99, 98] by Jockusch and Soare. They show, among other things, that there is a Π^0_1 class with no recursive members and such that any two members have mutually incomparable Turing degree.

The Cantor-Bendixson derivative which reduces a closed set to its perfect kernel, plays an important role here going back to the 1959 paper of Kreisel [116], who first noticed that the degree of a member x of a Π^0_1 class is related to the Cantor-Bendixson rank of x in P and that any countable class has a computable member. Countable Π^0_1 classes were closely examined by Soare and others [23, 46] in the 1980's. Π^0_1 classes are given an enumeration as $P_0, P_1, \dots$ and index sets for families of Π^0_1 classes are then studied in the manner that index sets for c.e. sets are studied in [198]. These can measure the complexity of certain properties of Π^0_1 classes, related in particular to cardinality and measure. Π^0_1 classes may be defined as sets of infinite paths through computable trees.

Part B presents some applications of Π^0_1 classes in logic, mathematics and theoretical computer science. The solution sets of many mathematical problems may be represented by Π^0_1 classes and the complexity of the problem can then be determined. The more difficult representation problem is to show that every Π^0_1 class (or every bounded or c. b. Π^0_1 class) can represent the solution set of a certain problem. For example, in 1960, Shoenfield [189] showed that the family of complete consistent extensions of an axiomatizable theory is a c. b. Π^0_1 class and Ehrenfeucht [69] showed that any c. b. Π^0_1 class can represent such a family.

The family of complete consistent extensions of an axiomatizable theory is of course closely related to the Lindenbaum algebra of the theory and Boolean algebras are an important topic for Π^0_1 classes. A number of articles in the area use the notion of a computably enumerable ideal of the computable dense Boolean algebra as an equivalent notion to that of a Π^0_1 class. This concept will be discussed in detail in the section on Boolean algebras.

Non-monotonic logic [135] is a general form of reasoning where certain "default" assumptions are made and may later be rescinded. The set of stable models of a logic program is a non-monotonic generalization of the (unique) closure under consequence of a set of axioms and rules. Different versions of a logic program may be used to represent c. b., bounded and unbounded Π^0_1 classes. Another area of theoretical computer science where Π^0_1 classes have application is the study of ω-languages. This refers to a sets of infinite words which is *accepted*, in some fashion, by a program.

The surjective matching problem of Philip and Marshall Hall [82] was analyzed by Manaster and Rosenstein, who showed that the set of bijective matchings in a symmetrically highly recursive society is always a c. b. Π^0_1 class, and can represent an arbitrary c. b. Π^0_1 class. Bean [12] showed in 1976 that the family of k-colorings of a highly computable graph is a c. b. Π^0_1 class and Remmel [174] showed that any c. b. Π^0_1 class can represent, up to a permutation of the colors, such a family.

The reason that Π^0_1 classes arise so naturally in the study of recursive combinatorics is that many combinatorial theorems about finite graphs and partially ordered sets (posets) can be extended to countably infinite graphs and posets by applying König's Lemma, which states that every infinite finitely branching tree T has an infinite path through it. Now König' Lemma, and also the so-called Weak König's Lemma play an important role in the Reverse Mathematics program of Friedman and Simpson [192]. Thus the study of Π^0_1 classes can be related to the study of König's Lemma. For example, Simpson [192] showed that Lindenbaum's lemma (that every countable consistent set of sentences has a complete consistent extension) and Gödel's completeness theorem are both equivalent to Weak König's Lemma over a certain subsystem (RCA_0) of second order arithmetic. For another example, Hirst [88] showed that a version of Hall's symmetric matching theorem is equivalent to König's Lemma over RCA_0.

The role of Π^0_1 classes in computable algebra and computable analysis is also presented.

Part C examines recent results on the family of Π^0_1 classes. One very important topic is the connection between effectively closed sets and algorithmic randomness, as developed by many researchers from Kucera [117, **?**, 119] to Lewis [**?**, 11, 10] and surveyed in the books of Downey-Hirschfeldt [63] and Nies [162]. The lattice $\mathcal{E}_\Pi$ of Π^0_1 classes under inclusion is compared and contrasted with the lattice $\mathcal{E}$ of c.e. sets under inclusion. This includes results of Downey and others [25, 51, 50] on thin classes and automorphisms and work of Cenzer and Nies [32, 33] on intervals and on definability in $\mathcal{E}_\Pi$. The degree of difficulty of a class was defined by Medvedev [149] and refers to the difficulty of finding a member of the class. The Medvedev lattice of degrees of difficulty was studied later by Sorbi [200] and then the study of the Medvedev and also the related Muchnik degrees of Π^0_1 classes was developed further by Simpson [193] and others. Here we also examine Π^0_1 classes which arise from trees with a specified complexity, such as polynomial time computable.

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Part A

Computability Theory and Π^0_1 Classes

Chapter I

Background

This chapter contains some of the definitions and notations needed for the study of effectively closed sets. We begin with objects under study: numbers, functions, sequences (or strings) and trees.

The set $\{0, 1, 2, ...\}$ of natural numbers is denoted by $\mathbb{N}$ and also by ω when we view $\mathbb{N}$ as an ordered set. Here $n = \{0, 1, \ldots, n-1\}$ is identified with the set of smaller natural numbers. Lower-case Latin letters $a, b, c, d, e, i, j, k, l, m, n$ denote integers; p, q, r, s, t denote rational numbers; u, v, w, x, y, z denote real numbers. The letters f, g, h (and occasionally other lower-case Latin letters) denote *total* functions from $\mathbb{N}^k$ to $\mathbb{N}$ for $k \geq 1$; the Greek letters ϕ, ψ, θ (and occasionally other lower-case Greek letters) denote (possibly) *partial* functions on $\mathbb{N}^k$ (functions whose domain is a subset of $\mathbb{N}^k$ for some k). Lower case Greek letters ρ, σ, τ, ν denote finite sequences of natural numbers; $\alpha, \beta, \delta, \gamma$ denote ordinals. Upper-case Latin letters $A, B, C, D, E, I, J, K, L, M$ denote subsets of $\mathbb{N}$; S, T denote trees; P, Q, U, V, W, X, Y, Z denote sets of real numbers. Upper-case Latin letters F, G, H denote *total* functions of real variables (with domain and range included in $\mathbb{N}^m \times \Re^n$); Upper-case Greek letters Φ, Ψ, Θ (and occasionally others) denote (possibly) partial functions of real variables. In our usage, a *set* usually refers to a set of natural numbers.

The composition of two functions f and g is denoted by $f \circ g$; f^n denotes the function f composed with itself n times. For a partial function ϕ, $\phi(x) \downarrow$ denotes that $\phi(x)$ is defined and $\phi(x) \uparrow$ denotes that $\phi(x)$ is not defined. $dom(\phi) = \{x : \phi(x) \downarrow\}$ and $ran(\phi) = \{\phi(x) : x \in dom(\phi)\}$ denote the domain and range of ϕ, respectively. If $F : X \rightarrow Y$, then $F[U]$ denotes $\{F(x) : x \in U\}$ for $U \subseteq X$ and $F^{-1}[V]$ denotes $\{x : F(x) \in V\}$ for $V \subseteq Y$. χ_A denotes the characteristic function of A, which is often identified with A and written simply as $A(x)$. $\phi\lceil m$ denotes the restriction of A to x.

For two sets X and Y, $X \times Y$ denotes the direct product of X and Y, that is, the set of ordered pairs (x, y) with $x \in X$ and $y \in Y$. The direct product $X_1 \times X_2 \times \ldots \times X_k$ of a sequence $X_1, \ldots X_k$ of sets is similarly defined. X^k is the product of k copies of X.

The power X^Y of two sets denotes the set of (total) functions with domain

Y and range a subset of X. In particular, $\{0,1\}^{\mathbb{N}}$ is the usual Cantor space and may be identified with the family of subsets of $\mathbb{N}$. $\mathbb{N}^{\mathbb{N}}$ is the Baire space. $\Re$ denotes the space of real numbers. The Cantor space may be identified with a (compact) subset of $\Re$ and the Baire space may be identified with the set of irrational numbers. For us a *class* refers to a subset of $\Re$ (or of the Cantor space or Baire space). A class in the Cantor space may be called a "class of sets" since its elements are the characteristic functions of sets of natural numbers.

I.1 Trees

Let Σ be a set of symbols (an *alphabet*), usually an initial segment of $\mathbb{N}$. Then for a natural number n, Σ^n denotes the set of strings $\sigma = (\sigma(0), \sigma(1), \ldots, \sigma(n-1))$ of n letters from Σ; the length n of σ is denoted by $|\sigma|$. The empty string has length 0 and will be denoted by $\emptyset$. Σ^* (or sometimes $\Sigma^{<\omega}$) denotes the set $\cup_{n\in\omega}\Sigma^n$ and Σ^ω denotes the set of infinite sequences. Strings may be coded by natural numbers in the usual fashion. First let $[x,y]$ denote the standard pairing function $\frac{1}{2}(x^2+2xy+y^2+3x+y)$ and in general $[x_0,x_1,\ldots,x_n] = [[x_0,\ldots,x_{n-1}],x_n]$. Then we can code strings of arbitrary length $n > 0$ by $\langle\sigma\rangle = [n,[\sigma(0),\sigma(1),\ldots,\sigma(n-1)]]$ and also $\langle\emptyset\rangle = 1$. A string may be identified with its code, so that functions on $\mathbb{N}^*$ are represented by functions on $\mathbb{N}$. A constant string σ of length n will be denoted k^n. For $m < |\sigma|$, $\sigma \restriction m$ is the string $(\sigma(0),\ldots,\sigma(m-1))$; σ is an *initial segment* of τ (written $\sigma \prec \tau$) if $\sigma = \tau|m$ for some m. Initial segments are also referred to as *prefixes*. Similarly τ is said to be a *suffix* of σ if $|\tau| \le |\sigma|$ and, for all $i < |\tau|$, $\sigma(|\sigma| - |\tau| + i) = \tau(i)$. The *concatenation* $\sigma^\frown\tau$ (or sometimes $\sigma * \tau$ or just $\sigma\tau$) is defined by $\sigma^\frown\tau = (\sigma(0),\sigma(1),\ldots,\sigma(m-1),\tau(0),\tau(1),\ldots,\tau(n-1))$, where $|\sigma| = m$ and $|\tau| = n$; in particular we write $\sigma^\frown a$ for $\sigma^\frown(a)$ and $a^\frown\sigma$ for $(a)^\frown\sigma$. Thus we may also say that σ is a prefix of τ if and only if $\tau = \sigma^\frown\rho$ for some ρ and that τ is a suffix of σ if and only if $\sigma = \rho^\frown\tau$ for some ρ.

For any $x \in \Sigma^*$ and any finite n, the *initial segment* $x\lceil n$ of x is $(x(0),\ldots,x(n-1))$. We write $\sigma \preceq x$ if $\sigma = x\lceil n$ for some n. For any $\sigma \in \Sigma^n$ and any $x \in \Sigma^*$, we have $\sigma^\frown x = (\sigma(0),\ldots,\sigma(n-1),x(0),x(1),\ldots)$.

For a sequence $a_0 < a_1 < \cdots < a_n$, we denote by $\lfloor a_0,\ldots,a_n\rfloor$ the string $\sigma \in \{0,1\}^{a_n}$ such that $\sigma(k) = 1$ if and only if $k = a_i$ for some $i < n$. Thus $\lfloor a_0,a_1,\ldots,a_n\rfloor = 0^{a_0}10^{a_1-a_0-1}1\cdots 0^{a_{n-1}-a_{n-2}-1}10^{a_n-a_{n-1}-1}$.

For any $x,y \in \mathbb{N}^{\mathbb{N}}$, the *join* $x \oplus y = z$, where $z(2n) = x(n)$ and $z(2n+1) = y(n)$. For two classes P and Q, the product $P \otimes Q = \{x \oplus y : x \in P \;\&\; y \in Q\}$. An infinite sequence $x_0, x_1, \ldots$ may be coded as $\langle x_0,x_1,\ldots\rangle = y$, where $y(\langle m,n\rangle) = x_m(n)$. For an infinite family $\{P_i : i \in \omega\}$ of sets, the product may then be defined as $\{\langle x_0,x_1,\ldots\rangle : (\forall i)x_i \in P_i\}$. We can also define the disjoint union $P \oplus Q = \{0^\frown x : x \in P\} \cup \{1^\frown y : y \in Q\}$.

A *tree* T over Σ is a set of finite strings from Σ^* which is closed under initial segments. The set Σ is sometimes called an *alphabet*. We say that $\tau \in T$ is an *immediate successor* of a string $\sigma \in T$ if $\tau = \sigma^\frown a$ for some $a \in \Sigma$. Since our alphabet will always be countable and effective, we may assume that $T \subseteq \mathbb{N}^*$.

For any tree T and any σ, $T(\sigma) = \{\tau : \sigma \preceq \tau \text{ or } \tau \preceq \sigma\}$.

A tree T is said to be a *shift* if it is also closed under suffixes.

Example I.1.1. *Define $T \subset \{0,1\}^*$ so that $\sigma \in T$ if and only if σ does not have 3 consecutive 0's, that is, if σ has no consecutive substring of the form (000). Clearly if σ does not have 3 consecutive 0's then no initial segment of σ can have 3 consecutive 0's either. Furthermore, if σ has no consecutive substring* (000)*, then no suffix of σ can have a consecutive substring* (000)*. Thus T is a shift.*

We say that a tree T is *finite-branching* if for every $\sigma \in T$, there are only finitely many immediate successors of σ in T. Certainly any tree T over a finite alphabet is finite-branching.

Example I.1.2. *Define the tree $T \subset \mathbb{N}^*$ so that for strings σ of length n, $\sigma \in T \iff \sigma(n-1) \leq 1 + \sigma(0) + \sigma(1) + \ldots \sigma(n-2)$. Then for any $\sigma \in T$, $\sigma(0) \leq 1$, $\sigma(1) \leq 2$, and by induction $\sigma(n) \leq 2^n$; it follows that σ can have at most 2^n immediate successors.*

We will see later that a tree T is finite-branching if and only if there is a function f such that for all strings $\sigma \in T$ of length n, σ has at most $f(n)$ immediate successors. The problem of computing the function f will be a very important one. More generally, we will look at the problems of computing list of these successors, or an upper bound on the size of the successors, or an upper bound on the number of successors.

I.2 Topology and Measure

The topology of the real line has a basis of open intervals $(x, y) = \{u : x < u < y\}$ where $x = -\infty$ and $y = \infty$ are allowed; $[x, y]$ denotes the closed interval $\{u : x \leq u \leq y\}$; $[x, y)$ and $(x, y]$ are similarly defined. The topology on the spaces $\Sigma^{\mathbb{N}}$, where Σ is either a finite alphabet or equals $\mathbb{N}$, is determined by a basis of intervals $I(\sigma) = \{x : \sigma \prec x\}$ and has a sub-basis of sets of the form $\{x : x(m) = n\}$ for fixed m, n. Notice that each interval is also a closed set and is therefore said to be *clopen* and that the clopen subsets of the Cantor space $\{0,1\}^{\mathbb{N}}$ are just the finite unions of intervals.

For a tree $T \subseteq \Sigma^*$, we define the set $[T]$ of infinite paths through T by letting

$$x \in [T] \iff (\forall n) x \restriction n \in T.$$

A subset P of $\mathbb{N}^{\mathbb{N}}$ is closed if and only if $P = [T]$ for some tree T. This justifies the description of a Π^0_1 class as an effectively closed subset of $\mathbb{N}^{\mathbb{N}}$. A function $F : X \to Y$ is *continuous* if $F^{-1}[V]$ is open for every open set $V \subseteq Y$. Then a function $F : \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ is continuous if, for all m, n, $\{x : F(x)(m) = n\}$ is open.

Let X be either $\Re$, $\mathbb{N}^{\mathbb{N}}$ or $\{0,1\}^{\mathbb{N}}$. A subset Y of X is *dense* in an interval I if it meets every subinterval of I; Y is *nowhere dense* if it is dense in no interval.

Y is *meager (first category)* if it is a countable union of nowhere dense sets; Y is *non-meager (second category)* if it is not meager. Y is *comeager (residual)* if $\overline{Y}$ is meager.

An element $x \in Y$ is *isolated* in Y if there exists an open set U such that $Y \cap U = \{x\}$. A closed, non-empty set Y is *perfect* if it has no isolated elements. Each of the spaces $\Re$, $\mathbb{N}^{\mathbb{N}}$ and $\{0,1\}^{\mathbb{N}}$ are perfect.

Definition I.2.1. *The* Cantor-Bendixson derivative $D(P)$ *of a compact set* P *is the set of nonisolated points in* P.

Note that $D(P)$ is empty if and only if P is finite.

The iterated Cantor-Bendixson derivative $D^\alpha(P)$ of a closed set P is defined for all ordinals α by the following transfinite induction.

$D^0(P) = P$; $D^{\alpha+1}(P) = D(D^\alpha(P))$ for any α; $D^\lambda(P) = \bigcap_{\alpha<\lambda} D^\alpha(P)$ for any limit ordinal λ.

The *Cantor-Bendixson (C.B.) rank* of a closed set P is the least ordinal α such that $D^{\alpha+1}(P) = D^\alpha(P)$. If α is the C-B rank of P, then $D^\alpha(P)$ is the *perfect kernel* of P and is a perfect closed set. For an element $x \in P$ which is not in the perfect kernel, the *Cantor-Bendixson (C.B.) rank* of x in P is the least ordinal α such that $x \notin D^{\alpha+1}(P)$.

The standard Lebesgue measure μ on $\{0,1\}^\omega$ is determined by letting $\mu(I(\sigma)) = 2^{-|\sigma|}$. A product measure on $\mathbb{N}^{\mathbb{N}}$ may be defined (with $\lambda(\mathbb{N}^{\mathbb{N}}) = 1$) by setting the measure of $\{x : x(m) = n\}$ to be 2^{-n-1}, so that $I(\sigma)$ has measure $2^{-(m_0+m_1+\cdots+m_{k-1}+k)}$.

I.3 Structures

We shall use the logical symbols &, $\vee$, $\neg$, $\rightarrow$ and $\iff$ to denote as usual "and", "or", "not", "implies" and "if and only if". The symbols $\exists$ and $\forall$ denote the quantifiers "there exists" and "for all". In addition, $(\exists m < p)$ and $(\forall m < p)$ denote *bounded quantifiers* where the range of the quantifier is restricted to numbers less than p, and $(\exists^\infty x)$ denotes "there exist infinitely many x such that".

As usual, a *first-order language* $\mathcal{L}$ is given by a set $\{R_i\}_{i\in S}$ of relation symbols, a set $\{f_j\}_{j\in T}$ of function symbols, and a set $\{c_i\}_{i\in U}$ of constant symbols, together with functions $m(i)$ and $n(i)$ such that R_i is an $m(i)$-ary relation symbol and f_i is an $n(i)$-ary function symbol. We assume here that S, T and U are subsets of ω. The language also includes variables and both existential and universal quantifiers using these variables. The set of terms of $\mathcal{L}$ and the set $Sent(\mathcal{L})$ of sentences of $\mathcal{L}$ are defined as usual by induction. A propositional language is given by a set of 0-ary relation symbols, or propositional variables. The reader is referred to Shoenfield [190] for details.

We shall consider structures over an effective first-order language

$$\mathcal{L} = \langle \{R_i^{m(i)}\}_{i\in S}, \{f_i^{n(i)}\}_{i\in T}, \{c_i\}_{i\in U} \rangle,$$

where S, T and U are initial segments of ω, for all $i \in U$, c_i is a constant symbol and there are partial recursive functions s and t such that, for all $i \in S$, R_i is an $s(i)$-ary relation symbol and, for all $i \in T$, f_i is a $t(i)$-ary function symbol.

Let Γ be some complexity class of sets (and functions), such as partial recursive, primitive recursive, exponential time, polynomial time (or p-time). We say that a set or function is Γ-computable if it is in Γ.

A *model* or *structure*, $\mathcal{A} = (A, \{R_i^{\mathcal{A}}\}_{i\in S}, \{f_i^{\mathcal{A}}\}_{i\in T}, \{c_i^{\mathcal{A}}\}_{i\in U})$, for the language $\mathcal{L}$ is given by a set A together with interpretations of the relation, function and constant symbols.

Definition I.3.1. *(a) A structure (where the universe A of $\mathcal{A}$ is a subset of Σ^*) is a Γ-structure if*

(i) A is a Γ-computable subset of Σ^*

(ii) for each $i \in S$, $R_i^{\mathcal{A}}$ is a Γ-computable relation on $A^{m(i)}$.

(iii) for each $j \in T$, $f_j^{\mathcal{A}}$ is a Γ-computable function from $A^{n(j)}$ into A.

(iv) If $S = \omega$, then there is a Γ-computable relation R such that, for all $i \in S$ and all $(x_0, \ldots, x_{m(i)})$,

$$R_i^{\mathcal{A}}(x_0, \ldots, x_{m(i)}) \iff R(i, \langle x_0, \ldots, x_{m(i)}\rangle).$$

(v) If $T = \omega$, then there is a Γ-computable function f such that, for all $j \in T$ and all $(x_0, \ldots, x_{n(j)})$,

$$f_i^{\mathcal{A}}(x_0, \ldots, x_{n(j)}) = f(i, \langle x_0, \ldots, x_{n(j)}\rangle).$$

For any complexity class Γ, we say that two structures $\mathcal{A}$ and $\mathcal{B}$ are Γ-isomorphic if there is an isomorphism f from $\mathcal{A}$ onto $\mathcal{B}$ and Γ-computable functions F and G such that $f = F\lceil A$ (the restriction of F to A) and $f^{-1} = G\lceil B$.

I.4 Orderings and Ordinals

The results of this book are all theorems of Zermelo-Fraenkel Set Theory with the Axiom of Choice. The (Generalized) Continuum is not assumed.

Our set-theoretic conventions are standard and we refer the reader to (for example) Jech [90] for further background. The inclusion relation $X \subseteq Y$ denotes $(\forall x)(x \in X \to x \in Y)$ and $X \subset Y$ denotes $X \subseteq Y$ and $X \neq Y$. The symbols $\cup$, $\cap$ and $\setminus$ denote the binary operations of union, intersection and difference; $\overline{A}$ denotes the complement of A.

A set X is *transitive* if $(\forall y)(y \in X \to y \subseteq X)$ and X is an *ordinal (number)* if X and all of its elements are transitive. For ordinals α and β, $\alpha < \beta$ if and only if $\alpha \in \beta$. For any ordinal α, $\alpha + 1 = \alpha \cup \{\alpha\}$ is the *successor ordinal* of α. α is a *limit ordinal* if it is neither 0 nor a successor, which implies that $(\forall \beta < \alpha)(\beta + 1 < \alpha)$. For any set X of ordinals, $\inf X$ denotes the least element of X and $\sup X$ denotes the least ordinal greater than or equal to every element of X.

An ordinal α is said to be a *recursive ordinal* if there is a recursive well-ordering of ω of order type α. The least non-recursive ordinal is denoted by $\omega_1^{\mathrm{C-K}}$, and was introduced by Church and Kleene [52].

The natural, or Hessenberg sum, $\alpha \oplus \beta$, of two ordinals α and β, may be defined as follows. Let $\alpha = \omega^{\gamma_1} a_1 + \omega^{\gamma_2} a_2 + \cdots + \omega^{\gamma_k} a_k$ and $\beta = \omega^{\gamma_1} b_1 + \omega^{\gamma_2} b_2 + \cdots + \omega^{\gamma_k} b_k$ be the Cantor normal forms of α and β, where we have inserted $a_i = 0$ and $b_j = 0$ to obtain expressions with the same powers of ω. Then

$\alpha \oplus \beta = \omega^{\gamma_1}(a_1 + b_1) + \omega^{\gamma_2}(a_2 + b_2) + \cdots + \omega^{\gamma_k}(a_k + b_k)$.

Thus we treat ordinals as polynomials over ω with natural number coefficients. This natural addition is commutative. For any ordinals α and β, $\alpha + \beta \leq \alpha \oplus \beta$. See [121] (p. 253) for details.

An ordinal κ is a *cardinal number* if there is no one-to-one correspondence between κ and any $\alpha < \kappa$. It follows from the Axiom of Choice that for every set X, there is a unique cardinal κ and a one-to-one correspondence between X and κ; κ is the *cardinality* ($Card(X)$)of X. The natural numbers are exactly the finite cardinals and ω is the least infinite cardinal. A set X is *countable* if $Card(X) \leq \omega$ and *countably infinite* if $Card(X) = \omega$. The infinite cardinal ω is also denoted by $\aleph_0$ and the least uncountable cardinal by $\aleph_1$.

For any set X, $\mathcal{P}(X)$ denotes the power set of X, the set of all subsets of X and 2^κ denotes $Card(\mathcal{P}(\kappa)$. Since there is a one-to-one correspondence between $\mathcal{P}(\mathbb{N})$ and the continuum $\Re$, $Card(\Re) = 2^{\aleph_0}$.

A relation R on a set X is a subset of $X \times X$; the domain of R is $dom(R) = \{x : (\exists y)(x, y) \in R\}$ and the range is $ran(R) = \{y : (\exists x)(x, y) \in R\}$. $R(x, y)$ and also xRy are sometimes used in place of $(x, y) \in R$. R is *reflexive* if $R(x, x)$ for all x and is *irreflexive* if $\neg R(x, x)$ for all x. R is *symmetric* if $R(x, y)$ implies $R(y, x)$ for all x, y and is *antisymmetric* if $R(x, y)$ & $R(y, x)$ implies $y = x$ for all x, y. R is *transitive* if $R(x, y)$ & $R(y, z))$ implies $R(x, z)$ for all x, y, z. R is *total* or *connected* if $R(x, y) \ \vee \ R(y, x)$ for all x, y. R is an *equivalence relation* if it is symmetric, reflexive and transitive.

R is a *pre-partial-ordering* if it is reflexive and transitive. A pre-partial-ordering R is a *pre-linear-ordering* if it is total. A pre-partial-(linear-)ordering is a partial (linear) ordering if it is antisymmetric.

R is is *well-founded* if every subset A of X has a *minimal element*, that is, some m such that for all x, $R(x, m) \rightarrow R(m, x)$. Assuming the Axiom of Dependent Choice (DC), this is equivalent to the following

$$(\forall f \in \mathbb{N}^X)[(\forall m)(R(f(m+1), f(m)) \rightarrow (\exists m)R(f(m), f(m+1)).$$

A (pre-)linear ordering is a *(pre-)well-ordering* if it is well-founded.

Chapter II

Computability Theory

In this chapter, we present some basic definitions and results from classical computability theory which are needed for the study of Π^0_1 classes. The key notion here is that of a computable functional, or function with domain a subset of $\mathbb{N}^{\mathbb{N}}$.

We begin with a brief review of computable functions and computably enumerable (c. e.) sets. Formal definitions of the set of computable functions have been given in many different ways. The computable functions are the functions mapping natural numbers (or more generally finite strings of symbols taken from a finite alphabet) which are computable by a Turing machine, register machine, or other idealized computer. These are the functions which can be computed by a program in Maple, or Matlab, or some other fixed programming language. The set of computable functions is the smallest which includes certain basic functions and is closed under primitive recursion, composition, and unbounded search.

All of these approaches are known to lead to the same family of functions, and Church's Thesis proclaims that any other attempt to formalize the notion of a computable function will lead to the same family of functions.

We refer the reader to Soare [198] and to Odifreddi [163] for full details on the basic definitions and results of computability theory.

II.1 Formal definitions of the computable functions

Since index sets will be a central topic in our work, we will give a definition in the spirit of Kleene [110] and Hinman [87] based on the index or code for a computable function. We will give the general definition for a computable function or functional with both natural number inputs and real number inputs (that is, functions from $\mathbb{N}^{\mathbb{N}}$). It is crucial that our functions may be *partial*, that is, defined on a proper subset of $\mathbb{N}^k \times (\mathbb{N}^{\mathbb{N}})^l$. The second crucial observation is that the (partial) computable functions may be enumerated as $\Phi_0, \Phi_1, \ldots$ so that the *universal* function $U(e, \overrightarrow{m}, \overrightarrow{x}) = \Phi_e(\overrightarrow{m}, \overrightarrow{x})$ is itself partial computable.

An index $e = \langle i, k, \ell, \ldots \rangle$ for a computable function is the code for a function Φ_e of k natural numbers and ℓ real numbers. Φ_e is a function on natural numbers if $\ell = 0$ and will then be denoted also by ϕ_e. Here $\overrightarrow{m} = (m_0, \ldots, m_{k-1})$ and $\overrightarrow{x} = (x_0, \ldots, x_{\ell-1})$.

The basic indices and functions are the following:

(0) Constant Functions: $\Phi_e(\overrightarrow{m}, \overrightarrow{x}) = n$ when $e = \langle 0, k, \ell, n \rangle$.

(1) Projection Functions: $\Phi_e(\overrightarrow{m}, \overrightarrow{x}) = m_i$ when $e = \langle 1, k, \ell, i \rangle$ and $i < k$.

(2) Successor Functions: $\Phi_e(\overrightarrow{m}, \overrightarrow{x}) = m_i + 1$ when $e = \langle 2, k, \ell, i \rangle$ and $i < k$.

(3) Application Functions: $\Phi_e(\overrightarrow{m}, \overrightarrow{x}) = x_j(m_i)$ when $e = \langle 3, k, \ell, i, j \rangle$, $i < k$ and $j < \ell$.

The primitive recursive functions are obtained from the basic functions by closure under composition and primitive recursion, which are defined as follows.

(4) Composition: $\Phi_e(\overrightarrow{m}, \overrightarrow{x}) = \Phi_a(\Phi_{b_1}(\overrightarrow{m}, \overrightarrow{x}), \ldots, \Phi_{b_r}(\overrightarrow{m}, \overrightarrow{x}))$ when $e = \langle 4, k, \ell, a, b_1, \ldots, b_r \rangle$ when $(a)_1 = r$, $(a)_2 = 0$ and, for each t, $(b_t)_1 = k$ and $(b_t)_2 = \ell$.

(5) Primitive Recursion: $\Phi_e(0, \overrightarrow{m}, \overrightarrow{x}) = \Phi_a(\overrightarrow{m}, \overrightarrow{x})$ and, for each n, $\Phi_e(n + 1, \overrightarrow{m}, \overrightarrow{x}) = \Phi_b(\Phi_e(n, \overrightarrow{m}, \overrightarrow{x}), n, \overrightarrow{m}, \overrightarrow{x}))$ when $e = \langle 5, k{+}1, \ell, a, b \rangle$, $(a)_1 = k$, $(a)_2 = \ell$, $(b)_1 = k + 2$ and $(b)_2 = \ell$.

A *set* $A \subseteq \mathbb{N}^k$ is primitive recursive if the characteristic function is primitive recursive. It is worth noting that the set of indices for primitive recursive functions is itself a primitive recursive set. Thus we may define an enumeration Π_e of the primitive recursive functions by letting $\Pi_e(\overrightarrow{m}, \overrightarrow{x}) = \Phi_e(\overrightarrow{m}, \overrightarrow{x})$ if e is a primitive recursive index and otherwise $\Pi_e(\overrightarrow{m}, \overrightarrow{x}) = 0$.

Lemma II.1.1. *There is a partial recursive function π such that for each e, $\Pi_e = \Phi_{\pi(e)}$.* $\square$

Details are left to the exercises.

The computable functions are obtained from the basic functions by closure under composition, primitive recursion and search, which is defined as follows. Here we let "(least p)$R(p)$" denote the least p such that $R(p)$.

(6) Search: $\Phi_e(\overrightarrow{m}, \overrightarrow{x}) = (\text{least } p)\Phi_a(p, \overrightarrow{m}, \overrightarrow{x}) = 0$ when $e = \langle 6, k, \ell, a \rangle$, where this means as usual that $\Phi_e(\overrightarrow{m}, \overrightarrow{x}) = q$ if $\Phi_a(q, \overrightarrow{m}, \overrightarrow{x}) = 0$ and for all $p < q$, $\Phi_a(p, \overrightarrow{m}, \overrightarrow{x})$ is defined and not equal to zero.

If $\Phi_e(\overrightarrow{m}, \overrightarrow{x})$ is defined by the above, we say that $\Phi_e(\overrightarrow{m}, \overrightarrow{x})$ *converges* and write $\Phi_e(\overrightarrow{m}, \overrightarrow{x}) \downarrow$. If $\Phi_e(\overrightarrow{m}, \overrightarrow{x})$ is not determined by this definition, then $\Phi_e(\overrightarrow{m}, \overrightarrow{x})$ is undefined. We say that $\Phi_e(\overrightarrow{m}, \overrightarrow{x})$ *diverges* and write $\Phi_e(\overrightarrow{m}, \overrightarrow{x}) \uparrow$.

If e is not an index of a computable function, then of course $\Phi_e(\overrightarrow{m}, \overrightarrow{x}) \uparrow$ for all $\overrightarrow{m}, \overrightarrow{x}$, so that Φ_e is the empty function.

If we replace the real variables x_j with finite sequences σ_j, then the definition of $\Phi_e(\overrightarrow{x}, \overrightarrow{\sigma})$ is obtained as above when we begin with $\Phi_e(\overrightarrow{m}, \overrightarrow{\sigma}) = \sigma_j(m_i)$ provided that $m_i < |\sigma_j|$.

Then the computation of $\Phi_e(\overrightarrow{m}, \overrightarrow{x}) = q$ is coded by $c = \langle e, \overrightarrow{m}, \overrightarrow{\sigma}, q\rangle$, where σ_j is the shortest initial segment of x_j needed.

We will next define the notions of a *computation tree* and a *derivation* for a computation.

For the constant, projection and successor functions, the computation tree of $\Phi_e(\overrightarrow{m}, \overrightarrow{x}) = n$ has a single node $\langle e, \overrightarrow{m}, \overrightarrow{\emptyset}, n\rangle$ and this is also the derivation.

For the application function, the computation tree for $\Phi_e(\overrightarrow{m}, \overrightarrow{x}) = x_i(m_j) = n$ also has a single node $\langle e, \overrightarrow{m}, \overrightarrow{\emptyset}, (x_i(0), \ldots, x_i(m_j)), \overrightarrow{\emptyset}, n\rangle$ and this is the derivation.

The other cases are more complicated.

(4) Composition:

The computation tree for $\Phi_e(\overrightarrow{m}, \overrightarrow{x}) = \Phi_a(\Phi_{b_0}(\overrightarrow{m}, \overrightarrow{x}), \ldots, \Phi_{b_{r-1}}(\overrightarrow{m}, \overrightarrow{x})) = q$ has a top node $c = \langle e, \overrightarrow{m}, \overrightarrow{\sigma}, q\rangle$ and has immediate predecessors $c_0, \ldots, c_{r-1}, c'$, where c_t is the top node of the computation tree for $\Phi_{b_t}(\overrightarrow{m}, \overrightarrow{x})$ for $t < r$, and c' is the top node of the computation tree for $\Phi_a(\Phi_{b_0}(\overrightarrow{m}, \overrightarrow{x}), \ldots, \Phi_{b_{r-1}}(\overrightarrow{m}, \overrightarrow{x}))$. For each j, σ_j is the union of the initial segments of x_j used in c_t. The derivation is $\langle d_1, \ldots, d_{r-1}, d, c\rangle$ where d_t is the derivation of $\Phi_{b_t}(\overrightarrow{m}, \overrightarrow{x})$ for $t < r$ and d is the derivation of $\Phi_a(\Phi_{b_0}(\overrightarrow{m}, \overrightarrow{x}), \ldots, \Phi_{b_{r-1}}(\overrightarrow{m}, \overrightarrow{x}))$.

(5) Primitive Recursion:

The computation tree for $\Phi_e(0, \overrightarrow{m}, \overrightarrow{x}) = \Phi_a(\overrightarrow{m}, \overrightarrow{x}) = q_0$ has top node $d_0 = \langle e, 0, \overrightarrow{m}, \overrightarrow{\sigma}, q_0\rangle$ with a single immediate predecessor $c_0 = \langle a, \overrightarrow{m}, \overrightarrow{\sigma}, q_0\rangle$. The derivation is $\langle c_0, d_0\rangle$.

The computation tree for $\Phi_e(n+1, \overrightarrow{m}, \overrightarrow{x}) = \Phi_b(\Phi_e(n, \overrightarrow{m}, \overrightarrow{x}), n, \overrightarrow{m}, \overrightarrow{x})) = q_{n+1}$ has top node $d_{n+1} = \langle e, n+1, \overrightarrow{m}, \overrightarrow{\sigma}, q_{n+1}\rangle$ with two immediate predecessors, the top node d_n of the computation tree for $\Phi_e(n, \overrightarrow{m}, \overrightarrow{x})$ and the top node c_n of the computation tree for $\Phi_b(q_n, n, \overrightarrow{m}, \overrightarrow{x}))$. For each j, σ_j is the union of the initial segments of x_j used in c_n and in d_n. The derivation is $\langle d_n, c_n, d_{n+1}\rangle$.

(6) Search:

The computation tree for $q = \Phi_e(\overrightarrow{m}, \overrightarrow{x}) = (\text{least } p)\Phi_a(p, \overrightarrow{m}, \overrightarrow{x}) = 0$ has top node $d = \langle e, \overrightarrow{m}, \overrightarrow{\sigma}, q\rangle$ with immediate predecessors $c_0, \ldots, c_p$ where c_t is the top node of the computation tree for $\Phi_a(t, \overrightarrow{m}, \overrightarrow{x})$ for $t \leq p$. For each j, σ_j is the union of the initial segments of x_j used in c_t for some $t \leq p$. The derivation is $\langle c_0, \ldots, c_p, d\rangle$.

We will often write $\Phi_e^y(\overrightarrow{m}, \overrightarrow{x})$ for $\Phi_e(\overrightarrow{m}, \overrightarrow{x}, y)$ and refer to the function Φ_e^y as being computable from the *oracle* y.

Lemma II.1.2. *The set of derivations is primitive recursive and, furthermore, the relation $T(e, \langle\overrightarrow{m}, \overrightarrow{\sigma}\rangle, d)$ which indicates that d is the derivation of $\Phi_e(\overrightarrow{m}, \overrightarrow{\sigma})$ is also primitive recursive.*

Sketch. The set of derivations may be defined by course-of-values recursion using coding and decoding of finite sequences, all of which is primitive recursive. Then the values of e, $\overrightarrow{m}$, $\overrightarrow{\sigma}$ and $\Phi_e(\overrightarrow{m}, \overrightarrow{\sigma})$ can be obtained from the last entry of the finite sequence coded by the derivation d. See Chapter II of Hinman [87] for details. □

II.1.1 Turing machines

The classic Turing machine, defined by Alan Turing, provides a very useful approach to computable functions. It has a simple elegant format but nevertheless has a strength equal to any other model of computing.

Our model of the Turing machine will be as follows. Let Σ_0 be a finite alphabet, let B denote the blank symbol (not included in Σ_0), and let $\Sigma = \Sigma_0 \cup \{B\}$. A Turing machine *tape* consists of a potentially infinite sequence of squares, on which symbols from the alphabet Σ may be stored, and possibly erased or written over during a computation.

Each tape comes equipped with a *pointer* or *reading head*, which will be pointing at one of the entries during any step of a Turing machine computation. The entries on a tape are ordered as $a_0, a_1, \ldots$ beginning with a leftmost square. Initially each reading head points at the leftmost square of its tape. Turing machine computations are based on two fundamental operations, the following. Say that the pointer on a tape is located over a_i. The Turing machine can replace the symbol a_i with any other symbol. Then it can move from the current square to a_{i+1} or to a_{i-1} (if $i > 0$) or remain at the current square.

A Turing machine M which defines a function $\varphi_M : \Sigma_0^k \to \Sigma$ for some finite k will have k *input tapes*, an *output tape*, and a fixed finite number m of *work* or *scratch* tapes. The inputs $\sigma_0, \ldots, \sigma_{k-1}$ are written on the input tapes at the start of the computation and the other tapes are initially empty. We will assume that the input tapes are *read-only*, that is, M does not ever write over any symbol on the input tapes and does not write any new symbols onto the empty squares of an input tape. The output tape is assumed to be *write-only*, that is, once a symbol is written onto the output tape, it cannot be changed.

The instructions for a Turing machine M to compute the function φ_M are given by a finite set Q of *states*, including some *initial* state s and a *halting state* h, together with a *transition function*

$$\delta_M : Q \times \Sigma^{k+m+1} \to Q \times \Sigma^{m+2} \times \{\leftarrow, \rightarrow, \vdash\}^{k+m+1}.$$

The state of the machine together with the symbols on the scanned squares, are used via the transition function to determine the operation of the machine as follows. Let the tapes be numbered so that tapes 0 through $k-1$ are the input tapes, tapes k through $k+m-1$ are the scratch tapes, and tape $k+m$ is the output tape. Suppose that M is in state q and that, for each $i < k + m$, pointer on tape i is scanning the symbol a_i. Let $\delta_M(s, a_0, \ldots, a_{k+m}) = (q', b_0, \ldots, b_{k+m}, X_0, \ldots, X_{k+m})$, where each $X_i \in \{\leftarrow, \rightarrow, \vdash\}$. Here we assume that, for $i < k$, $b_i = a_i$ and that, if $a_{k+m+1} \neq B$, then $b_{k+m+1} = a_{k+m+1}$. We also assume that if $b_{k+m} \neq B$, then $X_{k+m} = \rightarrow$ and otherwise $X_{k+m} = \vdash$. Then the symbol a_i is replaced on tape i by the symbol b_i. The pointer on tape i moves right if $X = \rightarrow$, moves left if $X = \leftarrow$ and it is not the leftmost square which is being scanned, and otherwise remains pointing at the same square. Finally, the machine transitions into state q'. If $q' = h$, then the computation is finished and the *output* $\varphi_M(\sigma_0, \dots, \sigma_{k-1})$ is the sequence of entries on the output tape. The *length* of the computation is the number of steps until the halting state is reached, if any, and also represents the amount of *time* used in the computation for the purpose of complexity theory. The amount of space used is the total number of squares on the work tapes which were ever written on during the computation.

Example II.1.3. *Natural numbers are usually represented in reverse binary form, so that* 6 *is represented as* 011. *(This is due to having a leftmost square on each tape.) The function* $\varphi(x) = x + 1$ *may be computed by the following Turing machine* M. M *has three states,* s, q *and* h *and just two tapes, the input tape and the output tape. The transition function has the following values.*

$$\begin{aligned}
\delta(s, 0, B) &= (q, 0, 1, \rightarrow) \\
\delta(s, 1, B) &= (s, 1, 0, \rightarrow) \\
\delta(s, B, B) &= (h, B, B, \vdash) \\
\delta(r, 0, B) &= (r, 0, B, \rightarrow) \\
\delta(r, 1, B) &= (r, 1, B, \rightarrow) \\
\delta(r, B, B) &= (h, B, B, \vdash)
\end{aligned}$$

Here we omit any transition where the output tape is not scanning a blank square, since that situation cannot occur.

The computation $\varphi(101) = 011$ *(that is,* $5+1 = 6$*) takes three steps, remaining in state* s *after the first step, moving to state* r *after the second step and finishing in the halting state* h *after scanning the blank at the third step.*

Frequently, we use computations to test whether a given input σ meets certain criteria, that is, belongs to some set A. Then our Turing machine M might output *Yes* or *No* if the input does or does not meet the criteria, or M might halt if σ meets the criteria and not halt otherwise. In the first case, M demonstrates that the set A is computable, and in the second case, M demonstrates that A is computably enumerable.

Example II.1.4. *Let* $A = \{\sigma \in \{0,1\}^* : (\exists n)\sigma(n) = 0 = \sigma(n+1)\}$. *We can show that* A *is computably enumerable with the following simple Turing* *machine. Here we do not need any work tapes or even an output tape.*

$$\begin{aligned}\delta(s,0) &= (q,\rightarrow)\\ \delta(s,1) &= (s,\rightarrow)\\ \delta(s,B) &= (s,\rightarrow)\\ \delta(q,0) &= (h,\vdash)\\ \delta(q,1) &= (s,\rightarrow)\\ \delta(q,B) &= (s,\rightarrow)\end{aligned}$$

If the input string σ is in A and n is the least such that $\sigma(n) = \sigma(n+1) = 0$, then the Turing machine takes $n+1$ steps to read through the first $n+1$ entries of σ and then halts. If σ is not in A, then the machine take $|\sigma| + 1$ steps to read through σ (without finding 00 and find the blank at the end of σ. Then it simply continues to read blanks and thus never halts.

Example II.1.5. *Let $A = \{0^n 1^n : n \in \mathbb{N}\}$. We will give an informal description of a Turing machine M which outputs Y if $\sigma \in A$ and otherwise outputs N. The machine M has one work tape where it copies the 0s from the input tape until either a 1 or a B is read. The reading head on the work tape will be pointing to the final 0. When a 1 is read on the input tape, M transitions to a new state and begins erasing the 0s from the work tape. When a B is now read in the input tape, M checks to see whether there is a B or a 0 on the work tape. If it is a B, then σ is accepted by writing Y on the output tape. If it is a 0, then σ is rejected by writing N on the output tape (in this case there are not enough 1s to match the initial sequence of 0s). If M finds a 0 after some sequence of 1s, then again σ is rejected. For the remaining case, if B is read on the input tape after a sequence of 0s but before any 1s are read, then σ is also rejected.*

Exercises

II.1.1. Show that the set of primitive recursive indices is itself a primitive recursive set. (You may assume here that the coding functions mapping $(a_0, a_1, \ldots, a_n)$ to $a = \langle a_0, a_1, \ldots, a_n\rangle$ and the decoding functions $(a)_i = a_i$ are primitive recursive.)

II.1.2. Prove Lemma II.1.1.

II.1.3. Show that the *universal* sequence $\{\Pi_e\}_{e\in\omega}$ is not uniformly primitive recursive, that is, the function f defined by $f(e,m) = \Pi_e(m)$, is not itself primitive recursive.

II.2 Basic results

In this section, we state a number of results about computable functions which will be needed later. Most proofs are omitted; the reader is referred to Hinman

[87] and Soare [198]. For simplicity of expression, we will generally write $\phi_e(m)$ for a function of k variables rather than $\phi_e(m_1, \ldots, m_k)$ or $\phi_e(\overrightarrow{m})$. Thus the results given here apply to functions taking any number of variables.

Lemma II.2.1 (Padding Lemma). *Each partial computable function ϕ_e has an infinite set of indices, and furthermore, there is a primitive recursive, one-to-one function f such that, for all e and n, $f(e,n)$ is an index for ϕ_e.*

Sketch. Let $f(e,n)$ be an index for the function which first computes $\phi_e(m)$, then adds n to the output, and finally subtracts n from the output. □

Theorem II.2.2 (Normal Form Theorem). *(Kleene) There is a primitive recursive predicate $T_1(e, \overrightarrow{m}, \overrightarrow{\sigma}, q)$ and a primitive recursive function U such that*

$$\Phi_e(\overrightarrow{m}, \overrightarrow{x}) = U(\textit{least } qT_1(e, \overrightarrow{m}, \overrightarrow{x}\lceil q, q))$$

Sketch. Let the T predicate be given by Lemma II.1.2 and define the predicate T_1 so that, for any $e, \overrightarrow{m}, \overrightarrow{\sigma}, q$, $T_1(e, \overrightarrow{m}, \overrightarrow{\sigma}, q)$ if and only if there exists initial segments τ_j of each σ_j such that $T(e, \langle \overrightarrow{m}, \overrightarrow{\tau} \rangle, q)$ and U outputs $\Phi_e(\overrightarrow{m}, \overrightarrow{\sigma})$ from the derivation q. □

Theorem II.2.3 (Enumeration Theorem). *For any $k, \ell < \omega$, there is a partial computable function Φ such that, for all e, $\overrightarrow{m}$ and $\overrightarrow{x}$, $\Phi(e, \overrightarrow{m}, \overrightarrow{x}) = \Phi_e(\overrightarrow{m}, \overrightarrow{x})$.*

Proof. Just let $\Phi(e, \overrightarrow{m}, \overrightarrow{x}) = U(\text{least } qT_1(e, \overrightarrow{m}, \overrightarrow{x}\lceil q, q))$, where T_1 and U are given by Theorem II.2.2. □

The finite approximation $\Phi_{e,s}$ at *stage s* of a partial computable function Φ_e is defined as follows.

Definition II.2.4. *(i)* $\Phi_{e,s}(\overrightarrow{m}, \overrightarrow{x}) = p$ *if and only if*

$$(\exists q < s)[T_1(e, \overrightarrow{m}, \overrightarrow{x}\lceil q, q) \ \&\ U(q)) = p].$$

(ii) $\Phi_{e,s}(\overrightarrow{m}, \overrightarrow{x})$ converges *(written $\Phi_{e,s}(\overrightarrow{m}, \overrightarrow{x})\downarrow$) if $\Phi_{e,s}(\overrightarrow{m}, \overrightarrow{x}) = p$ for some p and otherwise $\Phi_{e,s}(\overrightarrow{m}, \overrightarrow{x})$* diverges *($\Phi_{e,s}(\overrightarrow{m}, \overrightarrow{x})\uparrow$). Similar definitions apply for $\Phi_{e,s}(\overrightarrow{m}, \overrightarrow{\sigma})$.*

(iii) $\Phi_e(\overrightarrow{m}, \overrightarrow{\sigma}) = \Phi_{e,s}(\overrightarrow{m}, \overrightarrow{\sigma})$, *where* $s = |\overrightarrow{\sigma}|$.

The following results are immediate from the definitions and the Normal Form Theorem above. For simplicity of expression, the results are written only for a function of one real variable but applies to functions of several variables as well.

Theorem II.2.5 (Master Enumeration Theorem). $\{\langle e, \overrightarrow{m}, \sigma, s\rangle : \Phi_{e,s}(\overrightarrow{m}, \sigma)\downarrow\}$ *and* $\{\langle e, \overrightarrow{m}, \sigma, p, s\rangle : \Phi_{e,s}(\overrightarrow{m}, \sigma) = p\}$ *are both primitive recursive sets.*

Theorem II.2.6. *(Use Principle)*

(a) $\Phi_e(\overrightarrow{m}, x) = n \Longrightarrow (\exists s)(\exists \sigma \subset x)\Phi_{e,s}(\overrightarrow{m}, \sigma) = n.$

(b) $\Phi_{e,s}(\overrightarrow{m},\sigma) = n \Longrightarrow (\forall t \geq s)(\forall \tau \supset \sigma)\Phi_{e,t}(\overrightarrow{m},\tau) = n$.

(c) $\Phi_{e,s}(\overrightarrow{m},\sigma) = n \to (\forall x \supset \sigma)\Phi_e(\overrightarrow{m},x) = n$.

Theorem II.2.7 (s-m-n Theorem)**.** *For every $m, n \geq 1$, there exists a one-to-one primitive recursive function S^m_n such that, for all $e, i_1, \ldots, i_m, j_1, \ldots, j_n$,*

$$\Phi_{S^m_n(e,i_1,\ldots,i_m)}(j_1,\ldots,j_n,\overrightarrow{x}) = \Phi_e(i_1,\ldots,i_m,j_1,\ldots,j_n,\overrightarrow{x})$$

Proof. For $m = 1$, we want $S^1_1(e,i)$ to be the index for the function ϕ such that $\phi(j_1,\ldots,j_n,\overrightarrow{x}) = \phi_e(i,j_1,\ldots,j_n,\overrightarrow{x})$. Let u be given by the Enumeration Theorem so that $\phi_u(e,i,j_1,\ldots,j_n,\overrightarrow{x}) = \phi_e(i,j_1,\ldots,j_n,\overrightarrow{x})$. Let C_k denote the constant function $C_k(\overrightarrow{m},\overrightarrow{x}) = k$ and let P_i denote the projection function $P_i(\overrightarrow{m},\overrightarrow{x}) = m_i$, both with n number and ℓ real variables. Then

$$\begin{aligned}\phi(j_1,\ldots,j_n,\overrightarrow{x}) &= \phi_u(e,i,j_1,\ldots,j_n,\overrightarrow{x})\\ &= \phi_u(C_e(\overrightarrow{j},\overrightarrow{x}),C_i(\overrightarrow{j},\overrightarrow{x}),P_0(\overrightarrow{j},\overrightarrow{x}),\ldots,P_{n-1}(\overrightarrow{j},\overrightarrow{x})),\end{aligned}$$

so that

$$S^1_1(e,i) = \langle 4, n+1, \ell, u, \langle 0,n,\ell,e\rangle, \langle 0,n,\ell,i\rangle, \langle 1,n,\ell,0\rangle, \ldots, \langle 1,n,\ell,n-1\rangle\rangle.$$

Then S^{m+1}_n may be defined recursively by

$$S^{m+1}_n(e,i_0,\ldots,i_m) = S^m_n(S^1_{m+n}(e,i_0),i_1,\ldots,i_m).$$

□

This result is very useful. Here is an example.

Proposition II.2.8. *There is a primitive recursive function g such that, for all a and b, $W_{g(a,b)} = W_a \cup W_b$.*

Proof. Let $\phi(a,b,m) = U((\text{least } q)[T_1(a,m,q) \vee T_1(b,m,q)])$ and let ϕ have index e. Then let $g(a,b) = S^2_1(e,a,b)$. □

More importantly, we will need the following.

Theorem II.2.9 (Substitution Theorem)**.** *There is a primitive recursive function f such that, for all e, m, A such that Φ^A_b is total, $\Phi_e(m,\Phi^A_b) = \Phi_{f(b,e)}(m,A)$.*

Proof. Let $R(e,b,m,\sigma)$ if $\Phi_e(m,\sigma)\downarrow$; R is primitive recursive by the Master Enumeration Theorem. Now let $g(e,b,m,A) = (\text{least } s)R(e,b,m,A\lceil s)$ and

$$\Phi_c(e,b,m,A) = A\lceil g(e,b,m,A),$$

where we identify a finite sequence with its code. Then

$$\Phi_e(m,\Phi^A_b) = \Phi_e(m,\Phi_c(e,b,m,A)) = \Phi_d(e,b,m,A),$$

where

$$d = \langle 4,3,1,\langle 1,3,1,2\rangle,c\rangle.$$

Now apply the s-m-n Theorem to get $f(b,e) = S^2_1(d,e,b)$. □

Theorem II.2.10 (Recursion Theorem)**.** *For any partial computable function Φ, there exists an index e such that, for all $\overrightarrow{m}$, $\Phi_e(\overrightarrow{m}, \overrightarrow{x}) = \Phi(e, \overrightarrow{m}, \overrightarrow{x})$. Furthermore, there is a primitive recursive function g such that if $\Phi = \Phi_i$, then $e = g(i)$.*

Proof. Given Φ, let $\Phi_b(a, \overrightarrow{m}, \overrightarrow{x}) = \Phi(S_1^{k+1}(a,a), \overrightarrow{m}, \overrightarrow{x})$ and let $e = S_1^{k+1}(b,b)$. Then

$$\Phi_e(\overrightarrow{m}, \overrightarrow{x}) = \Phi_b(b, \overrightarrow{m}, \overrightarrow{x}) = \Phi(S^{k+1}(b,b), \overrightarrow{m}, \overrightarrow{x}) = \Phi(e, \overrightarrow{m}, \overrightarrow{x}).$$

□

This leads to the following.

Theorem II.2.11 (Fixed Point Theorem)**.** *For any computable function f, there exists an index e such that $\Phi_e = \Phi_{f(e)}$. Furthermore, there is a primitive recursive function h such that if $f = \Phi_i$, then $e = h(i)$.*

Proof. Let $\Phi(a, \overrightarrow{m}, \overrightarrow{x}) = \Phi_{f(a)}(\overrightarrow{m}, \overrightarrow{x})$ and let e be given by the Recursion Theorem such that $\phi_e(\overrightarrow{m}, \overrightarrow{x}) = \phi(e, \overrightarrow{m}, \overrightarrow{x})$. □

Corollary II.2.12. *For any computable function f, there exists an index e such that $W_e = W_{f(e)}$. Furthermore, there is a primitive recursive function h such that if $f = \phi_i$, then $e = h(i)$.*

Definition II.2.13. *A function $F : (\mathbb{N}^{\mathbb{N}})^{\ell} \to \mathbb{N}^{\mathbb{N}}$ is (partial) computable (or* computably continuous*) if there is a (partial) computable functional Φ such that, for all $\overrightarrow{x}$ and n, $\Phi(n, \overrightarrow{x}) = F(\overrightarrow{x})(n)$.*

Theorem II.2.14. *Let $F : (\mathbb{N}^{\mathbb{N}})^{\ell} \to \mathbb{N}^{\mathbb{N}}$ be total. Then F is continuous if and only if F is computable in some oracle $A \subseteq \mathbb{N}$.*

Proof. $(\longleftarrow)$. We give the proof for $\ell = 1$. Let A be the given oracle and suppose that $F(x)(m) = \Phi(m, x, A)$. It suffices to show that, for any m and n, $\{x \in \mathbb{N}^{\mathbb{N}} : F(x)(m) = n\}$ is an open set. Suppose that $F(x)(m) = n$. By the Use Principle (Theorem II.2.6) there is some s and some finite $\sigma \subset x$ and finite $\tau \subset \chi_A$ such that $\Phi_{e,s}(m, \sigma, \tau) = n$. It follows that for all $x \in I(\sigma)$, $\Phi_{e,s}(m, x, \tau) = n$ and hence $I(\sigma) \subseteq F^{-1}(\{y : y(m) = n\})$.

$(\longrightarrow)$: Let $F : \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ be continuous. Then for each m and n, $U_{m,n} = \{x : F(x)(m) = n\}$ is open and thus for each $x \in U_{m,n}$, there exists a finite $\sigma \prec x$ such that $I(\sigma) \subseteq U_{m,n}$. Let

$$A = \{\langle m, n, \sigma \rangle : I(\sigma) \subseteq U_{m,n}\}.$$

To compute $F(x)(m)$ from x simply fix m and search for the least $\langle m, n, \sigma \rangle \in A$ such that $\sigma \prec x$; then $F(x)(m) = n$. □

Exercises

II.2.1. Use the s-m-n Theorem to prove Lemma II.1.1.

II.2.2. Use induction to prove Theorem II.2.6 (a).

II.2.3. Show that there is a primitive recursive function g such that, for all a and m, $m \in W_{g(a)} \iff (\exists p)\langle m, p\rangle \in W_a$.

II.2.4. Use the Recursion Theorem to show that the Fibonacci sequence $1, 1, 2, 3, 5, 8, \ldots$ is computable.

II.3 Computably enumerable sets

Definition II.3.1. *(i) A subset A of $\mathbb{N}$ is* computably enumerable *(*c. e.*) if A is the domain of some partial computable function.*

(ii) The c. e. sets can be enumerated in the form

$$W_e = \{m : \phi_e(m)\downarrow\} = \{m : (\exists q)T(e, m, q)\}.$$

(iii) $W_{e,s} = \{m : \phi_{e,s}(m)\downarrow\}$.

The following lemma is immediate from Theorem II.2.5.

Lemma II.3.2. $\{\langle e, m, s\rangle : m \in W_{e,s}\}$ *is primitive recursive.* □

There are several equivalent definitions.

Definition II.3.3. *A set $A \subseteq \mathbb{N}^k$ is Σ^0_1 (resp. Σ^B_1) if there is a computable relation R (resp. computable in B) such that, for all $\vec{m}$, $\vec{m} \in A \iff (\exists p)R(p, \vec{m})$.*

Theorem II.3.4 (Normal Form Theorem for c. e. sets)**.** *A set A is c. e. if and only if it is Σ^0_1.* □

The proof is left as an exercise. Observe that any computable set is trivially Σ^0_1 and hence also is computably enumerable.

Theorem II.3.5 (Quantifier Contraction Theorem)**.** *If W is a c. e. set, then $\{m : (\exists p)\langle p, m\rangle \in W\}$ is a c. e. set.*

Proof. Let $V = \{m : (\exists p)\langle m, p\rangle \in W\}$. Then $m \in V \iff (\exists q)\langle m, (q)_0\rangle \in W_{e,(q)_1}$. Thus V is c. e. by the Normal Form Theorem for c. e. sets. □

The intended meaning of the term "computably enumerable set" is that there is an effective listing $a_0, a_1, \ldots$ of the set.

Theorem II.3.6 (Listing Theorem)**.** *A set A is c. e. if and only if either $A = \emptyset$ or A is the range of a total computable function.*

Proof. ($\Longleftarrow$): If $A = \emptyset$, then A is c. e. If $A = \{\phi_e(m) : m \in \mathbb{N}\}$ where ϕ is a total computable function, then

$$n \in A \iff (\exists n)(\exists s)\phi_{e,s}(p) = n.$$

Thus A is c. e. by Theorem II.2.5 and the Quantifier Contraction Theorem.

($\Longrightarrow$): Let $A = W_e \neq \emptyset$ and choose $a \in A$. Then A is the range of the following computable function.

$$f(\langle m, s\rangle) = \begin{cases} m, & \text{if } m \in W_{e,s+1} \setminus W_{e,s}; \\ a, & \text{otherwise.} \end{cases}$$

□

Theorem II.3.7 (Complementation Theorem)**.** *A set A is computable if and only if both A and $\mathbb{N} \setminus A$ are c. e.*

Proof. ($\Longleftarrow$): If A is computable, then $\mathbb{N} \setminus A$ is also computable and hence both sets are c. e..

($\Longrightarrow$): Suppose that $A = W_a$ and $\mathbb{N} \setminus A = W_b$ and let $\phi(m) = (\text{least } s)[m \in W_{a,s} \lor m \in W_{b,s}]$. Then ϕ is a total computable function and $m \in A \iff m \in W_{a,\phi(m)}$, so that A is computable. □

There are natural noncomputable c. e. sets.

Definition II.3.8. *(a)* $K = \{e : e \in W_e\}$*;*

(b) $K_0 = \{\langle m, e\rangle : m \in W_e\}$.

Proposition II.3.9. *K and K_0 are noncomputable c. e. sets.*

Proof. It follows from Lemma II.3.2 that K_0 and K are c. e. sets. Suppose now that K were computable, so that $\mathbb{N} \setminus K$ is c. e., by the Complementation Theorem, and choose a such that $\mathbb{N} \setminus K = W_a$. Then, for any m,

$$m \in W_m \iff m \in K \iff m \notin W_a$$

and when $m = a$ we obtain the contradiction

$$a \in W_a \iff a \in K \iff a \notin W_a.$$

Now $a \in K \iff \langle a, a\rangle \in K_0$, so that K would be computable if K_0 were computable. □

Exercises

II.3.1. Prove lemma II.3.2 and the Normal Form Theorem for c.e. sets. Hint: Use the corresponding results for partial computable functions.

II.3.2. Show that a partial function is partial computable function if and only if the graph is Σ^0_1.

II.3.3. Use the s-m-n Theorem to obtain a primitive recursive function such that for any e, $\{m : (\exists p)\langle m, p\rangle \in W_e\} = W_{f(e)}$.

II.3.4. Show that if A is a Σ^0_1 relation and $B = \{\langle m, p\rangle : (\forall n < p)\langle m, n\rangle \in A\}$, then B is also Σ^0_1.

II.4 Computability of real numbers

Any set A of natural numbers represents a real number $r_A \in [0,1]$ where $r_A = \sum_{n\in A} 2^{-n-1}$. For every real r in $[0,1]$, there exists $A \subseteq \mathbb{N}$ such that $r = r_A$ and r has a unique representation except for dyadic rationals r, which have exactly two such representations. The real r_A is said to be computable if A is a computable set. For an arbitrary real x, we have $x = i + r$, where i is an integer and $r \in [0,1]$, so we will say that x is computable if and only if r is computable.

The unit interval $[0,1] \subset \mathbb{R}$ has a natural linear ordering and this corresponds to the lexicographic ordering on $\{0,1\}^{\mathbb{N}}$.

Definition II.4.1. *For $x, y \in \mathbb{N}^{\mathbb{N}}$, $x <_{lex} y$ if $x(n) < y(n)$ where n is the least such that $x(n) \neq y(n)$.*

It is easy to see that $<_{lex}$ is a linear ordering on $\mathbb{N}^{\mathbb{N}}$. We sometimes say that "x is left of y" if $x <_{lex} y$, since this fits the picture of the tree $\mathbb{N}^*$. For $x, y \in \{0,1\}^{\mathbb{N}}$, if $r_x \neq r_y$, then $r_x < r_y \iff x <_{lex} y$. If r is a dyadic rational, then there are two representations $x \neq y$ such that $r_x = r_y = r$ and these are successors under $<_{lex}$.

Another useful way of determining the complexity of a real number is by means of Dedekind cuts of rationals. Rational numbers may be represented as quotients of integers and thereby as finite sequences of natural numbers. Thus we may view the set $\mathbb{Q}$ of rational numbers as a computable structure equipped with a computable ordering and computable operations of addition, subtraction, multiplication and division. The Dedekind cut $L(r)$ of a real number is defined by

$$L(r) = \{q \in \mathbb{Q} : q \leq r\}.$$

It turns out that the complexity of the Dedekind cut is quite useful in computable analysis.

Proposition II.4.2. *For any real r, r is computable if and only if $L(r)$ is computable.*

Proof. It suffices to consider $r \in [0,1]$, so let $r = r_x$ for some $x \in \{0,1\}^{\mathbb{N}}$. If r is rational, then both r and $L(r)$ are computable. So let r be irrational and suppose first that r is computable. Then, for any rational q,

$$q < r \iff (\exists n) q < \sum_{i=0}^{n} x(i) 2^{-i-1}$$

and

$$q > r \iff (\exists n) q > 2^{-n-1} + r \iff (\exists n) q > 2^{-n-1} + \sum_{i=0}^{n-1} x(i) 2^{-i-1}.$$

Next suppose that $L(r)$ is computable. Then we can recursively define x so that $r = r_x$ as follows. Let $x(0) = 0$, if $r < \frac{1}{2}$ and $x(0) = 1$ otherwise. Given $x(n)$, let

$$x(n+1) = \begin{cases} 0, & \text{if } r < 2^{-n-2} + \sum_{i=0^n} x(i) 2^{-i-1}, \\ 1, \text{otherwise.} \end{cases}$$

□

There is a nice characterization for Σ^0_1 and Π^0_1 Dedekind cuts.

Proposition II.4.3. *(a) $L(r)$ is Σ^0_1 if and only if $r = \lim_n q_n$, where $\{q_n\}_{n\in\omega}$ is a computable, increasing sequence of rationals.*

(b) $L(r)$ is Π^0_1 if and only if $r = \lim_n q_n$, where $\{q_n\}_{n\in\omega}$ is a computable, decreasing sequence of rationals.

Proof. (a) Suppose first that $r = lim_n q_n$ where $\{q_n\}_{n\in\omega}$ is a computable, increasing sequence of rationals. Then, for any rational q, $q < r \iff (\exists n) q < q_n$.

Suppose now that $L(r)$ is Σ^0_1 and let $L(r)$ have a computable enumeration as $p_0, p_1, \ldots$. Then we can define a computable nondecreasing sequence q_n of rationals with limit r by

$$q_n = max\{p_0, p_1, \ldots, p_n\}.$$

It is routine to convert this into an increasing sequence.

(b) If $L(r)$ is Π^0_1, then $L(1-r)$ is Σ^0_1, since $q < 1 - r \iff r < 1 - q$. Thus $1-r = lim_n p_n$ where $\{p_n\}_{n\in\omega}$ is a computable, increasing sequence of rationals. It follows that $r = lim_n(1-p_n)$ is the limit of a computable, decreasing sequence. Conversely, if $r = lim_n q_n$ where $\{q_n\}_{n\in\omega}$ is a computable, increasing sequence of rationals, then $1 - r$ is the limit of a decreasing sequence so that $L(1-r)$ is Σ^0_1 and hence $L(r)$ is Π^0_1. □

The following notions are important in computable analysis.

Definition II.4.4. *Let r be a real number. Then*

(a) r is lower semicomputable *if it is the limit of an increasing computable sequence of rationals;*

(b) r is upper semicomputable *if it is the limit of a decreasing computable sequence of rationals;*

(c) r is weakly computable *if it is either lower semicomputable or upper semicomputable.*

It would be natural to say that r_x is Σ^0_1 if the set with characteristic function x is Σ^0_1, but there is no corresponding equivalence as in Proposition II.4.2. One direction only holds. In example II.5.3 below we will construct a lower semicomputable real r which is not the characteristic function of a c. e. set.

Proposition II.4.5. *(a) If A is Σ^0_1, then $L(r_A)$ is Σ^0_1;*

(b) If A is Π^0_1, then $L(r_A)$ is Π^0_1.

Proof. (a) If A is finite, then of course r_A is rational and therefore $L(r_A)$ is computable. Suppose therefore that A is Σ^0_1 and infinite and let A have computable enumeration $a_0, a_1, \ldots$ without repetition. Then for any rational q,

$$q < r_A \iff (\exists n) q < \sum_{i=0}^{n} 2^{-a_i-1}.$$

(b) If A is Π^0_1, then $\mathbb{N} \setminus A$ is Σ^0_1 and $r_{\mathbb{N}\setminus A} = 1 - r_A$, so that $L(1 - r_A)$ is Σ^0_1 and therefore $L(r_A)$ is Π^0_1. □

Exercises

II.4.1. Show that a real number r is computable if and only if there is a computable sequence q_n of rationals such that $|q_n - r| < 2^{-n}$ for all n.

II.5 Turing, many-one, and truth-table reducibility

Definition II.5.1. *(i) A is* many-one reducible *(m-*reducible*) to B ($A \leq_m B$ if there is a computable function f such that $a \in A \iff f(a) \in B$*

(ii) A is one-one reducible *to B ($A \leq_1 B$ if there is a one-to-one computable function f such that $a \in A \iff f(a) \in B$*

*(iii) C is m-*complete *(or Σ^0_1* complete*) if $A \leq_m C$ for all c. e. sets A.*

For example, any c. e. set is m-reducible to K_0, since $m \in W_e \iff \langle m, e\rangle \in K_0$; here the function f is given by $f(m) = \langle m, e\rangle$. Thus K_0 is m-complete. The following useful lemma is left as an exercise.

Lemma II.5.2. *If every Σ^0_1 (respectively Π^0_1) set is m-reducible to A, then A is not Π^0_1 (resp. Σ^0_1).* □

Example II.5.3. *Let K be a noncomputable c. e. set and let $A = \{2n : n \in K\} \cup \{2n+1 : n \notin K\}$. Then A is a difference of c. e. sets and is m-complete for both Π^0_1 and Σ^0_1 sets. It follows from Lemma II.5.2 that A is not Σ^0_1, but r_A is the limit of the nondecreasing computable sequence $\{q_s\}_{s\in\omega}$ defined as follows. Let*

$K = \cup_s K_s$ where K_s is a uniformly computable finite subset of $\{0, 1, \ldots, s-1\}$ and let

$$q_s = \sum\{2^{-2n-1} : n \in K_s\} + \sum\{2^{-2n-2} : n < s \ \&\ n \notin K_s\}.$$

Observe that if $n \in K_s \setminus K_{s-1}$, then 2^{-2n-1} is added to the first part of q_s and 2^{-2n-2} is subtracted from the second part, so that $q_{s-1} < q_s$.

Definition II.5.4. *(i) $A \equiv_m B$ if $A \leq_m B$ and $B \leq_m A$.*

(ii) $A \equiv_1 B$ if $A \leq_1 B$ and $B \leq_1 A$.

Proposition II.5.5. *Suppose that $A \leq_m B$. If B is c. e., then A is c. e. and if B is computable, then A is computable.*

The proof is left as an exercise.

Definition II.5.6. *A is* computably isomorphic *to B (written $A \equiv B$) if there is a computable permutation π of $\mathbb{N}$ such that $\pi[A] = B$.*

The following is an effective version of the classic Cantor-Schröder-Bernstein Theorem.

Theorem II.5.7 (Cantor-Schröder-Bernstein Theorem)**.** *Let A and B be sets and let f and g be injections, $f : A \to B$ and $g : A \to B$; then there exists an isomorphism $h : A \to B$.* □

Banach's version of the Cantor-Schröder-Bernstein Theorem adds the requirement that, for all $a \in A$, either $h(a) = f(a)$ or $h(a) = g^{-1}(a)$.

Theorem II.5.8 (Myhill Isomorphism Theorem)**.** $A \equiv B \iff A \equiv_1 B$.

Proof. The direction ($\Longrightarrow$) is trivial. Suppose therefore that $A \leq_1 B$ via f and $B \leq_1 A$ via g. We will define π in stages $\pi_s = \{\langle m_0, n_0\rangle, \ldots, \langle m_{2s-1}, n_{2s-1}\rangle\}$ so that for all $m < s$, $m \in Dom(\pi_s)$ and $m \in Ran(\pi_s)$ and such that

$$m_i \in A \iff n_i \in B.$$

We begin with $\pi_0 = \emptyset$.

Stage $s+1$: Let π_s be given as above and let $m = m_{2s}$ be the least $m \notin Dom(\pi_s)$. $\pi(m)$ is computed as follows. First compute $b_0 = f(m)$ and check if $b_0 \in Ran(\pi_s)$. If not, then $b_0 = \pi_{s+1}(m)$. If so, then compute $b_1 = f(\pi_s^{-1}(b_0))$ and again check whether $b_1 \in Ran(\pi_s)$ and let $b_1 = \pi_{s+1}(m)$ if not and $b_2 = f(\pi_s^{-1}(b_0))$ if so. Observe that after we reach b_{2s-1}, $Ran(\pi_s) = \{b_0, b_1, \ldots, b_{2s-1}\}$ is exhausted, so that $b_{2s} = \pi_{s+1}(m)$.

Next let $n = n_{2s+1}$ be the least not in $Ran(\pi_s) \cup \{\pi_{s+1}(m)\}$ and similarly define $a_0 = g(n)$, $a_1 = g(\pi_s(a_0))$, and so on to obtain $\pi_{s+1}^{-1}(n)$. □

We now show that, in the setting of m-reduciblity, Banach's version of the Cantor-Schröder-Bernstein Theorem is not effective,

Theorem II.5.9. *There exist computable injections $f : \mathbb{N} \to \mathbb{N}$ and $g : \mathbb{N} \to \mathbb{N}$ such that, for any computable permutation h of $\mathbb{N}$, there is some i such that $h(i) \neq f(i)$ and $h(i) \neq g^{-1}(i)$.*

Proof. Let K be a noncomputable c. e. set and let ψ be a one-one computable function with range K. We define injections f and g as follows.

$$\textbf{(1)}\quad f(2^m(2n+1)) = \begin{cases} 2n+1, & \text{if } \psi(m-2) = n, \\ 2^m(2n+1), & \text{if } (\exists j < m-2)\psi(j) = n, \\ 2^{m+1}(2n+1), & \text{otherwise.} \end{cases}$$

$$\textbf{(2)}\quad g(2^m(2n+1) = \begin{cases} 2^{m+1}(2n+1), & \text{if } (\exists j \leq m-2)\psi(j) = n, \\ 2^m(2n+1) & \text{otherwise.} \end{cases}$$

Now by way of contradiction, let h be a computable permutation such that, for all i, either $h(i) = f(i)$ or $h(i) = g^{-1}(i)$. We claim that

$$n \notin K \iff h(2n+1) = 2n+1.$$

This would contradict the assumption that K is not computable. It remains to verify the claim. Suppose first that $n \notin K$. Then $g(2n+1) = 2n+1$ and $2n+1$ is not in the range of f, so that $h(2n+1) = g^{-1}(2n+1) = 2n+1$. Next suppose that $n \in K$ and let $n = \psi(m)$. Then $2^{m+2}(2n+1)$ is not in the range of g, so $h(2^{m+2}(2n+1)) = f(2^{m+2}(2n+1) = 2n+1$, so that $h(2n+1) \neq 2n+1$.

□

Let *Sent* be the set of propositional sentences on variables $a_0, a_1, \ldots$. There is a an effective enumeration $\psi_0, \psi_1, \ldots$ of these sentences, so that we may identify the sentence ψ_n with n in context. For any set $B \subseteq \mathbb{N}$ and any sentence $\psi \in Sent$, we say that $B \models \psi$ if ψ is true under the truth assignment which makes a_i true if and only if $i \in B$. Let B^{tt} denote the set of $\psi \in Sent$ such that $B \models \psi$. Then we say that A is truth-table reducible to B ($A \leq_{tt} B$) if $A \leq_m B^{tt}$. Equivalently, $A \leq_{tt} B$ if and only if there exists a computable relation R and a computable function f such that for any n, $n \in A \iff R(\langle B \restriction f(n) \rangle)$. It is immediate that many-one reducibility implies truth-table reducibility.

Theorem II.5.10. *(Trakhtenbrot-Nerode [211, 157]) $A \leq_{tt} B$ if and only if there is a total, computable function $\Phi : \{0,1\}^{\mathbb{N}} \to \{0,1\}^{\mathbb{N}}$ such that $\Phi(B) = A$.*

Proof. Suppose first that there is a total computable function Φ such that $\Phi(B) = A$. We will define a function $f : \mathbb{N} \to Sent$ such that $a \in A \iff B \models f(a)$. Given a, use the Master Enumeration Theorem II.2.5 to compute the least n such that $\Phi(a, \sigma) \downarrow$ for all $\sigma \in \{0,1\}^n$. For each σ such that $\Phi(a, \sigma) = 1$, let φ_σ be the conjunction $\bigwedge_{i=0}^{n-1} b_i$, where $b_i = a_i$ if $\sigma(i) = 1$ and $b_i = \neg a_i$ if $\sigma(i) = 0$. Thus $B \models \varphi_\sigma$ if and only if, for all $i < n$, $\sigma(i) = 1 \iff i \in B$. Finally, let φ be the disjunction of $\{\varphi_\sigma : \Phi(a, \sigma) = 1\}$. Then

$$a \in A \iff \Phi(a, B) = 1 \iff B \models \varphi.$$

Next suppose that $A \leq_{tt} B$ and let f be given so that $a \in A \iff B \models f(a)$. Then in general, define Φ so that

$$\Phi(c, X) = 1 \iff X \models f(c)$$

Here we can compute for each propositional variable a_i occuring in $f(c)$, whether $X \models a_i$ immediately from X and then use truth tables to check whether $X \models f(c)$. □

This leads naturally to *Turing reducibility*, where we allow *partial* computable functions.

Definition II.5.11. *(i) A is* Turing reducible *to B (written $A \leq_T B$) if there is a functional Φ such that, for all m, $A(m) = \Phi(m, B)$.*

(ii) A is Turing equivalent *to B if both $A \leq_T B$ and $B \leq_T A$.*

(iii) The Turing degree $\mathbf{a}$ *of A is the equivalence class of A under Turing equivalence.*

Informally, this means that $A \leq_T B$ if A can be computed using B as an oracle. It follows from Theorem II.5.10 that truth-table reducibity implies Turing reducibility. It is easy to see that $\equiv_T$ is an equivalence relation. The Turing degrees are partially ordered by $\leq_T$ with least element $\mathbf{0}$, which is the Turing degree of a recursive set.

It is possible that $A \leq_T B$ but A is not truth-table reducible to B. (See the exercises below.) However, there is a family of sets B for which the two reducibilities are equivalent.

Definition II.5.12. *A function $g \in \mathbb{N}^{\mathbb{N}}$ is* almost computable *(or* hyperimmune-free*) if for all $f \leq_T g$, there exists a computable function h such that $f(n) \leq h(n)$ for all n.*

Theorem II.5.13. *Suppose g is almost computable. Then for all f, if $f \leq_T g$, then $f \leq_{tt} g$.*

Proof. Suppose that y is almost computable and that $f(n) = \Phi(n, g)$ where Φ is computable. Define $u(n)$ to be the least k such that $\Phi(n, g\lceil k) \downarrow$. Then u is computable from g and hence there is a computable function h such that $g(n) \leq h(n)$ for all n. It follows that $f(n) = \Phi(n, g\lceil h(n))$ for all n, so that Φ can be extended to a total function Ψ by letting $\Psi(n, x) = 0$ whenever $\Phi(n, x\lceil h(n)) \uparrow$. Hence f is truth-table reducible to g, as desired. □

Exercises

II.5.1. Prove Lemma II.5.2 and show that A is Σ^0_1 complete if and only if $K \leq_m A$.

II.5.2. Prove Proposition II.5.5.

II.5.3. Show that $\leq_m$ and $\leq_T$ is transitive.

II.5.4. Show that the two definitions of truth-table reducibility are equivalent.

II.5.5. Give an example to show that $A \leq_T B$ but not $A \leq_{tt} B$. (Hint: let A be the set of e such that Φ_e defines a total functional F_e, where $y = F_e(x)$ means that $y(m) = \Phi_e(m, x)$. Then let $e \in B \iff (e \in A \;\&\; \Phi_e(e, A) = 0)$.

II.6 The jump and the arithmetical hierarchy

Definition II.6.1. *1.* $W_e^A = \{m : \Phi_e(m, A)\downarrow\}$.

2. B is c. e. in A if $B = W_e^A$ for some e.

3. The jump *of A is $K_0^A = \{\langle e, m\rangle : m \in W_e^A\}$ and is denoted by A'.*

4. $A^{(n)}$ is the nth jump of A, that is, $A^{(0)} = A$ and $A^{(n+1)} = (A^{(n)})'$.

The following two theorems generalize from results of Section II.3 and II.5.

Theorem II.6.2. *The following are equivalent:*

(a) B is c. e. in A;

(b) $B = \emptyset$ or $B = Ran(\phi_e^A)$ for some e;

(c) B is Σ_1^A. □

Theorem II.6.3. *$B \leq_T A$ if and only if B and $\mathbb{N} \setminus B$ are both c. e. in A.* □

Theorem II.6.4 (Jump Theorem)**.** *(a) $A' \not\leq_T A$.*

(b) B is c. e. in A if and only if $B \leq_1 A'$.

(c) $B \leq_T A$ if and only if $B' \leq_1 A'$.

Proof. Parts (a) and (b) relativize from results of sections II.3 and II.5. For part (c), suppose first that $B \leq_T A$ and let $B(n) = \Phi_b(n, A)$. Then for any e, $\Phi_e(m, B) = \Phi_e(m, \Phi_b^A)$ and by Theorem II.2.9, there is a primitive recursive function such that $\Phi_e(m, B) = \Phi_{f(e)}(m, A)$. Thus $\langle e, m\rangle \in B' \iff \langle f(e), m\rangle \in A'$. For the other direction, suppose that $B' \leq_1 A'$. Then B and $\mathbb{N} \setminus B$ are both c. e. in A and therefore $B \leq_T A$ by Theorem II.6.3. □

In particular, there is an infinite hierarchy of degrees $\mathbf{0^{(n)}} = deg(\emptyset^{(n)})$.
The arithmetical hierarchy of sets of natural numbers may be defined as follows.

Definition II.6.5. *Let $R \subseteq \mathbb{N}^k \times (\mathbb{N}^{\mathbb{N}})^\ell$ and let $n > 0$ be a natural number.*

1. R is Σ_0^0 if it is computable.

2. R is Π_n^0 if $\mathbb{N}^k \times (\mathbb{N}^{\mathbb{N}})^\ell \setminus R$ is Σ_n^0.

3. *R is Σ^0_{n+1} if it is the projection of a Π^0_n set, that is, if there exists a Π^0_n relation B such that, for all $\overrightarrow{m}$ and $\overrightarrow{x}$:*

$$R(\overrightarrow{m}, \overrightarrow{x}) \iff (\exists j)B(j, \overrightarrow{m}, \overrightarrow{x}).$$

4. *R is Δ^0_n if it is both Σ^0_n and Π^0_n.*

Note of course that the Σ^0_1 sets are just the computably enumerable sets. These definitions can be relativized to any oracle C to define the $\Sigma^0_n[C]$, $\Pi^0_n[C]$ and $\Delta^0_n[C]$ sets and relations.

Here are some basic facts about the arithmetical hierarchy. Part (e) refers to *bounded quantification.* See [198] for proofs.

Theorem II.6.6. *(a) $A \in \Sigma^0_n \cup \Pi^0_n$ and $m > n$ implies $A \in \Delta^0_m$;*

(b) $A, B \in \Sigma^0_n(\Pi^0_n) \Longrightarrow A \cap B, A \cup B \in \Sigma^0_n(\Pi^0_n)$;

(c) If $R \in \Sigma^0_n$ for $n > 0$ and $A = \{m : (\exists p)R(m,p)\}$, then R is Σ^0_n;

(d) If $B \leq_m A$ and $A \in \Sigma^0_n$, then $B \in \Sigma^0_n$;

(e) If $R \in \Sigma^0_n$ and $A = \{\langle m,p\rangle : (\forall i < p)R(i,m,p)\}$, then $A \in \Sigma^0_n$. □

An important result is the following.

Theorem II.6.7 (Post's Theorem)**.** *For any subset A of $\mathbb{N}$:*

(a) A is Σ^0_{n+1} if and only if it is c. e. in $\emptyset^{(n)}$.

(b) A is $\Delta^0_{n+1} \iff A \leq_T \emptyset^{(n)}$.

Proof. The proofs are by induction on n. For $n = 0$, both parts are immediate. Now suppose by induction that (a) and (b) are true for n and for all subsets of $\mathbb{N}$.

First we show that $\emptyset^{(n+1)}$ is Σ^0_{n+1}. That is, by induction assume that $\emptyset^{(n)}$ is Σ^0_n. Then

$$\langle e, m\rangle \in \emptyset^{(n+1)} \iff (\exists s)(\exists \sigma)[\sigma \subset \emptyset^{(n)} \ \& \ \Phi_e(m,\sigma)\downarrow].$$

But in general, $\sigma \subset C$ if and only if $(\forall i < |\sigma|)[\sigma(i) = 1 \iff i \in C]$, so that for $C = \emptyset^{(n)}$, this condition is a disjunction of Σ^0_n and Π^0_n clauses. It follows that the quantified condition is Δ^0_{n+1} and hence $\emptyset^{(n+1)}$ is Σ^0_{n+1} by Theorem II.6.6.

Now suppose that A is c. e. in $\emptyset^{(n)}$. Then $A \leq_1 \emptyset^{(n+1)}$ and therefore A is Σ^0_{n+1}.

Conversely, suppose that A is Σ^0_{n+1} and let R be Π^0_n such that $m \in A \iff (\exists p)R(m,p)$. Then $\mathbb{N} \setminus R$ is Σ^0_n and hence c. e. in $\emptyset^{(n-1)}$ by induction. It follows from the Jump Theorem that $\mathbb{N} \setminus R \leq_1 \emptyset^{(n)}$ and therefore $R \leq_T \emptyset^{(n)}$. Since A is c. e. in R, it follows that A is also c. e. in $\emptyset^{(n)}$.

For part (b), A is Δ^0_{n+1} if and only if both A and $\mathbb{N} \setminus A$ are Σ^0_{n+1}, which is if and only if A and $\mathbb{N} \setminus A$ are c. e. in $\emptyset^{(n)}$ (by (a)). But this is if and only if $A \leq_T \emptyset^{(n)}$ by the Jump Theorem. □

Definition II.6.8. *A is said to be Σ^0_n (Π^0_n)* complete *if A is Σ^0_n (Π^0_n) and every Σ^0_n (Π^0_n) set is m-reducible to A.*

It follows from Post's Theorem that $\emptyset^{(n)}$ is Σ^0_n complete for all $n > 0$.
Δ^0_2 sets and functions will be of particular interest.

Definition II.6.9. *Let $\{f_s\}_{s\in\omega}$ be a sequence of total functions from $\mathbb{N}^{\mathbb{N}}$.*

(i) $\lim_s f_s = f$ *means that, for all m, there exists s such that $f(m) = f_t(m)$ for all $t \geq s$;*

(ii) h is a modulus of convergence *for $\{f_s\}_{s\in\omega}$, if, for all m and all $s \geq h(m)$, $f_s(m) = f(m)$.*

Given a uniformly computable sequence $\{f_s\}_{s\in\omega}$ of functions with limit f and modulus of convergence h, f is always computable in h.

Lemma II.6.10 (Modulus Lemma)**.** *If A is c. e. and $f \leq_T A$, then there is a uniformly computable sequence $\{f_s\}_{s\in\omega}$ such that* $\lim_s f_s = f$ *and a modulus of convergence $h \leq_T A$.*

Proof. Let $A = W_i$ be c. e., let $\sigma_s = W_{i,s}\lceil s$ and let $f = \Phi^A_e$. Now let

$$f_s(m) = \begin{cases} \Phi_{e,s}(m, \sigma_s), & \text{if convergent,} \\ 0, & \text{otherwise.} \end{cases}$$

and

$$h(m) = (\text{least } s)(\exists z \leq s)[\Phi_{e,s}(m, \sigma_s\lceil z)\downarrow \ \& \ \sigma_s\lceil z = A\lceil z].$$

Then $\{f_s\}_{s\in\omega}$ is a computable sequence with limit f and h is a modulus of convergence which is computable from A. □

In particular, this implies that any Δ^0_2 function is the limit of a computable sequence.

Lemma II.6.11 (Limit Lemma)**.** *$f \leq_T A'$ if and only if there exists an A-computable sequence $\{f_s\}_{s\in\omega}$ such that $f = \lim_s f_s$.*

Proof. ($\Longrightarrow$): This follows from the Modulus Lemma relativized to A, since $f \leq_T A'$ if and only if f is c. e. in A.

($\Longleftarrow$): Let $f = \lim_s f_s$ and let $h(m) = (\text{least } s)[(\forall t \geq s) f_t(m) = f_s(m)]$. Since $\{f_s\}_{s\in\omega}$ is computable in A, it follows that $h \leq_T A'$, so that $f \leq_T A'$ as well. □

For real numbers, we say that r_x is *computably approximable* if x is Δ^0_2, so that r is computably approximable if and only if it is the limit of a computable sequence of rationals.

For any Δ^0_2 set A, it follows from the Jump Theorem that $\emptyset' \leq_T A' \leq_T \emptyset''$.

Definition II.6.12. *Let $A \leq_T \emptyset'$; A is* low *if $A' = \emptyset'$; A is* high *if $A' = \emptyset''$.*

Clearly any computable set is low, whereas $\emptyset'$ is high. A c.e., noncomputable low set is constructed in Soare [198] (p. 111).

Exercises

II.6.1. The difference $B \setminus C$ of two c. e. sets is said to be a d. r. e. set. More generally, a set C is n-r.e. if there is a computable sequence $\{A_s\}_{s\in\mathbb{N}}$ such that $A = \lim_s A_s$ and such that (i) $A_0 = \emptyset$ and (ii) for each m, $card(\{s : A_{s+1}(m) \neq A_s(m)\}) \leq n$. Show that for each n, there is an $(n+1)$-r.e. set which is not n-r.e.

II.7 The lattice of c. e. sets

The lattice $\mathcal{E}$ of c. e. sets is ordered by inclusion and has the natural operations of union and intersection. The lattice $\mathcal{E}^*$ is the quotient of $\mathcal{E}$ under equality modulo finite difference.

In this section, we consider properties of c. e. sets related to the lattice.

Definition II.7.1. *(i) A set is* immune *if it is infinite but contains no infinite c. e. set;*

(ii) A c. e. set A is simple *if $\mathbb{N} \setminus A$ is immune.*

Simple sets were first constructed by Post [168] as a partial solution to *Post's Problem*, which was to find natural intermediate c. e. degrees. It is easy to see that simple sets are neither computable nor m-complete.

Definition II.7.2. *Let R be a property of c. e. sets, that is $R \subseteq \mathcal{E}$.*

(i) R is lattice-theoretic *or* invariant *in $\mathcal{E}$ ($\mathcal{E}^*$) if it is invariant under all automorphisms of $\mathcal{E}$ ($\mathcal{E}^*$).*

(ii) R is elementary lattice-theoretic *or* definable *in $\mathcal{E}$ (respectively, $\mathcal{E}^*$) if there is a first-order formula φ with one free variable in the language $\{\leq, \vee, \wedge, 0, 1\}$ of lattice theory such that $R(A)$ if and only if $\mathcal{E} \models \varphi(A)$ (resp. $\mathcal{E}^* \models \varphi(A)$).*

Clearly any definable property is also invariant.

Lemma II.7.3. *The properties of computability and of finiteness are both definable in $\mathcal{E}$.*

Proof. A is computable if and only if

$$\mathcal{E} \models (\exists y)[A \vee y = 1 \ \&\ A \wedge y = 0].$$

A is finite if and only if

$$\mathcal{E} \models (\forall y)[y \subseteq A \longrightarrow A \text{ is computable}].$$

□

In the lattice $\mathcal{E}^*$, A is simple if, for all $B \neq 0$, $A \cap B \neq 0$. Thus, the property of being simple is elementary lattice-theoretic in $\mathcal{E}^*$. The following lemma will imply that simplicity is also definable in $\mathcal{E}$.

Lemma II.7.4. *If a property R is preserved under finite differences, then R is definable in $\mathcal{E}$ if and only if R is definable in $\mathcal{E}^*$.*

Proof. Let R be preserved under finite differences. If R is definable in $\mathcal{E}$ by a formula φ, then the same formula works in $\mathcal{E}^*$. Suppose next that R is definable in $\mathcal{E}^*$. Since finiteness is definable in $\mathcal{E}$, the relation $=^*$ of equality modulo finite difference is also definable in $\mathcal{E}$. Thus the definition from $\mathcal{E}^*$ may be rewritten in $\mathcal{E}^*$ by replacing all occurrences of $=$ with $=^*$. □

Definition II.7.5. *(i) If $e = \sum_{i=0}^{k} e_i 2^i$, then $D_e = \{i \leq k : e_i = 1\}$. (Thus $D_0, D_1, \ldots$ effectively enumerates the finite sets of natural numbers.)*

(ii) A sequence $\{F_n\}_{n\in\mathbb{N}}$ of finite sets is a strong (weak) array *if there is a computable function f such that $F_n = D_{f(n)}$ ($F_n = W_{f(n)}$).*

(iii) An infinite set B is hyperimmune (h-immune) *(respectively,* hyperhyperimmune (hh-immune)*) if for any pairwise disjoint strong (respectively, weak) array, $F_n \cap B = \emptyset$ for some n.*

(iv) A c. e. set A is hypersimple (h-simple) *(respectively,* hyperhypersimple (hh-simple)*) if $\mathbb{N} \setminus A$ is h-immune (respectively, hh-immune).*

It is easy to see that hh-simple implies h-simple and that h-simple implies simple.

Definition II.7.6. *(i) A function f* majorizes *a function g if $f(n) \geq g(n)$ for all n and f* dominates *g if $f(n) \geq g(n)$ for all but finitely many n.*

(ii) The principal function *p_A of an infinite set A is defined by $p_A(n) = a_n$, where $a_0 < a_1 < \cdots$ enumerates A is increasing order.*

The following result is easy to prove.

Theorem II.7.7. *An infinite set A is hyperimmune if and only if no computable function majorizes p_A.* □

Dekker used this characterization to show that every nonzero c. e. degree contains a h-simple (hence simple) set.

A degree **a** is said to be *hyperimmune-free* if it does not contain any hyperimmune sets. A set A is sometimes said to be *almost recursive* if its degree is hyperimmune-free, that is, any function computable in A is dominated by some recursive function.

We will need the following result from Soare [198] (p. 85) in our study of Π_1^0 classes.

Theorem II.7.8. *For any noncomputable c. e. set B, there is a simple, nonhypersimple c. e. set $A \equiv_T B$. Furthermore, for any c. e. set C and any infinite set $D \leq_T C$, if $A \cap D = \emptyset$, then $B \leq_T C$.*

Proof. (based on [198], p. 85). The requirements are fourfold:

(i) A is simple;

(ii) $A \leq_T B$;

(iii) A is not hypersimple;

(iv) $B \leq_T A$.

Let f be a one-to-one computable function with range B and let $B_s = \{f(0), \dots, f(s)\}$. A is enumerated in stages A_s, beginning with $A_0 = \emptyset$. Let $\mathbb{N} \setminus A_s = \{a_0^s < a_1^s < \dots\}$. There are two actions which may be taken at stage $s+1$.

Step 1. Here we take action to satisfy requirements (i) and (ii). For all $e \leq s$, attention is required if $W_{e,s} \cap A_s = \emptyset$ and

$$(\exists n)[n > 3e \;\&\; n \in W_{e,s} \;\&\; f(s+1) < n].$$

In this case, put into A_{s+1} the least such n corresponding to e. If no such e exists, do nothing.

Step 2 Here we take action to satisfy requirements (iii) and (iv) by putting $a_{3f(s+1)+1}$ into A.

We show that the requirements are satisfied.

(i) Suppose by way of contradiction that W_e is infinite and $W_e \cap A = \emptyset$. Then here is an algorithm for testing $m \in B$, that is, find s and $n > \max\{m, 3e\}$ such that $n \in W_{e,s}$; then by Step 1, $f(t) \geq n > m$ for all $t > s$, so that $m \in B \iff m \in B_s$.

(ii) We claim that $B_s\lceil n = B\lceil n$ implies $A_s\lceil n = A_s\lceil n$. To see this, let $m < n$ and $m \in A_{t+1} \setminus A_t$. Then in Step 1, we have $m > f(t+1)$, and in Step 2, we have $m = a^s_{3f(t+1)+1} > f(t+1)$, so that in either case $f(t+1) < n$ and $f(t+1) \in B_{t+1} \setminus B_t$. Thus to test $m \in A$, just compute from B a stage s such that $B_s\lceil m+1 = B\lceil m+1$ and then $m \in A \iff m \in A_s$.

(iii) Note that $|A \cap [0, 3e]| \leq 2e$, since at most e elements $\leq 3e$ are put into A under Step 1 (one from each W_i, $i < e$) and at most e elements under Step 2 (one for each $i \in B$, $i < e$, since $a_{3f(s+1)+1} = a_{3i+1} > 3i$ for $i = f(s+1) \in B$). Thus $\mathbb{N} \setminus A$ is majorized by the function $3x$ and A is not h-simple.

(iv) To test $m \in B$, use A to compute s such that $a^s_{3m+1} = \lim_t a^t_{3m+1}$; then $m \in B \iff m \in B_s$. Thus $B \leq_T A$.

Now let C be a c. e. set and let D be any infinite set such that $D \leq_T C$ and $A \cap D = \emptyset$. Let $D = \{d_0 < d_1 < \dots\}$. By the Modulus Lemma, there exists a uniformly computable double sequence $\{d_i^s\}_{i,s \in \mathbb{N}}$ such that $\lim_s d_i^s = d_i$ for all i with a modulus of convergence computable in C.

Let $s(e) = f^{-1}(e)$ and use the Recursion Theorem to define a computable function $h(e)$ such that $W_{h(e)} = \emptyset$ if $e \notin B$ and otherwise

$$W_{h(e)} = \{d_0^{s(e)}, d_1^{s(e)}, \dots, d^{s(e)}_{3(h(e))}\}.$$

We may assume without loss of generality that $h(e) > e$ for all e. Use C to compute a function $r(e)$ such that $d_i^{r(e)} = d_i$ for all $i \leq g(h(e))$. Let

$$\hat{B} = \{e \in B : e \notin B_{r(e)}\}.$$

There are two cases.

Case 1. $\hat{B}$ is finite. Then clearly $B \leq_T C$.

Case 2. $\hat{B}$ is infinite. Here is the procedure to test whether $b \in B$ using the function r. First find $e \in \hat{B}$ such that $3(h(e)) > b$. Since $e \in \hat{B}$, $r(e) < s(e)$ and therefore $d_i^{s(e)} = d_i$ for all $i \leq g(h(e))$, so that $W_{h(e)} \subseteq D \subseteq \mathbb{N} \setminus A$. It follows that $W_{h(e)}$ contains an element $u > 3h(e)$; let s_b be a stage such that $u \in W_{h(e),s_k}$. It follows from Step 1 that $f(s) \geq u > b$ for all $s \geq s_k$, so that

$$b \in B \iff b \in B_{s_b}.$$

Since s_b can be computed from C uniformly in b, it follows that $B \leq_T C$. □

We will obtain a lattice-theoretic characterization due to Lachlan [123] of hhsimple using the following two lemmas. For any c. e. set C, let $\mathcal{L}(C)$ denote the lattice (under inclusion) of c. e. supersets of C.

Lemma II.7.9. *(Lachlan) For any c. e. set C, if $\mathcal{L}(C)$ is a Boolean algebra, then C is hh-simple.*

Proof. Suppose that C is not hhsimple as witnessed by the disjoint weak array $\{W_{f(n)}\}_{n \in \mathbb{N}}$ and let $A = C \cup \bigcup_n (W_n \cap W_{f(n)})$. Suppose by way of contradiction that W_n is the complement of A in $\mathcal{L}(C)$ and choose $m \in W_{f(n)} \setminus C$. Since $m \notin C$, it follows that $m \in A \iff m \in W_n$, a contradiction. □

Theorem II.7.10 (Owings Splitting Theorem)**.** *Let $C \subseteq B$ be c. e. sets such that $B \setminus C$ is not co-c. e. Then there exists disjoint c. e. sets A_0 and A_1 (whose indices may be obtained uniformly from those of B and C) such that*

(i) $B = A_0 \cup A_1$;

(ii) $A_i \setminus C$ *is not co-c.e. for* $i = 0, 1$;

(iii) For any c. e. set W, if $C \cup (W \setminus B)$ is not c. e., then $C \cup (W \setminus A_i)$ is not c. e. for $i = 0, 1$.

Proof. Let f be a 1:1 computable function with range B and let $\{C_s\}_{s \in \mathbb{N}}$ be any computable enumeration of C. We try to meet the requirements

$$P_s : f(s) \in A_{0,s} \cup A_{1,s}, s \in \mathbb{N}$$

and

$$R_{\langle e,i \rangle} : A_i \setminus C \neq \mathbb{N} \setminus W_e, \text{ for } i = 0, 1.$$

Requirement $R_{\langle e,i \rangle}$ requires attention at stage $s + 1$ if $f(s + 1) \in W_{e,s}$ and $f(s + 1) \leq g(e, i, s)$.

Stage $s = 0$: Put $f(0) \in A_0$ and set $g(e, i, 0 = 0$ for all e, i.

Stage $s + 1$:

Step 1: If there exists $x \le g(e, i, s)$ such that $x \in W_{e,s} \cap (A_{i,s} \setminus C_s)$, set $g(e, i, s+1) = g(e, i, s)$. Otherwise $g(e, i, s+1) = s+1$.

Step 2: Let $y = f(s+1)$ and choose the least $\langle e, i\rangle$ such that $R_{\langle e,i\rangle}$ requires attention at stage $s+1$. Then put $y \in A_{i,s+1}$. If no such e, i exist, then put $y \in A_{0,s+1}$.

Let $A_i = \cup_s A_{i,s}$. Clearly $B = A_0 \cup A_1$.

To prove (ii), assume that $A_i \setminus C = \mathbb{N} \setminus W_e$. We must show that $B \setminus C$ is co-c. e. to obtain a contradiction. For $\langle e', i'\rangle < \langle e, i\rangle$, there are two possibilities. Either $\lim_s g(e', i', s) = z_{e',i'} < \infty$ so that after some stage s, we never put any $y \in A_i$ for the sake of $R_{\langle a,i\rangle}$, or $\lim_s g(e, i, s) = \infty$. Let z be the maximum of the $z_{e',i'}$ and choose s_0 large enough so that for all $\langle e, i\rangle$ of the first type, $g(e', i', s)$ has already converged to $z_{e',i'}$ and such that $f(s) > z$ for all $s \ge s_0$. Define the c. e. set

$$V_e = \{m : (\exists s \ge s_0)[m \in W_{e,s} \setminus B_s \;\&\; m \le g(e, i, s)\}.$$

Now $V_e \setminus B = W_e \setminus B$ since $\lim_s g(e, i, s) = \infty$, so that in fact $\mathbb{N} \setminus B \subseteq V_e$, that is, if $m \notin B$, then also $m \notin A_i$, so that by our assumption, $m \in W_e$ and thus $m \in V_e \setminus B$.

Also $V_e \cap (B \setminus C) = \emptyset$ (and hence $V_e \subseteq C \cup (\mathbb{N} \setminus B)$) by the following. Let $m \in V_e \cap B$ and take $s \ge s_0$ such that $m \le g(e, i, s)$ and $m \in W_{e,s} \setminus B_s$. Then $m \in B \setminus B_s$ so that $m = f(t)$ for some $t > s_0$. Now at stage $t+1$, there exists $\langle e', i'\rangle \le \langle e, i\rangle$ such that $\lim_r g(e', i', r) = \infty$ and m is put into $A_{i'}$ at stage $t+1$. But m cannot be a permanent witness for $\langle e', i'\rangle$, and therefore $m \in C$.

It follows that

$$\mathbb{N} \setminus (B \setminus C) = C \cup (\mathbb{N} \setminus B) = C \cup V_e.$$

□

Theorem II.7.11. *(Lachlan) For any c. e. set C, C is hhsimple if and only if $\mathcal{L}(C)$ is a Boolean algebra.*

Proof. The direction ($\longleftarrow$) follows from Lemma II.7.9. For the other direction, suppose that B is not complemented in $\mathcal{L}(C)$, that is $B \setminus C$ is not co-c. e. Apply Theorem II.7.10 to obtain A_0 and A_1. Let $W_{g(0)} = A_0$ and apply Theorem II.7.10 to A_1 and $C \cap A_1$ to obtain A_0^1 and A_1^1. Set $W_{g(1)} = A_0^1$ and continue in this fashion to obtain a pairwise disjoint sequence of c. e. sets $W_{g(n)}$ such that $W_{g(n)} \setminus C$ is not co-c. e. and hence is non-empty for all n. Finally, it is easy to uniformly define finite subsets $W_{f(n)} \subseteq W_{g(n)}$ such that $W_{f(n)} \setminus C \neq \emptyset$ for each n. That is, given a c. e. set $W = W_{g(n)}$ such that $W \setminus C \neq \emptyset$, let $W_{f(n),s}$ contain i if $i \in W_s$ and $(\forall j < i)[j \in W_s \to j \in C_s]$. (Thus if $j \in W_s \setminus C$, then no $i > j$ can enter $W_{f(n)}$ after stage s, so that $W_{f(n)}$ is finite, as desired. If j is the least element of $W \setminus C$, then j will be put into $W_{f(n)}$ as soon as all elements $i < j$ of W have come into C.) □

Definition II.7.12. *An infinite set C is* cohesive *if there is no c. e. set W such that $W \cap C$ and $C \setminus W$ are both infinite. A c. e. set A is* maximal *if for any c. e. set $B \supseteq A$, either B is cofinite or $B \setminus A$ is finite.*

Thus A is maximal if and only if its complement is cohesive.

From the lattice viewpoint, A is maximal if it is as large as possible in $\mathcal{E}^*$ without being trivial. Friedberg first constructed a maximal c. e. set in [70]. The proof is based on the following notion.

Definition II.7.13. *The* e-state *of a number m is $\{i \leq e : m \in W_i\}$ and the e-state at stage s is $\{i \leq e : m \in W_{i,s}\}$.*

The e-states are ordered lexicographically so that m has a higher e-state than n if there is some $j < e$ such that $m \in W_j$ but $n \notin W_j$ and for all $i < j$, $m \in W_i \iff n \in W_i$.

Theorem II.7.14 (Friedberg)**.** *There exists a maximal c. e. set A.*

Proof. Let $\sigma(e, m, s)$ denote the e-stage of m at stage s. We define the maximal set A in stages A_s so that

$$\mathbb{N} \setminus A_s = \{a_0^s < a_1^s < \cdots <\}.$$

Then $a_i = \lim_s a_i^s$ and $A = \mathbb{N} \setminus \{a_i : i \in \mathbb{N}\}$.

Initially $a_i^0 = i$ for all i, since $A_0 = \emptyset$. The construction proceeds in stages with the goal of making $\sigma(e, a_i) \leq \sigma(e, a_j)$ for all $e < i < j$.

At stage $s + 1$, choose the least e such that for some i with $e < i \leq s$, $\sigma(e, a_i^s, s{+}1) > \sigma(e, a_e^s, s{+}1)$ For this e, choose the least such i and let $a_e^{s+1} = a_i^s$. For $j < e$, $a_j^{s+1} = a_j^s$ and for all j, $a_{e+j}^{s+1} = a_{i+j}^s$. If no such e exists, then $a_i^{s+1} = a_i^s$ for all i.

Claim 1: For every e, $\lim_s a_e^s = a_e$ exists.

Proof of Claim 1: The proof is by induction on e, so we may suppose that it holds for all $i < e$ and take s_0 so that $a_i^s = a_i$ for all $i < e$ and all $s \geq s_0$. Then for any $s > s_0$, if $a_e^{s+1} > a_e^s$, it follows that $\sigma(e, a_e^{s+1}, s + 1) > \sigma(e, a_e^s, s)$. But there are only 2^{e+1} different e-states, so this can happen at most $2^{e+1} - 1$ times after stage s_0.

It follows from Claim 1 that A is coinfinite. It is clear from the construction that $a \in A \iff (\exists a) a \in A_s$, so that A is a c. e. set.

Claim 2: For all $e < i$, $\sigma(e, e) \geq \sigma(a_e, a_i)$.

Proof of Claim 2: Assume by induction that the Claim holds for all $d < e$. Fix $e < i$ and let s be large enough so that $a_j^s = a_j$ for all $j \leq i$. Suppose by way of contradiction that $\sigma(e, e) < \sigma(e, i)$ and choose $t > s$ such that $\sigma(e, e, t) = \sigma(e, e)$ and $\sigma(e, i, t) = \sigma(e, i)$. Then at stage $t + 1$, the construction will force $a_e^{t+1} \neq a_e^t$, a contradiction.

Claim 3: For all e such that $A \subseteq W_e$, either $W_e \setminus A$ is cofinite or $W_e \setminus A$ is finite.

Proof of Claim 3: Fix e such that $A \subseteq W_e$ and suppose that $W_e \setminus A$ is infinite. Now given any $i > e$, we may choose $j > e$ such that $a_j \in W_e$ and it follows from Claim 2 that $\sigma(i, i) \geq \sigma(i, j)$, so that $a_i \in W_e$. Hence W_e is cofinite. $\square$

Martin proved the following result connecting high degrees and maximal sets in [143].

Theorem II.7.15. *(Martin) A degree* **d** *is high if and only if there exists a maximal c. e. set* A *of degree* **d**. □

Exercises

II.7.1. Prove Theorem II.7.7.

II.7.2. Show that any hypersimple set is simple.

II.7.3. Prove that any maximal set must have high degree. Hint: Show that the principal function p_a is *dominant*, that is, for any computable function f, $f(m) \leq p_a(m)$ for almost all m.

II.8 Computable ordinals and the analytical hierarchy

Definition II.8.1. *(i) A relation* P *is said to be* Π^1_0 *(and also* Σ^1_0*) if* P *is arithmetical.*

(ii) A relation P *is said to be* Π^1_{n+1} *if there is a* Σ^1_n *relation* R *such that*

$$P(\overrightarrow{m}, \overrightarrow{x}) \iff (\exists y) R(\overrightarrow{m}, \overrightarrow{x}, y).$$

(iii) A relation S *is* Σ^1_n *if the complement if* Π^1_n.

(iv) S *is* Δ^1_n *if it is both* Σ^1_n *and* Π^1_n.

The relativized notions of $\Sigma^1_n[z]$ and $\Pi^1_n[z]$ are similarly defined. A relation is said to be *analytic* (resp. coanalytic, Borel) if it is $\Sigma^1_1[z]$ (resp. $\Pi^1_1[z]$, $\Delta^1_1[z]$) for some z.

There are two classic examples here.

Example II.8.2. *A set* $A \subseteq \mathbb{N}$ *may code a partial ordering* $\leq_A$*, where* $m \leq_A n \iff \langle m, n \rangle \in A$. *Also, we write* $m <_A n \iff m \leq_A n \ \& \ \neg n \leq_A m$.

$$WO = \{A : \leq_A \text{ is a well-ordering}\}.$$

Let LO *be the set of linear orderings. It is easy to see that* LO *is* Π^0_1. *Note for example that* $\leq_A$ *is transitive if and only if*

$$(\forall i)(\forall j)(\forall k)[(i \leq_A j \ \& \ j \leq_A k) \to i \leq_A k)],$$

which is a Π^0_1 *condition. Now a linear ordering is a well-ordering if and only if it is well-founded, that is, has no infinite descending chain. Thus* WO *is a* Π^1_1 *class, since* $\leq_A$ *is well-founded if and only if*

$$(\forall x)[(\forall m)(x(m+1) \leq_A x(m)) \to (\exists m)(x(m) \leq_A x(m+1))].$$

We similarly define the Π^0_1 class

$$PWO = \{A : \leq_A \text{ is a pre-well-ordering}\}.$$

We may also define the following Σ^1_1 relation $A \precsim (\prec) B$ to mean that $\leq_B$ is a (pre)-linear ordering and $\leq_A$ is isomorphic to a (proper) subordering of $\leq_B$. Here the subordering property can be expressed as

$$(\exists x)(\forall p)(\forall q)[p \leq_A q \iff x(p) \leq_B x(q)].$$

For the proper subordering add the following clause

$$(\exists r)(\forall p)(x(p) <_B r).$$

Observe that if $\leq_A$ and $\leq_B$ are linear orderings and $B \in WO$ and $A \precsim B$, then $A \in WO$. A similar result holds for pre-orderings.

The order type $\|R\|$ of a well-ordering in WO is the unique ordinal ρ such that $(Fld(R), \leq_R)$ is isomorphic to the standard ordering $(\rho, \in)$, where $Fld(R) = dom(R) \cup ran(R)$. For a pre-well-ordering, the norm *$\|R\|$ of R is the unique ordinal α such that there is an order-preserving map from $Fld(R)$ onto α.*

In a certain sense, the ordering relation $\precsim$ on PWO is Δ^1_1. Let $\|A\| = \aleph_1$ if $A \notin WO$. The proof of the following result, sometimes known as the Prewellordering Theorem, is left as an exercise.

Theorem II.8.3. *For all pre-linear orderings A and B, if either A or B is in WO, then*

(i) $A \precsim B \vee B \notin W \iff [A \in W \ \& \ \|A\| \leq \|B\|] \iff \neg B \precsim A$;

(ii) $A \prec B \vee B \notin W \iff [A \in W \ \& \ \|A\| < \|B\|] \iff \neg B \prec A$. □

A related example is the following.

Example II.8.4. *$T \subseteq \mathbb{N}$ is said to be a* tree *if $\{\sigma \in \mathbb{N}^* : \langle\sigma\rangle \in T\}$ is closed under initial segments.*

Then the set WF of well-founded trees is Π^1_1 since

$$T \in WF \iff (\forall x)(\exists n) x\lceil n \notin T.$$

An ordinal may be associated with a well-founded tree by means of the Brouwer-Kleene linear ordering $\leq_{KB}$ on $\mathbb{N}^*$, where

$$\sigma \leq_{KB} \tau \iff (\tau \preceq \sigma) \vee (\exists j)[\sigma(j) < \tau(j) \ \& \ (\forall i < j)\sigma(i) = \tau(i)].$$

Now given a well-founded tree T, let

$$F(T) = \{\langle\langle\sigma\rangle, \langle\tau\rangle\rangle : \sigma \in T \ \& \ \tau \in T \ \& \ \sigma \leq_{KB} \tau\}.$$

Lemma II.8.5. *T is a well-founded tree if and only if $F(T)$ is a well-ordering.*

Proof. It is easy to see that $\leq_{KB}$ is a linear ordering on $\mathbb{N}^*$ (see the exercises). If T is not well-founded, then there is an infinite path y through T and $\{y\lceil n : n \in \mathbb{N}\}$ provides an infinite descending $\leq_{KB}$ chain in $F(T)$. On the other hand, suppose that $F(T)$ is not well-founded and let $\{\sigma_i\}_{i\in\omega}$ be a descending $\leq_{KB}$ chain in $F(T)$. Then we can define by recursion an infinite path through T. For all $i > 0$, $|\sigma| > 0$ and $\sigma_{i+1}(0) \leq \sigma_i(0)$ since $\sigma_{i+1} \leq_{KB} \sigma_i$. Thus $y(0) = \lim_i \sigma_i(0)$ exists. Now if $\sigma_j(0) = y(0)$, then we have $\sigma_{j+1}(1) \leq \sigma_j(1)$, so that $y(1) = \lim_i \sigma_i(1)$ also exists. Proceeding by recursion we can define a sequence $y(n) = \lim_i \sigma_i(n)$. Now for each n, there is some j such that $y(i) = \sigma_j(i)$ for all $i < n$, so that $y\lceil n \preceq \sigma_j$ and therefore $y\lceil n \in T$ for all n. □

For a computable well-founded tree, $F(T)$ is of course a computable ordinal, since $\leq_{KB}$ is a computable relation.

The well-ordering $F(T)$ of a well-founded tree T is closely related to the *height* $ht(T)$, defined inductively as follows.

Definition II.8.6. *For any non-empty well-founded tree $T \subseteq \mathbb{N}^*$ and any $\sigma \in T$, the height $ht_T(\sigma)$ of $\sigma \in T$ is given by*

$$ht_T(\sigma) = sup\{ht_T(\sigma^\frown i) + 1 : \sigma^\frown i \in T\}.$$

Then the height $ht(T) = ht_T(\emptyset)$. If T is not well-founded, then $ht(T) = \infty$.

Lemma II.8.7. *For any non-empty well-founded tree T, $ht(T) \leq \|F(T)\| \leq \omega^{ht(T)} + 1$.*

Proof. These inequalities are proved by induction on $ht(T)$. For the base case, $ht(T) = 0$ if and only if $T = \{\emptyset\}$, which is if and only if $\|F(T)\| = 1$. Now for $h(T) > 0$, the ordering $F(T)$ consists of ω blocs $T((0)), T((1)), \ldots$ followed by the largest element $\emptyset$. By induction $ht(T((i))) \leq \|F(T((i))\| \leq \omega^{ht(T((i)))} + 1$ for each i. The first inequality is immediate. For the second, we have

$$\|F(T)\| \leq (\omega^{ht(T((0)))} + 1 + \omega^{ht(T((1)))} + 1 + \ldots) + 1.$$

There are two cases.

(Case 1): There is a fixed m such that $ht((T((i))) \leq ht(T((m)))$ for each i, so that $ht(T) = ht(T((m))) + 1 = \alpha + 1$. Then $\|F(t((m)))\| \leq \omega^\alpha + 1$ for each m and hence

$$\|F(T)\| \leq \omega \cdot \omega^\alpha + 1 \leq \omega^{\alpha+1} + 1.$$

(Case 2): There is no maximum $ht(T((m)))$. Let $ht(T) = \alpha$ and, for each m, let $ht(T((m))) = \alpha_m$, so that $\alpha = \sup\{\alpha_n : n \in \mathbb{N}\}$. For each m, there exists $n > m$ such that $\alpha_i < \alpha_n$ for all $i < n$ and thus

$$\|F(T((0)))\| + \|F(T((1)))\| + \cdots + \|F(T((n)))\| \leq \omega^{\alpha_0} + 1 + \cdots + \omega^{\alpha_n} = \omega^{\alpha_n} < \omega^\alpha,$$

so that

$$\|F(T)\| \leq \omega^\alpha + 1 = \omega^{ht(T)} + 1.$$

□

The next result follows from the Enumeration Theorem II.2.5.

Theorem II.8.8. *For each n, there is universal Π^1_{n+1} and a universal Σ^1_{n+1} set of numbers.* □

The sets WO and WF are both many-one complete for Π^1_1 sets. To see this, we first need a normal form for Σ^1_1 and Π^1_1 relations.

Lemma II.8.9. *For any Π^1_1 relation S, there is a computable relation R such that*

$$S(\vec{m}, \vec{x}) \iff (\exists y)(\forall n)R(y\lceil n, \vec{m}, \vec{x}).$$

Proof. It is clear that the family of relations expressible in this form includes the computable relations and it will suffice to show that this family is closed under number quantification and under existential function quantification. Given S in this form,

$$(\exists i)S(i, \vec{m}, \vec{x}) \iff (\exists z)(\forall n)R(z(0), \langle z(1), \dots, z(n)\rangle, \vec{m}, \vec{x}).$$

Also

$$(\forall i)S(i, \vec{m}, \vec{x}) \iff (\exists z)(\forall n)R(i, \langle z(2^i), \dots, z(2^i(2n+1))\rangle, \vec{m}, \vec{x}).$$

Finally,

$$(\exists u)S(\vec{m}, \vec{x}, u) \iff (\exists z)(\forall n)R(\langle z(1), \dots, z(2n-1)\rangle, \vec{m}, \vec{x}, (z(0), z(2), \dots)).$$

□

Definition II.8.10. *An ordinal α is* computable *if there is a computable well-ordering A such that $\|A\| = \alpha$. $W = \{e : \phi_e = \chi_A \quad \text{for some } A \in WO\}$ and $PW = \{e : \phi_e = \chi_A \quad \text{for some } A \in PWO\}$. The least noncomputable ordinal is denoted by ω_1^{CK} or just ω_1.*

Later on, we will need the concept of a *system of notations* for a computable ordinal.

Definition II.8.11. *A system of notations for a computable ordinal α is a map o from $\omega \setminus \{0\}$ to $\kappa + 1$ such that each of the following relations is recursive:*

(i) $o(a)$ *is a limit ordinal;*

(ii) $o(b) = o(a) + 1$;

(iii) $o(a) < o(b)$.

Lemma II.8.12. *Any computable ordinal α possesses a system of notations.*

Proof. Note that a computable well-ordering induces a mapping from $\omega \to \kappa$ with property (iii) but not with the other two properties. First observe that the computable ordinals form an initial segment of the ordinals (See the exercises.)

Now let α be an infinite countable ordinal and let λ be the largest limit ordinal $< \alpha$. Let $\lambda = \omega \cdot \gamma$ and $\alpha = \lambda + n$ for some ordinal γ and some finite n. Let $\leq_G$ denote a computable well-ordering of type γ with domain $G \subseteq \omega$ and let $a \in G$ such that $|a|_G = \gamma$. Let

$$C = \{\langle g, i\rangle : g \in G \ \&\ i \in \mathbb{N} \ \&\ (a = \lambda \ \rightarrow\ i \leq n)\}.$$

C is an infinite computable set, so computably isomorphic to ω. Thus it suffices to define the desired map o from C to α by

$$o(\langle g, i\rangle) = \omega \cdot |g|_G + i.$$

To verify that this defines a system of notations, observe that for $a = \langle g, i\rangle$ and $b = \langle h, j\rangle$ in C,

(i) $o(a)$ is a limit ordinal if and only if $i = 0$.

$o(b) = o(a) + 1$ if and only if $g = h$ and $j = i + 1$.

$o(a) < o(b)$ if and only if either $g = h$ and $i < j$ or if $g <_G h$.

□

Theorem II.8.13. *For all relations P, P is Π^1_1 if and only if $P \leq_m WO$, which is if and only if $P \leq_m WF$. Furthermore, if $P \subseteq \mathbb{N}$, then P is Π^1_1 if and only if $P \leq_m W$.*

Proof. It is clear that the Π^1_1 relations are closed under $\leq_m$, which gives one direction. Now let R be Π^1_1 and let P be a computable relation so that

$$P(\overrightarrow{m}, \overrightarrow{x}) \iff (\forall y)(\exists n) R(y \restriction n, \overrightarrow{m}, \overrightarrow{x}).$$

We may assume without loss of generality that

$$R(\tau, \overrightarrow{m}, \overrightarrow{x}) \ \&\ \sigma \preceq \tau \rightarrow R(\sigma, \overrightarrow{m}, \overrightarrow{x}).$$

by using bounded quantification if necessary. Then we may define a computable functional F such that

$$F(\overrightarrow{m}, \overrightarrow{x}) = \{\sigma : \neg R(\sigma, \overrightarrow{m}, \overrightarrow{x})\},$$

and we claim that

$$P(\overrightarrow{m}, \overrightarrow{x}) \iff F(\overrightarrow{m}, \overrightarrow{x}) \in WF.$$

It is clear that a witness y to the fact that $\neg P(\overrightarrow{m}, \overrightarrow{x})$ will also be a witness to the fact that $F(\overrightarrow{m}, \overrightarrow{x})$ is not well-founded, and vice versa. Thus $P \leq_m WF$.

It remains to check that $WF \leq_m WO$. It follows from Lemma II.8.5 that T is well-founded if and only if $F(T)$ is a well-ordering. Now for $P \subseteq \mathbb{N}$, let F be a computable function such that $m \in P \iff R_m \in WO$, where $R_m(p) \iff F(p, m) = 1$ and otherwise $F(p, m) = 0$. Let f be a computable function such that $\phi_{f(m)}(p) = F(p, m)$. Then $m \in P \iff f(m) \in W$. □

The proof of this theorem also shows that PWO is also m-complete for Π^1_1 sets.

Corollary II.8.14. *None of the sets W, PW, WO, PWO and WF are Σ^1_1.*

Proof. Since these sets are m-complete, if they were Σ^1_1, then all Π^1_1 sets would be Σ^1_1. (See exercise 4 below. □

Theorem II.8.15 (Selection Theorem)**.** *Any Π^1_1 relation R has a partial selection function Sel_R with a Π^1_1 graph such that*
$(\exists n)R(n, \vec{m}, \vec{x}) \iff R(Sel_R(\vec{m}, \vec{x}), \vec{m}, \vec{x}) \iff Sel_R(\vec{m}, \vec{x})\downarrow.$

Proof. Let R be reducible to W by the function F and let

$$Sel_R(\vec{m}, \vec{x}) = b \iff R(b, \vec{m}, \vec{x}) \;\&\; (\forall a)[F(a, \vec{m}, \vec{x}) \preceq F(b, \vec{m}, \vec{x}) \to b \leq a].$$

□

Theorem II.8.16 (Boundedness Theorem)**.** *(Spector, [201])*

(i) If S is a Σ^1_1 subset of PWO, then $sup\{\|A\| : A \in S\} < \omega_1$;

(ii) If S is a Σ^1_1 subset of PW, then $sup\{\|\phi_e\| : e \in S\} < \omega_1$.

Proof. Suppose that S were a counterexample to (i). Then

$$e \in W \iff (\exists c)[c \in S \;\&\; \phi_e \preceq \phi_c],$$

contradicting Corollary II.8.14.

If S were a counterexample to (ii), then $\{\phi_c : c \in S\}$ would be a counterexample to (i). □

Let $WO_\alpha = \{R \in WO : \|R\| < \alpha\}$ and $W_\alpha = \{c : \phi_c \in WO_\alpha\}$. Similarly, $PWO_\alpha = \{R \in PWO : \|R\| < \alpha\}$ and $PW_\alpha = \{c : \phi_c \in PWO_\alpha\}$

Theorem II.8.17. *For all $R \subseteq \mathbb{N}^k \times \mathbb{N}^{\mathbb{N}^l}$,*

(i) R is Δ^1_1 if and only if $R \leq_m WO_\alpha$ for some $\alpha < \omega_1$.

(ii) R is Δ^1_1 if and only if $R \leq_m W_\alpha$ for some $\alpha < \omega_1$.

Proof. Let $B \in WO$ such that $\|B\| = \alpha$. Then

$$A \in WO_\alpha \iff A \precsim B \iff (A \in WO \;\&\; \neg(B \precsim A).$$

Thus WO_α is Δ^1_1.

Now let $R \subseteq \mathbb{N}$ be Δ^1_1, let F be a computable functional such that $R(x) \iff F(x) \in WO$ and define the Σ^1_1 set Q by

$$Q = \{F(x) : R(x)\}.$$

Then $Q \subseteq WO$, so by the Boundedness Theorem, there exists $\alpha < \omega_1$ such that $Q \subseteq WO_\alpha$.

The proof of (ii) is similar. □

Here is a surprising corollary to the Boundedness Principle.

Theorem II.8.18. *(i) For any Σ^1_1 pre-well-ordering relation R, $\|R\| < \omega_1$;*

(ii) There is a Π^1_1 well-ordering of order type ω_1.

Proof. (i) Let $\|R\| = \alpha$. Then $A \in W_{\alpha+1} \iff A \precsim R$ and it is easy to see that this is a Σ^1_1 definition of $W_{\alpha+1}$, implying that $\alpha < \omega_1$ by the Boundedness Principle.

(ii) Define the Π^1_1 set W^* to contain a unique index c with $\|\phi_c\} = \alpha$ for each $\alpha < \omega_1$. That is,

$$c \in W^* \iff c \in W \;\&\; (\forall d < c)[\neg\phi_c \precsim \phi_d \vee \neg\phi_d \precsim \phi_c].$$

Then let

$$R(c, d) \iff c \in W^* \;\&\; d \in W^* \;\&\neg\phi_d \precsim \phi_c.$$

□

Exercises

II.8.1. Show that the computable ordinals form an initial segment of the countable ordinals.

II.8.2. Show that the property of coding a linear ordering is in fact Π^0_1.

II.8.3. Show that the Brouwer-Kleene ordering is a linear ordering.

II.8.4. Prove the Enumeration Theorem II.8.8 for Π^1_1 sets and show that the universal Π^1_1 set cannot be Σ^1_1.

II.8.5. Prove Theorem II.8.3.

II.8.6. The definition of a computable ordinal may be relativized to computability from a fixed oracle A. Give appropriate definitions for W^A and ω_1^A and prove relativized versions of Theorems II.8.16, II.8.17 and II.8.18.

II.9 Inductive Definability

Inductive definitions play a fundamental role in many areas of mathematics. We have already seen that the set of computable functions is defined inductively and of course the set of terms and formulas of a given language are also defined inductively. The formal notion of inductive definability was first given by Spector [203] and is fully developed by Moschovakis in his book [154].

An *operator* Γ over a set X is a a function from $\mathcal{P}(X)$ to $\mathcal{P}(X)$. Γ is said to be *inclusive* if $Y \subseteq \Gamma(Y)$ for all $Y \subseteq X$. Γ is said to be *monotone* if $\Gamma(Y) \subseteq \Gamma(Z)$ for all $Y \subseteq Z \subseteq X$. Γ is said to be *inductive* if it is either inclusive or monotone. The operator Γ inductively defines a subset $Cl(\Gamma)$ as follows. A sequence Γ^α of subsets of X is defined recursively by $\Gamma^0 = \emptyset$, $\Gamma^{\alpha+1} = \Gamma(\Gamma^\alpha)$ and $\Gamma^\lambda = \bigcup_{\beta<\lambda} \Gamma^\beta$ for limit ordinals λ. The closure of Γ is $Cl(\Gamma) = \bigcup_\alpha \Gamma^\alpha$. A set Y is said to be a *fixed point* of Γ if $\Gamma(Y) = Y$.

Lemma II.9.1. *For any inductive operator* Γ,

(i) *For any ordinal* α, *if* $\Gamma^{\alpha+1} = \Gamma^\alpha$, *then* $\Gamma^\beta = \Gamma^\alpha = Cl(\Gamma)$ *for all* $\beta \geq \alpha$.

(ii) *There exists* α *such that* $Card(\alpha) \leq Card(X)$ *such that* $\Gamma^{\alpha+1} = \Gamma^\alpha$.

Proof. (i) is easily proved by induction on β.

For (ii), let κ be a cardinal and suppose that $\Gamma^{\alpha+1} \setminus \Gamma^\alpha$ is non-empty for all $\alpha < \kappa$. Then clearly $Card(\kappa) \leq Card(\Gamma^\kappa) \leq Card(X)$. □

Now we can define the *closure ordinal* $|\Gamma|$ of Γ to be the least ordinal α such that $\Gamma^{\alpha+1} = \Gamma^\alpha$. The following is immediate.

Corollary II.9.2. *For any inductive operator* Γ *over* X, $Card(|\Gamma|) \leq Card(X)$ *and* $Cl(\Gamma) = \Gamma^{|\Gamma|}$.

In particular, for $X = \mathbb{N}$, $|\Gamma| < \aleph_1$.

Example II.9.3. *The* Π^1_1 *set* W *can be given by a* Π^0_1 *monotone inductive definition. First define the computable function* ν *so that* $\phi_{\nu(c,n)}$ *defines a restriction of* ϕ_c *to elements* below n *in the following sense. Recall that* $i \leq_c j$ *means that* $\phi_c(i,j) = 1$ *and* $i <_c j$ *means that* $i \leq_c j$ *but not* $j \leq_c i$. *Let*

$$\phi_{\nu(c,n)}(i,j) = \begin{cases} \phi_c(i,j), & \text{if } j <_c n \\ 0, & \text{otherwise.} \end{cases}$$

Now let

$$c \in \Gamma(A) \iff W_c = \emptyset \ \vee \ (\forall n)\nu(c,n) \in A$$

It is clear that $c \in \Gamma^1$ *if and only if* $W_c = \emptyset$, *which is if and only if* $c \in W$ *and* $\|c\| = 0$. *It follows by induction that* $c \in \Gamma^{m+1} \iff \|c\| \leq m$ *and hence* $c \in \Gamma^\omega \iff \|c\| < \omega$ *and in general,* $c \in \Gamma^\alpha \iff (c \in W \ \& \ \|c\| < \alpha)$.

Thus we see that $|\Gamma| = \omega_1$ *and* $Cl(\Gamma) = W$.

Note that $Cl(\Gamma)$ is a *fixed point* of Γ.

Theorem II.9.4. *For any monotone operator* Γ *over a set* X, $Cl(\Gamma)$ *is the least fixed point of* Γ, *that is,*

$$x \in Cl(\Gamma) \iff (\forall Z)[\Gamma(Z) = Z \to x \in Z].$$

Proof. Let $U = \{x : (\forall Z)[\Gamma(Z) = Z \to x \in Z]\}$. Then $x \in U$ implies $x \in Cl(\Gamma)$, since $Z = \Gamma$ satisfies $\Gamma(Z) = Z$. For the other direction, let Z be any set such that $\Gamma(Z) = Z$. It can be seen by induction that $\Gamma^\alpha \subseteq Z$ for all α. That is, certainly $\Gamma^0 = \emptyset \subseteq Z$. Then supposing $\Gamma^\alpha \subseteq Z$, we have $\Gamma^{\alpha+1} = \Gamma(\Gamma^\alpha \subseteq \Gamma(Z) = Z$. Finally, if $\Gamma^\beta \subseteq Z$ for all $\beta < \lambda$, then $\Gamma^\lambda \subseteq Z$. It follows that $\Gamma^\alpha \subseteq U$ for all α and hence $Cl(\Gamma) \subseteq U$. □

The monotone operator Γ is said to be *finitary* if for all x and Y, if $x \in \Gamma(Y)$, then there is a finite $Z \subseteq Y$ such that $x \in \Gamma(Z)$. For example, the operator which defines the set of formulas of propositional logic is finitary, since each new formula is generated by either one or two previously generated formulas.

Lemma II.9.5. *If Γ is a finitary monotone operator on X, then $|\Gamma| \leq \omega$.*

Proof. Suppose $x \in \Gamma(\Gamma^\omega)$ and let $Z \subseteq \Gamma^\omega$ be a finite set such that $x \in \Gamma(Z)$. Then there exists $n < \omega$ such that $Z \subseteq \Gamma^n$ and hence $x \in \Gamma^{n+1}$. Thus $\Gamma^{\omega+1} = \Gamma^\omega$. □

The complexity of an operator Γ on $\mathbb{N}$ is given by the complexity of the relation $\{\langle m, A\rangle : m \in \Gamma(A)\}$. We will also consider operators on $\mathbb{N}$ with real *parameters*. That is, for example, a family $\{\Gamma_x : x \in \mathbb{N}^\mathbb{N}\}$ of operators over $\mathbb{N}$ will be computable if $\{\langle m, x, A\} : m \in \Gamma_x(A)\}$ is a computable relation.

Lemma II.9.6. *Any Σ^0_1 operator is finitary.*

Proof. Let Γ be a Σ^0_1 operator and let R be a computable relation so that

$$m \in \Gamma(A) \iff (\exists k)R(m, A\lceil k).$$

Suppose $m \in \Gamma(A)$ and let k be given as above so that $R(m, A\lceil k)$. Now let $Z = \{i : i < k \;\&\; i \in A\}$. Then $R(m, Z\lceil k)$ so that $m \in \Gamma(Z)$. □

Theorem II.9.7. *If Γ is a Σ^0_1 monotone operator over $\mathbb{N}$, then $|\Gamma| \leq \omega$ and $Cl(\Gamma)$ is a c. e. set.*

Proof. Let Γ be a Σ^0_1 operator and let R be a computable relation so that

$$m \in \Gamma(A) \iff (\exists k)R(m, A\lceil k).$$

It follows from lemmas II.9.5 and II.9.6 that $|\Gamma| \leq \omega$.

For the closure, we have the following

CLAIM: $m \in \Gamma(W_e) \iff (\exists k, s)R(m, W_{e,s}\lceil k)$.

Proof of Claim: If $m \in \Gamma(W_e)$, then $(\exists k)R(m, W_e\lceil k)$ and thus $R(m, W_{e,s}\lceil k)$ where s is large enough so that $W_{e,s}\lceil k = W_e\lceil k$. If $R(m, W_{e,s}\lceil k)$, then $m \in \Gamma(W_{e,s})$ and since Γ is monotone, $m \in \Gamma(W_e)$.

It follows that there is a computable function ϕ such that $\Gamma(W_e) = W_{\phi(e)}$. Now we can recursively define a function ψ such that $\Gamma^n = W_{\psi(n)}$ by letting $\psi(0) = 0$ and then $\psi(n+1) = \phi(\psi(n))$. Finally, we have $m \in Cl(\Gamma) \iff (\exists n)m \in W_{\psi(n)}$. □

Theorem II.9.8. *For any $n > 0$ and any Π^1_n monotone operator Γ over $\mathbb{N}$, $Cl(\Gamma)$ is Π^1_n.*

Proof. By the improved version of Theorem II.9.4 (see Exercise 1 below), we have

$$\begin{aligned} m \in Cl(\Gamma) &\iff (\forall Z)[\Gamma(Z) \subseteq Z \to m \in Z] \\ &\iff (\forall Z)[(\forall m)(m \in \Gamma(Z) \to m \in Z) \to m \in Z], \end{aligned}$$

which is a Π^1_1 definition. □

In particular, the closure of any Π^0_1 monotone operator is Π^1_1. This can be reversed up to many-one reduction.

Theorem II.9.9. *For any Π^1_1 $P \subseteq \mathbb{N} \times \mathbb{N}^{\mathbb{N}}$, there is a uniformly Π^0_1 monotone operator Γ_x and a computable function f such that*

$$P(m,x) \iff f(m) \in Cl(\Gamma_x).$$

Proof. Let R be a computable relation such that

$$P(m,x) \iff (\forall y)(\exists k)R(m, y\lceil k, x).$$

Let

$$\langle m, \sigma\rangle \in \Gamma_x(A) \iff R(m,\sigma,x) \ \vee \ (\forall n)\langle m, \sigma\lceil n\rangle \in A.$$

Then it is easy to check that $P(m,x) \iff \langle m, \emptyset\rangle \in Cl(\Gamma_x)$. □

Theorem II.9.10. *For any Π^0_1 inductive operator Γ, $|\Gamma| \leq \omega_1$ and $Cl(\Gamma)$ is Π^1_1.*

Proof. Let R be a computable relation so that, for all m and A,

$$m \in \Gamma(A) \iff (\forall k)R(m, A\lceil k).$$

We want to uniformly define the levels Γ^α of the inductive definition Γ as follows. Let $i <_c j$ denote $\phi_c(i,j) = 1$ & $\phi_c(j,i) = 0$, so that if $c \in W$ and ϕ_c is the characteristic function of the set C, then $i <_c j$ means that $i <_C j$; we also let $\|i\|_c = \|i\|_C$. Now define

$$S(c,P) \iff (\forall m,i)(P(m,i) \iff (\exists j)[j <_c i \ \& \ m \in \Gamma(\{n : P(n,j)\})]).$$

It follows by induction that if $c \in W$ and $S(c,P)$, then, for all $i \in Fld(R)$,

$$P(m,i) \iff m \in \Gamma^{\|i\|_c}.$$

Now we have

$$m \in \Gamma^{\omega_1} \iff (\exists i,c)[c \in W \ \& \ (\forall P)(S(c,P) \ \rightarrow P(m,i))].$$

Thus Γ^{ω_1} is a Π^1_1 set, which we denote as Q. For each $c \in W$, let Q_c denote $\Gamma^{\|c\|}$, so that as above, we have a Π^1_1 definition

$$m \in Q_c \iff (\exists i)(\forall P)[S(c,P) \ \rightarrow \ P(m,i)],$$

and we also have a Σ^1_1 definition

$$m \in Q_c \iff (\exists i,P)[S(c,P) \ \& \ P(m,i)].$$

We note that this part of the argument applies to any Δ^1_1 operator.

It remains to show that $\Gamma(Q) \subseteq Q$. Suppose therefore that $m \in \Gamma(Q)$, so that

$$(\forall k)R(m, Q\lceil k).$$

and define the Σ^1_1 set V by

$$a \in V \iff (\exists k)(\forall c)[(c \in W \ \& \ R(m, Q_c)) \rightarrow \phi_a \precsim \phi_c].$$

It follows that $V \subseteq W$ and hence $sup\{\|\phi_a\| : a \in V\} = \alpha < \omega_1$. Then $(\forall k)R(m, \Gamma^\alpha\lceil k)$, so that $m \in \Gamma^{\alpha+1}$ and thus $m \in Q$. □

A natural problem in connection with Theorem II.9.4 is the nature of the family of fixed points of an inductive operator. Of course we have $\Gamma(\mathbb{N}) = \mathbb{N}$ by the assumption that $X \subseteq \Gamma(X)$, so that we will not have a unique fixed point. However, we can refine the previous result to show that

The Boundedness Principle for inductive definability is needed for the analysis of the Cantor-Bendixson derivative in Chapter V. The following prewellordering theorem for inductively definable sets is due to Kunen; see [154], p. 27.

The prewellordering associated with any inductive operator Γ is induced by the following norm:

$$|x|_\Gamma = \begin{cases} (\text{least } \alpha) x \in \Gamma^{\alpha+1}, & \text{if } \ x \in Cl(\Gamma), \\ \infty, & \text{otherwise.} \end{cases}$$

The following Stage Comparison Theorem is due independtly to P. Aczel and K. Kunen; see [154] for a more general result.

Theorem II.9.11. *Let Δ be a Δ^1_1 monotone inductive operator. Then the following relations are both Π^1_1.*

$$R(m,n) \iff |m|_\Delta \le |n|_\Delta \ \& \ m \in Cl(\Delta);$$
$$S(m,n) \iff |m|_\Delta < |n|_\Delta \ \& \ m \in Cl(\Delta).$$

Proof. Define a Δ^1_1 monotone operator Λ by

$$(0,m,n) \in \Lambda(A) \iff m \in \Delta(\{i : 1,i,n) \in A\});$$
$$(1,m,n) \in \Lambda(A) \iff n \notin \Delta(\{j : (0,m,j) \notin A\}).$$

It can be checked that $R(m,n) \iff (0,m,n) \in Cl(\Lambda)$ and $S(m,n) \iff (1,m,n) \in Cl(\Lambda)$. □

Theorem II.9.12. *For any Δ^1_1 monotone inductive operator Δ and any Σ^1_1 set $A \subseteq Cl(\Delta)$, there is a computable ordinal α such that $A \subseteq \Delta^\alpha$.*

Proof. Let A be a Σ^1_1 subset of $Cl(\Delta)$ as described. Then by Theorem II.9.11, the prewellordering R_A defined by

$$R_A(m,n) \iff |m|_\Delta \le |n|_\Delta \ \& \ n \in A$$

is a Σ^1_1 prewellordering. It now follows from Theorem II.8.18(a) that $|R_A| < \omega_1^{CK}$. □

Corollary II.9.13. *For any Δ^1_1 monotone inductive operator Δ, $|\Delta| \le \omega_1^{C-K}$.*

Exercises

II.9.1. Improve Theorem II.9.4 by showing that $Cl(\Gamma) = \bigcap\{Z \subseteq X : \Gamma(Z) \subseteq Z\}$.

II.9.2. Show that a non-monotone Σ^0_1 operator need not have a Σ^0_1 closure and in particular can have a Π^0_n complete closure for any n.

II.9.3. Show that a non-monotone Π^0_2 operator may have a closure which is not Π^1_1.

II.9.4. Show that there is a Σ^0_1 inductive operator Γ such that the set of true sentences of arithmetic is many-one reducible to $Cl(\Gamma)$.

II.9.5. Show that if Γ is a Δ^1_1 inductive operator and $A \subseteq \Gamma^{\omega_1}$ is Σ^1_1, then for some $\alpha < \omega_1$, $A \subseteq \Gamma^\alpha$. Then show that if Γ is Π^0_1, then $Cl(\Gamma)$ is Δ^1_1 if and only if $|\Gamma| < \omega_1$.

II.10 The hyperarithmetical hierarchy

In our study of the Cantor-Bendixson derivative $D(P)$ of a Π^0_1 class P, we will see that the iterated derivative $D^\alpha(P)$ for an infinite, computable ordinal α is a hyperarithmetical, or effectively Borel set. Since index sets for Π^0_1 classes and for hyperarithmetical sets will be important in this work, we will present a definition of the hyperarithmetical sets based on indices. This approach is based on that given by Hinman ([87], p. 163).

Informally, a set is Σ^0_ω if it is the union of an effective sequence A_n of arithmetical sets and more generally a set is Σ^0_λ for a computable ordinal λ if it is the union of an effective sequence of sets, each of which is Σ^0_α for some $\alpha < \lambda$. As for the arithmetical hierarchy, a set is Π^0_α if its complement is Σ^0_α and is $\Sigma^0_{\alpha+1}$ if it is the union of an effective sequence of Π^0_α sets.

The following is an inductive definition of the hyperarithmetic sets H_e, taken essentially from Hinman ([87], p. 163).

First we define a set of ordinal notations.

Definition II.10.1. *H is the smallest subset of $\mathbb{N}$ such that for all a,*

(i) $\langle 7, a\rangle \in H$;

(ii) if $\phi_a(n) \in H$ *for all* n, *then* $a \in H$.

We observe that H is the closure of a Π^0_1 monotone inductive operator Γ_H and thus each $a \in H$ is assigned an ordinal $\alpha = \|a\|_H$ so that $a \in H^{\alpha+1} - H^\alpha$. It follows from Theorem II.9.8 that H is Π^1_1 and that each ordinal $\|a\|_H$ is computable.

Then each $a \in H$ is assigned a hyperarithmetic set by the following. (For $a \notin H$, we let $H_a = \emptyset$.)

Definition II.10.2. *Let* $a \in H$. *Then*

(i) If $a = \langle 7, b\rangle$, *then* $H_a = W_b$;

(ii) if ϕ_a *is total, then* $H_a = \cup_n \mathbb{N} \setminus H_{\phi_a(n)}$.

For example, let B be a Σ^0_2 set and let R be a computable relation such that $i \in B \iff (\exists n)(\forall m)R(i,m,n)$. Let $A_n = \{(i,m) : \neg(\forall m)R(i,m,n)$ so that $B = \cup_n \mathbb{N} \setminus A_n$. Let $\psi(i,m) =$ (least $n)\neg R(i,m,n)$ and define ϕ by the s-m-n

Theorem so that $\phi_{\phi(m)}(i) = \psi(i, m)$. Then $A_n = W_{\phi(m)}$ for each n, so that if $\phi_a(m) = \langle 7, \phi(m) \rangle$, then $H_a = B$.

A subset of $\mathbb{N}$ is said to be *hyperarithmetical* if it equals H_a for some index $a \in H$. The hyperarithmetical hierarchy is defined as follows.

Definition II.10.3. *For all α and all $A \subseteq \mathbb{N}$,*

(i) A is Σ^0_α if $A = H_a$ for some $a \in H^\alpha$;

(ii) A is Π^0_α if $\mathbb{N} \setminus A$ is Σ^0_α;

(iii) $\Delta^0_\alpha = \Sigma^0_\alpha \cap \Pi^0_\alpha$.

It follows that for limit ordinals λ, A is Σ^0_λ if and only if A is Σ^0_α for some $\alpha < \lambda$. It is easy to see that this definition agrees with the arithmetic hierarchy for $\alpha < \omega$. Note that for infinite ordinals α, some authors (e.g. Hinman [87]) denote this hierarchy as the $\Sigma^0_{(\alpha)}$ sets and let the Σ^0_λ sets be effective unions of $\Sigma^0_{(\lambda)}$ sets. The definition we use here is for uniformity of results concerning Π^0_1 classes.

There is a natural set $P_H = \{\langle a, m \rangle : a \in H \ \& \ m \in H_a\}$ which is complete for the family of hyperarithmetical sets. However, this set is Π^1_1 and not hyperarithmetical.

Lemma II.10.4. *Every hyperarithmetical set is Δ^1_1.*

Proof. We give a monotone Π^0_2 inductive definition of the following set:

$$V = \{\langle 0, a, m \rangle : a \in H \ \& \ m \in H_a\} \cup \{\langle 1, a, m \rangle : a \in H \ \& \ m \notin H_a\}.$$

That is, let

(0) $\langle 0, a, m \rangle \in \Gamma(X) \iff [(a = \langle 7, b \rangle \ \wedge \ m \in W_b) \ \vee \ (\exists n)(\phi_a(n) \downarrow \ \wedge \ \langle 1, \phi_a(n), m \rangle \in X)]$.

(1) $\langle 1, a, m \rangle \in \Gamma(X) \iff [(a = \langle 7, b \rangle \ \wedge \ m \notin W_b) \ \vee \ (\forall n)(\phi_a(n) \downarrow \ \wedge \ \langle 0, \phi_a(n), m \rangle \in X)]$.

We then show by induction on α that $\Gamma^\alpha = V^\alpha$. For $\alpha = 0$, both sets are empty and for $\alpha = 1$,

$$\Gamma^1 = V^1 = \{\langle \langle 7, b \rangle, 0, m \rangle : m \in W_b\} \cup \{\langle \langle 7, b \rangle, 1, m \rangle : m \notin W_b\}.$$

Now suppose that $\Gamma^\alpha = V^\alpha$ for all $\alpha < \beta$. If β is a limit ordinal, then clearly

$$\Gamma^\beta = \cup_{\alpha<\beta} \Gamma^\alpha = \cup_{\alpha<\beta} V^\alpha = V^\beta.$$

Next suppose that $\beta = \alpha + 1$ for some α.

If $\langle a, i, m \rangle \in \Gamma^{\alpha+1}$, then, for all p, $(\exists j \leq 1)\langle \phi_a(p), j, \rangle \in \Gamma^\alpha$, so that by induction $\phi_a(p) \in H^\alpha$ for all p and hence $a \in H^{\alpha+1}$. For $i = 0$, there exists p such that $\langle \phi_a(p), 1, m \rangle \in \Gamma^\alpha$ and hence by induction $m \notin H_{\phi_a(p)}$. It follows that $m \in H_a$ and therefore $\langle a, 0, m\rangle \in V^{\alpha+1}$. For $i = 1$, it follows similarly that $m \notin H_a$.

Now suppose that $\langle a, i, m\rangle \in V^{\alpha+1}$. Then $a \in H^{\alpha+1}$, so that for all p, $\phi_a(p) \in H^{\alpha}$. For $i = 0$, $m \in H_a$ and therefore there exists p such that $m \notin H_{\phi_a(p)}$ and hence by induction, $\langle \phi_a(p), 1, m\rangle \in \Gamma^{\alpha}$. It follows that $\langle a, 0, m\rangle \in \Gamma^{\alpha+1}$. The argument for $i = 1$ is similar.

Since Γ is an arithmetical monotone inductive operator, it follows from Theorem II.9.8 that V is a Π^1_1 set and hence the hyperarithmetical set H_a is Δ^1_1 for each a. □

It follows from the proof of Lemma II.10.4 that the set P_H is Π^1_1. We can now prove the Spector-Gandy Theorem, due independently to Spector [202] and Gandy [76]. We say that a relation $R \subseteq \mathbb{N}^k \times \mathbb{N}^{\mathbb{N}^\ell}$ is ${\Sigma^1_1}^{HYP}$ if and only if there is an arithmetical relation P such that

$$R(\overrightarrow{m}, \overrightarrow{x}) \iff (\exists y \in \Delta^1_1[\overrightarrow{x}])P(\overrightarrow{m}, \overrightarrow{x}, y).$$

Theorem II.10.5 (Spector-Gandy Theorem). *A relation $R \subseteq \mathbb{N}^k \times \mathbb{N}^{\mathbb{N}^\ell}$ is ${\Sigma^1_1}^{HYP}$ if and only if it is Π^1_1.*

Proof. We give the proof without the real parameters $\overrightarrow{x}$ and with just one number variable m. We may assume that the $\exists y$ quantifier ranges over $\{0, 1\}^{\mathbb{N}}$. It follows from the proof of Lemma II.10.4 that we can define a Σ^1_1 relation S such that for any $a \in H$, $S(a, y) \iff y = H_a$. That is,

$$y = H_a \iff (\forall n)(\forall i < 2)[(a, i, n) \in V \rightarrow y(n) = i].$$

Now we have

$$(\exists y \in \Delta^1_1)(P(m, y)) \iff (\exists a)[a \in H \;\&\; (\forall y)(y = H_a \rightarrow P(m, y))].$$

This demonstrates that any ${\Sigma^1_1}^{HYP}$ relation R is in fact Π^1_1.

For the reverse direction, it clearly suffices to show that the Π^1_1 complete relation W is ${\Sigma^1_1}^{HYP}$.

It follows immediately from the first part of our proof that the set $\{0, 1\}^{\mathbb{N}} \cap \Delta^1_1$ is itself Π^1_1. Then by Theorem II.8.13, there is a computable function F such that, for all z,

$$z \in \Delta^1_1 \iff F(z) \in W.$$

Now $\Delta^1_1 \cap \{0, 1\}^{\mathbb{N}}$ cannot be a Δ^1_1 set, by the following argument. Choose $a_0 \in W$ and let

$$Q(a, y) \iff a \in W \;\&\; [(y \in \Delta^1_1 \;\&\; F(y) = a] \;\vee\; (y \notin \Delta^1_1 \;\&\; a = a_0)].$$

Then Q is Π^1_1 and therefore has a selector Sel_Q with Π^1_1 graph by Theorem II.8.15. Since Sel_Q is total, the graph is actually Δ^1_1. Now if Δ^1_1 were itself Δ^1_1, then the image of Δ^1_1 would be a Σ^1_1 subset of W and hence bounded by Theorem II.8.16.

It follows from Theorem II.8.17 that the range of F is unbounded in W and therefore

$$a \in W \iff (\exists z \in \Delta^1_1)\phi_a \preceq F(z).$$

□

Before establishing the reverse implication of lemma II.10.4, we will need several technical lemmas from [87].

Lemma II.10.6. *For all α, Σ^0_α and Π^0_α are effectively closed under many-one reduction, that is, there is a primitive recursive function g such that for all m, e:*

$$m \in H_{g(a,e)} \iff \phi_e(m) \in H_a.$$

Proof. Let h be a primitive recursive function such that

$$g(a,e) = \langle 7, b\rangle, \text{ where } \phi_b(m) = \phi_{(a)_1}(\phi_e(m)), \text{ if } a = \langle 7, b\rangle;$$

and otherwise define g by the Recursion Theorem so that

$$\phi_{g(a,e)}(p) = g(\phi_a(p), e).$$

□

Lemma II.10.7. *For all $\alpha > 0$, the family of Σ^0_α relations is effectively closed under computably enumerable union and finite intersection; that is, there exists primitive recursive functions f and g such that*

(i) If $\phi_a(p) \in H^\alpha$ for all p, then $f(a) \in H^\alpha$ and

$$m \in H_{f(a)} \iff (\exists p) m \in H_{\phi_a(p)}.$$

(ii) if $a, b \in H_\alpha$, then $g(a,b) \in H_\alpha$ and

$$H_{g(a,b)} = H_a \cap H_b.$$

Proof. (i) We will define $\phi_{f(a)}(r)$ in two cases and then use the s-m-n theorem to define $f(a)$. In either case,

$$m \in H_{f(a)} \iff (\exists r) m \notin H_{f(a)}(r).$$

Let $(r)_0 = p$ and $(r)_1 = q$.

Case I: If $\phi_a(p) \notin H^1$, then

$$\phi_{f(a)}(r) = \phi_{\phi_a(p)}(q).$$

If there exists such an r with $m \notin H_{f(a)}(r)$, then

$$(\exists p)[(\exists q) m \notin \phi_{\phi_a(p)}(q)],$$

so that as desired

$$(\exists p) m \in H_{\phi_a(p)}.$$

This argument is clearly reversible for $\phi_a(p) \notin H^1$.

Case II: If $\phi_a(p) = \langle 7, b \rangle$, then define $h(a, r)$ so that

$$W_{h(a,r)} = \mathbb{N} \setminus W_{b,q}$$

and let $\phi_{f(a)}(r) = \langle 7, h(a, r) \rangle$.

If $m \notin H_{\phi_{f(a)}(r)}$, then $m \notin W_{h(a,r)}$, so that $m \in W_b = H_{\phi_a(p)}$ and hence $(\exists p) m \in H_{\phi_a(p)}$ as desired. Again the reverse direction is clear.

(ii) There are four cases.

(1) If $a = \langle 7, d \rangle$ and $b = \langle 7, e \rangle$, then $g(a, b) = \langle 7, c \rangle$, where

$$\phi_c(n) = \phi_d(n) + \phi_e(n).$$

(ii) If $a = \langle 7, d \rangle$ and $b \notin H^1$, then $g(a, b) = c$ may be defined by part (i) so that

$$H_{\phi_c(p)} = H_{\phi_a(p)} \cup (\mathbb{N} \setminus W_d).$$

(iii) The case when $a \notin H^1$ and $b = \langle 7, e \rangle$ is similar to (ii).

(iv) If $a, b \notin H^1$, then $g(a, b)$ is defined by (i) so that

$$H_{\phi_{g(a,b)}(p)} = H_{\phi_a(p)} \cup H_{\phi_b(p)}.$$

□

Lemma II.10.8. *For all $\alpha > 0$, the family of Σ^0_α relations is effectively closed under existential number quantification (and thus the family of Π^0_α relations is effectively closed under universal number quantification); that is, there is a primitive recursive function h such that for all $a \in H_\alpha$, $h(a) \in H_\alpha$ and $m \in H_{h(a)} \iff (\exists p)\langle p, m \rangle \in H_a$.*

Proof. Let h be a computable function such that $\phi_{h(p)}(m) = \langle p, m \rangle$. Then

$$(\exists p)\langle p, m \rangle \in H_a \iff (\exists p)\phi_{h(p)}(m) \in H_a.$$

Then taking g from Lemma II.10.6, we have

$$(\exists p)\langle p, m \rangle \in H_a \iff (\exists p) m \in H_{g(a,h(p))}.$$

Now letting $\phi_{\pi(a)}(p) = g(a, h(p))$ and taking f from Lemma II.10.7, we have

$$(\exists p)\langle p, m \rangle \in H_a \iff m \in H_{f(\pi(a))},$$

so we let $h(a) = f(\pi(a))$. □

Theorem II.10.9. *Let Γ be a Π^0_k (resp. Σ^0_k) inductive operator, let λ be a limit ordinal and let $n < \omega$. Then Γ^n is Π^0_{kn} (resp. Σ^0_{kn}), Γ^λ is $\Sigma^0_{\lambda+1}$ and $\Gamma^{\lambda+n}$ is $\Pi^0_{\lambda+kn+1}$ (resp. $\Sigma^0_{\lambda+kn+1}$).*

Proof. We give the proof for a Π^0_n operator and leave the other case to the reader. First we show that there is a primitive recursive function g such that $\mathbb{N} \setminus \Gamma(H_e) = \mathbb{N} \setminus H_{g(e)}$ and furthermore if $e \in H^\alpha$, then $g(e) \in H^{\alpha+n}$. We give the proof for $k = 1$ and leave the general result as an exercise. Let R be a computable relation such that

$$i \in \Gamma(A) \iff (\forall j) R(i, A\lceil j).$$

Then

$$i \notin \Gamma(H_e) \iff (\exists j) \neg R(i, (\mathbb{N} \setminus H_e)\lceil j).$$

Now

$$R(i, (\mathbb{N} \setminus H_e)\lceil j) \iff (\exists \sigma \in \{0,1\}^j)[R(i,\sigma) \;\&\; (\forall t < j)(\sigma(t) = 0 \iff t \in H_e)].$$

Since the last clause makes both positive and negative reference to H_e, it follows that we can define primitive recursive functions f_p and f_n such that

$$R(i, (\mathbb{N} \setminus H_e)\lceil j) \iff (i \in H_{f_p(e,j)} \;\wedge i \notin H_{f_n(e,j)}).$$

Then

$$i \notin \Gamma(H_e) \iff (\exists j)(i \notin H_{\phi_{f_p(e)}(j)}) \vee (\exists j)(i \in H_{\phi_{f_n(e)}(j)}).$$

It follows that

$$\mathbb{N} \setminus \Gamma(H_e) = H_{f_p(e)} \cup H_{f(f_n(e))},$$

where f is the function from Lemma II.10.7. Now the set $H_{g(e)} = H_{f_p(e)} \cup H_{f(f_n(e))}$ is $\Sigma^0_{\alpha+1}$ by Lemma II.10.7 and thus $\Gamma(H_e)$ is $\Pi^0_{\alpha+1}$ as desired.

Now fix $c \in W$ with $\|c\| = \alpha$ and use the Recursion Theorem to define a primitive recursive function h such that

$$\Gamma^{\|i\|_c} = \mathbb{N} \setminus H_{h(i)}.$$

(1) If $\|i\|_c = 0$, then $H_{h(i)} = \mathbb{N}$;

(2) If $\|j\|_c = \|i\|_c + 1$, then $H_{h(j)} = H_{g(h(i)}$;

(3) If $\|j\|_c$ is a limit, then $H_{h(j)} = \bigcap\{H_{h(i)} : i <_c j\}$.

It follows by induction that if n is finite and λ is a limit, then Γ^n is Π^0_n, that Γ^λ is $\Sigma^0_{\lambda+1}$ and that $\Gamma^{\lambda+n}$ is $\Pi^0_{\lambda+n+1}$. □

The next result gives a uniform inductive definition of the hypararithmetic sets and can be used to define the transfinite jumps $\mathbf{0}^\alpha$.

Theorem II.10.10. *There is a Π^0_1 inductive definition Γ such that, for all a and m, $m \notin H_a \iff \langle 4, a, m\rangle \in Cl(\Gamma)$ and furthermore, if $a \in H^\alpha$, then $m \notin H_a \iff \langle 4, a, m\rangle \in \Gamma^\alpha$.*

Proof. There are several clauses in the definition. We assume that ϕ_0 is the empty function and omit the inclusive part of each clause (that i must be in $\Gamma(A)$ if it is in A.)

(1) $\langle 1,a,m\rangle \in \Gamma(A) \iff m \notin W_a$.

(2) $\langle 2,a\rangle \in \Gamma(A) \iff \langle 1,0,0\rangle \in A \ \& \ (\forall m)\langle 1,a,m\rangle \notin A$.

The result of these two clauses is that ϕ_a is total if and only if $\langle 2,a\rangle \in Cl(\Gamma)$, which is if and only if $\langle 2,a\rangle \in \Gamma^2$.

(3a) $\langle 3,\langle 7,a\rangle\rangle \in \Gamma(A)$

(3b) $\langle 3,b \in \Gamma(A) \iff \langle 2,b\rangle \in \Gamma(A) \ \& \ (\forall n)\langle 3,\phi_a(n)\rangle \in A$.

These two clauses ensure that, for all a and for all ordinals α,

$\langle 3,a\rangle \in \Gamma^\alpha \iff a \in H^\alpha$.

(4a) $\langle 4,\langle 7,a\rangle,m\rangle \in \Gamma(A) \iff m \notin W_a$.

(4b) $\langle 4,b,m\rangle \in \Gamma(A) \iff \langle 3,b\rangle \in \Gamma(A) \ \& \ (\forall n)\langle 4,\phi_b(n),m\rangle \notin A$.

These final two clauses complete the definition. The theorem follows by induction on α as follows.

For $\alpha = 1$, if $b = \langle 7,a\rangle \in H^1$, then $\langle 3,b\rangle \in \Gamma^1$ by clause (3a) and, by clause (4a):

$$m \notin H_b \iff m \notin W_a \iff \langle 4,b,m\rangle \in \Gamma^1.$$

For $\alpha \geq 1$ and $b \in H^{\alpha+1} - H^\alpha$, ϕ_b must be total, so we have $\langle 2,b\rangle \in \Gamma^{\alpha+1}$ and then

$$\langle m \notin H_b \iff (\forall n) m \in H_{\phi_b(n)} \iff (\forall n)\langle 4,\phi_b(n),m\rangle \in \Gamma^\alpha \iff \langle 4,b,m\rangle \in \Gamma^{\alpha+1}.$$

□

This theorem has two important corollaries. Note that we have already defined $\mathbf{0}^{(n)}$ for finite n.

Definition II.10.11. *For any computable ordinal $\alpha \geq \omega$, let*

$$\mathbf{0}^{(\alpha)} = \{\langle a,m\rangle : a \in H^\alpha \ \& \ m \in H_a\}.$$

Theorem II.10.12. *For each computable ordinal $\alpha \geq \omega$,*

1. *$\mathbf{0}^{(\alpha+1)}$ is $\Sigma^0_{\alpha+1}$ complete, and*
2. *any set A is $\Sigma^0_{\alpha+1}$ if and onlyl if it is Σ^0_1 in $\mathbf{0}^{(\alpha)}$.*

Proof. (1) It follows from Theorems II.10.9 and II.10.10 that $\mathbf{0}^{(\alpha+1)}$ is $\Sigma^0_{\alpha+1}$. The completeness is immediate from Theorem II.10.10.

(2) is left as an exercise. □

The Σ^0_α and Π^0_α sets may be characterized in terms of inductive definability. The next result follows directly from Theorems II.10.9 and II.10.10.

Theorem II.10.13. *For any $A \subseteq \mathbb{N}$ and any computable ordinal α, A is Π^0_α if and only if A is m-reducible to Γ^α for some Π^0_1 inductive definition Γ.* $\square$

Finally, we can characterize the $\Sigma^0_{\alpha+1}$ sets as relative c. e. over the jumps.

Theorem II.10.14. *For any computable ordinal α and any $A \subseteq \mathbb{N}$, A is $\Sigma^0_{\alpha+1}$ if and only if A is c. e. in $\mathbf{0}^{(\alpha)}$ and A is $\Delta^0_{\alpha+1}$ if and only if A is computable in $\mathbf{0}^{(\alpha)}$.*

Proof. Let $B = \mathbf{0}^{(\alpha)} = H_b$ for some $b \in H^\alpha$. Suppose first that A is $\Sigma^0_{\alpha+1}$. Then $A = H_a$ for some $a \in H^{\alpha+1}$. Thus we have

$$m \in A \iff (\exists n) m \in H_{\phi_a(n)} \iff (\exists n)\langle \phi_a(n), m\rangle \in B.$$

For the other direction, suppose that A is c. e. in B. Then for some e, we have

$$m \in A \iff \phi_e(m, B)\downarrow \iff (\exists t)\phi_e(m, B\lceil t)\downarrow \iff (\exists \sigma)[\sigma \prec B \ \&\ \phi_e(m,\sigma)\downarrow].$$

By Lemma II.10.7, it suffice to show that $\sigma \prec B$ is $\Sigma^0_{\alpha+1}$. But we have

$$\sigma \prec B \iff (\forall i < |\sigma|)(\sigma(i) = 0 \to i \in B) \ \&\ (\forall i < |\sigma|)(\sigma(i) = 1 \to i \notin B).$$

Now "$i \in B$" is Σ^0_α and therefore $\Sigma^0_{\alpha+1}$ and $i \notin B$ is clearly $\Sigma^0_{\alpha+1}$, so the result follows, again by Lemma II.10.7. $\square$

Monotone inductive definitions are frequently used and there is a finer result for the complexity of the levels.

Theorem II.10.15. *For any ordinal α, any $n \in \mathbb{N}$ and any monotone Π^0_1 inductive operator Γ, $\Gamma^{\omega\cdot\alpha}$ is $\Sigma^0_{2\alpha}$ and $\Gamma^{\omega\cdot\alpha+n+1}$ is $\Pi^0_{2\alpha+1}$.*

Proof. The proof is similar to that of Theorem II.10.9, with the additional idea that if A is Π^0_β, then $\Gamma(A)$ is also Π^0_β, since by monotonicity,

$$i \in \Gamma(A) \iff (\forall j)R(i, A\lceil j) \iff (\forall j)(\forall C \subseteq n)[A \subseteq C \to R(i, C)].$$

It follows that Γ^n is Π^0_1 for all n and that if Γ^λ is Σ^0_β, then $\Gamma^{\lambda+n}$ is $\Pi^0_{\beta+1}$. Making use of Lemmas II.10.6, II.10.7, II.10.8 and the Recursion Theorem, the proof follows as above. Details are left to the reader. $\square$

Theorem II.10.16. *(Souslin-Kleene) A subset of $\mathbb{N}^k \times (\mathbb{N}^\mathbb{N})^l$ is Δ^1_1 if and only if it is hyperarithmetical.*

Proof. The direction $(\longleftarrow)$ is a routine generalization of Lemma II.10.4. For the other direction, it suffices by Lemma II.10.6 and Theorem II.8.17 to show that W_α is hyperarithmetical for any computable ordinal α. Recall the Π^0_1 monotone inductive definition Γ of W given in Example II.9.3 such that such that $W_\alpha = \Gamma^\alpha$, It now follows from Theorem II.10.9 each W_α is hyperarithmetic. $\square$

The next application of inductive definability is part of a theorem of Chen [48].

Theorem II.10.17. *Let $\alpha > 1$ be a computable ordinal and let $n \geq 1$ be a natural number. Then $PW_{\omega \cdot \alpha}$ is $\Sigma^0_{2\alpha}$ complete and $PW_{\omega \cdot \alpha + n}$ is $\Pi^0_{2\alpha+1}$ complete.*

Proof. We will just demonstrate the upper bound on the complexity. A partial computable function ϕ_c represents the pre-ordering R_c if it is the characteristic function, that is

$$R_c(i,j) \iff \phi_c(\langle i,j \rangle) = 1$$

and

$$\neg R_c(i,j) \iff \phi_c(\langle i,j \rangle) = 0.$$

The restriction of R_c to elements below n may be given by

$$\phi_{h(c,n)}(i,j) = \phi_c(i,j) \cdot (1 - \phi_c(n,j)).$$

Note that if ϕ_c is total, then $\phi_{h(c,n)}$ is total for all n.

There is a natural Π^0_1 monotone inductive definition of PW given by

$$c \in \Gamma(A) \iff (\forall n) h(c,n) \in A.$$

Then it is easy to see that for a total function ϕ_c which represents a pre-linear-ordering,

$$c \in PW_\beta \iff c \in Tot \ \& \ c \in \Gamma^\beta.$$

The condition that ϕ_c is total is Π^0_2 and the condition that ϕ_c is the characteristic function of a pre-linear-ordering is Π^0_1. Thus the upper bound on the complexity follows from Theorem II.10.15.

The proof of the other direction is omitted. □

Theorem II.10.18. *Let T be a computable tree and define a prewellordering R on T so that $\|\sigma\|_R = ht_T(\sigma)$ (where $ht(\sigma) = \infty$ if $\sigma \notin Ext(T)$.) Then for any ordinal α, and any $n \in \mathbb{N}$, $\{\sigma \in T : ht_T(\sigma) < \omega \cdot \alpha\}$ is $\Sigma^0_{2\alpha}$ and $\Gamma^{\omega\cdot\alpha+n+1}$ is $\Pi^0_{2\alpha+1}$.*

Proof. The proof is left as an exercise. □

Computable trees with a unique infinite branch were studied by Clote [53]. It is well-known that for every hyperarithmetic set A, there exists a computable tree with a unique infinite branch x such that A is Turing reducible to x. We will prove this below in Chapter V.

Theorem II.10.19. *([53]) Let T be a computable tree T with a unique infinite branch x. Then for any $\sigma \in T - Ext(T)$, $ht_T(\sigma) < \omega_1$. If $ht_T(\sigma) < \omega \cdot \alpha$ for all $\sigma \notin Ext(T)$, then x is Turing reducible to a $\Sigma^0_{2\alpha}$ set.*

Proof. The first part follows from Lemma II.8.7. For the second part, note that the set $A = \{\sigma \in T : ht_T(\sigma) < \omega \cdot \alpha\}$ is $\Sigma^0_{2\alpha}$ by Lemma II.10.18. Then x may be computed recursively from A by

$$x(n+1) = (\text{least } i)[(x(0), \ldots, x(n), i) \notin A].$$

□

Exercises

II.10.1. Show that for finite n, the definition of the hyperarithmetical sets agrees with the previous definition of the arithmetical sets.

II.10.2. Give an alternate proof of Lemma II.10.4 using the Prewellordering Theorem II.9.11.

II.10.3. Give the following improvement of Theorem II.10.9. Suppose that Γ is a Π^0_k (respectively, Σ^0_k) inductive operator and that Γ^1 is Δ^0_k for some $m < k$. Show that for each n, Γ^{n+1} is Π^0_{kn+m} (resp. Σ^0_{kn+m}).

II.10.4. For computable ordinals $\alpha < \beta$, $\Sigma^0_\alpha \cup \Pi^0_\alpha \subset \Delta^0_\beta$.

II.10.5. Show that a set A is $\Sigma^0_{\alpha+1}$ if and onlyl if it is Σ^0_1 in $\mathbf{0}^{(\alpha)}$.

II.10.6. Give the details in the proof of Theorem II.10.15.

II.10.7. Let T be a computable tree and define a prewellordering R on T so that $\|\sigma\|_R = ht_T(\sigma)$ (where $ht(\sigma) = \infty$ if $\sigma \notin Ext(T)$.) Show that for any ordinal α, and any $n \in \mathbb{N}$, $\{\sigma \in T : ht_T(\sigma) < \omega \cdot \alpha\}$ is $\Sigma^0_{2\alpha}$ and $\Gamma^{\omega\cdot\alpha+n+1}$ is $\Pi^0_{2\alpha+1}$.

II.10.8. Show that P_H is Π^1_1.

II.10.9. Show that there can be no *universal* hyperarithmetical set (so in particular P_H is not hyperarithmetical.)

II.10.10. For each computable ordinal α, both the Σ^0_α and the Π^0_α relations are closed under $\leq_m$ reducibility.

II.10.11. The definition of the hyperarithmetical hierarchy is easily extended to subsets of $\mathbb{N}^\mathbb{N}$ and in general to relations $R \subseteq \mathbb{N}^k \times (\mathbb{N}^\mathbb{N})^\ell$. Give the details.

Chapter III

Fundamentals of Π^0_1 Classes

This chapter contains the formal definition of a Π^0_1 class as well as the set of infinite paths through a computable tree, well as some equivalent formulations. In particular, the notation "Π^0_1" indicates that a Π^0_1 class may be represented in arithmetic by a formula having one universal quantifier, ranging over natural numbers. We will explain the connection between these notions. Some notions of boundedness for trees are examined, including highly computable and finite-branching trees. This leads to computably bounded and bounded Π^0_1 classes. Decidable Π^0_1 classes are defined, corresponding to computable trees with no dead ends. Products and disjoint unions of trees and classes are studied. The notion of compactness is examined together with König's Lemma. The family of Π^0_1 classes is shown to have the dual reduction and separation properties. Strong Π^0_n classes are defined. Computable and continuous functions on $\mathbb{N}^{\mathbb{N}}$ and $\{0,1\}^{\mathbb{N}}$ are studied in connection with Π^0_1 classes. Classes of separating sets for a pair of disjoint c.e. sets are studied and in particular diagonally non-computable sets. The connection between retraceable and hyperimmune sets and the Π^0_1 class of initial subsets of a co-c.e. set is given. Several notions of reducibility between various classes are examined. For example, every c. b. class is computably homeomorphic to a subclass of $\{0,1\}^{\mathbb{N}}$, every bounded Π^0_1 class is computably homeomorphic to a strong Π^0_2 class of sets and every Π^0_2 class can be put in one-to-one degree-preserving correspondence with a Π^0_1 class.

We also introduce the applications of Π^0_1 classes by considering the representation of logical theories.

III.1 Computable trees and notions of boundedness

Recall that a tree T is a subset of $\mathbb{N}^*$ which is closed under initial segments. Such a tree is said to be *ω-branching*, since each node has potentially a countably infinite number of immediate successors. We identify an element σ of $\mathbb{N}^*$ with its code $\langle\sigma\rangle \in \mathbb{N}$ and say that T is computable if the set of codes $\langle\sigma\rangle$ such that $\sigma \in T$ is a computable set.

Definition III.1.1. *(i) For a given function $g : \mathbb{N}^* \to \mathbb{N}$, a tree $T \subseteq \mathbb{N}^*$ is said to be g-bounded* if for every $\sigma \in T$ and every $i \in \omega$, if $\sigma^\frown i \in T$, then $i < g(\sigma)$.

(ii) T is *finite branching* if each node σ of a tree T has finitely many immediate successors $\sigma^\frown i$.

There are other equivalent formulations.

Lemma III.1.2. *For any tree $T \subseteq \mathbb{N}^*$, the following are equivalent:*

1. *T is finite branching;*
2. *T is g-bounded for some g.*
3. *There is a function $f : \mathbb{N}^* \to \mathbb{N}^*$ such that, for each $\sigma \in T$, $f(\sigma) = (i_1, \ldots, i_k)$, where $i_1 < \cdots < i_k$ enumerates $\{i : \sigma^\frown i \in T\}$;*
4. *There is a function $f' : \mathbb{N}^* \to \mathbb{N}$ such that, for each $\sigma \in T$, σ has at most $f'(\sigma)$ immediate successors;*
5. *There is a function h such that $\sigma(i) < h(i)$ for all $\sigma \in T$ and all $i < |\sigma|$.* □

The proof of this lemma is left as an exercise.

Definition III.1.3. *(i) A tree T is* computably bounded (c. b.) *if it is g-bounded for some computable function g.*

(ii) A computable tree T is said to be highly computable *if it is also computably bounded.*

(iii) T is highly computable in z *if it is computable in z and also g-bounded by some function g computable in z.*

Lemma III.1.4. *For any computable tree $T \subseteq \mathbb{N}^*$, the following are equivalent:*

(a) T is highly computable;

(b) There is a computable function h such that $\sigma(i) < h(i)$ for all $\sigma \in T$ and all $i < |\sigma|$.

(c) There is a computable function $f : \mathbb{N}^* \to \mathbb{N}^*$ such that, for each $\sigma \in T$, $f(\sigma) = (i_1, \ldots, i_k)$, where $i_1 < \cdots < i_k$ enumerates $\{i : \sigma^\frown i \in T\}$.

Proof. (a) → (b): Given the function g, let $h(0) = g(\emptyset)$ and recursively compute h by $h(n+1) = max\{g(\sigma) : \sigma \in \{0, 1, \ldots, h(n)\}^n \cap T\}$.

(b) → (c): Given the function h, the sequence $i_1 < \cdots < i_k$ can be computed from σ by testing in turn whether $\sigma^\frown i \in T$ for $i < h(|\sigma| + 1)$.

(c) → (a): For a given σ, use the function f to compute $i_1 < \cdots < i_k$ as indicated and then $g(\sigma) = i_k + 1$ will be an upper bound for $\{i : \sigma^\frown i \in T\}$. □

Note that the argument that (c) implies (a) does not use the computability of T but the other two arguments do. We leave it as an exercise to define (noncomputable) trees which possess computable bounding functions of type (a) but do not possess bounding functions of type (b) and similarly for (b) and (c).

Observe that we have omitted here the condition from Lemma III.1.2 that there is a computable upper bound on the number of successors of σ. It is an exercise to show that any highly computable tree possesses such a function, but there exists a computable tree which is not highly computable but possesses such a bounding function.

Example III.1.5. *Here is an example of a computable tree which is bounded but not computably bounded. Put $0^e \frown s+1 \in T$ if and only if $\phi_e(e)$ converges at stage s. Then each node $\sigma \in T$ has at most two extensions, so that T is finite branching. Now suppose by way of contradiction that T were computably bounded by the function g. Then $\phi_e(e)\downarrow \iff \phi_{e,g(0^e)}(e)\downarrow$, which would make $\{e : \phi_e(e)\downarrow\}$ a computable set.*

There is a further notion of boundedness.

Definition III.1.6. 1. *T is* almost bounded *by g if there is some $k \in \mathbb{N}$ such that for all σ with $|\sigma| > k$ and for all i, if $\sigma\frown i \in T$, then $i < g(\sigma)$.*

2. *T is* almost bounded (a. b.) *if it is almost bounded by some g.*

3. *T is* almost computably bounded (a. c. b.) *if it is almost bounded by some computable function g.*

Note that these notions are not equivalent to the existence of a (computable) function h and a finite k such that $\sigma(i) < h(i)$ for all $\sigma \in T$ with $|\sigma| > k$ and all $i < |\sigma|$.

Example III.1.7. *Let $T = \{n^k : n, k \in \omega\}$. Then T is a. c. b. but has, for each k and n, strings $\sigma = n^{k+1} \in T$ with $\sigma(k) = n$. It is clear that T is not finite branching.*

Two trees may be joined together in various ways.

Definition III.1.8. *For two trees $S, T \subset \mathbb{N}^*$*

1. *$S \oplus T = \{0\frown\sigma : \sigma \in S\} \cup \{1\frown\tau : \tau \in T\}$.*

2. *For strings σ and τ with $|\sigma| = |\tau| = n$, $\sigma \oplus \tau = (\sigma(0), \tau(0), \ldots, \sigma(n-1), \tau(n-1))$; if $|\sigma| = n+1$ and $|\tau| = n$, then $\sigma \oplus \tau = (\sigma(0), \tau(0), \ldots, \sigma(n-1), \tau(n-1), \sigma(n))$.*

3. *$S \otimes T = \{\sigma \oplus \tau : \sigma \in S \;\&\; \tau \in T\}$.*

Clearly $S \oplus T$ is bounded if and only if both S and T are bounded and similarly for the other notions of boundedness.

Exercises

III.1.1. Prove Lemma III.1.2.

III.1.2. Find a (non-computable) tree T with a computable function h such that $\sigma(i) < h(i)$ for all $\sigma \in T$ and all $i < |\sigma|$, such that there is no computable function f which enumerates the immediate successors of $\sigma \in T$.

III.1.3. Given a highly computable tree T, find a computable function f such that, for each $\sigma \in T$, σ has $\leq f(\sigma)$ immediate successors in T.

III.1.4. Show that $S \oplus T$ and $S \otimes T$ are trees.

III.2 Definition and basic properties of Π^0_1 Classes

In this section, we examine Π^0_1 classes with various boundedness conditions. We begin with some general facts.

Lemma III.2.1. *A subset K of $\mathbb{N}^{\mathbb{N}}$ is closed if and only if $K = [T]$ for some tree T.*

Proof. Suppose first that K is closed and let $T = \{\sigma \in \mathbb{N}^* : K \cap I[\sigma] \neq \emptyset\}$. We will verify that $K = [T]$. If $x \in K$, then for any n, $x \in K \cap I[x \upharpoonright n]$, so that $x \upharpoonright n \in T$. It follows that $x \in [T]$. Conversely, suppose that $x \notin K$. Since K is closed, there must be some basic interval $I[x \upharpoonright n]$ such that $K \cap I[x \upharpoonright n] = \emptyset$. But then $x \upharpoonright n \notin T$ and hence $x \notin [T]$. □

Definition III.2.2. *1. For any closed set $P \subseteq \mathbb{N}^{\mathbb{N}}$, let T_P denote the tree $\{\sigma \in \mathbb{N}^* : P \cap I[\sigma] \neq \emptyset\}$.*

2. For any tree T, an infinite path *through T is a sequence $(x(0), x(1), \dots)$ such that $x \upharpoonright n \in T$ for all n. Let $[T]$ be the set of infinite paths through T.*

3. A subset P of $\mathbb{N}^{\mathbb{N}}$ is a Π^0_1 class *(P is* effectively closed *if $P = [T]$ for some computable tree T.*

4. A subset P of $\mathbb{N}^{\mathbb{N}}$ is a decidable *Π^0_1 class if T_P is a computable set.*

It is important to note that for a Π^0_1 class P, the set T_P need not be computable.

Example III.2.3. *Let A be an arbitrary c.e. set and let $P = \{0^n 1^\omega : n \notin A\}$. Then P is a Π^0_1 class but T_P is not computable. (Details left as an exercise.)*

The notions of boundedness for trees from the previous section carry over to notions of boundedness for closed sets.

Definition III.2.4. *A subset K of $\mathbb{N}^{\mathbb{N}}$ is* (topologically) bounded *if there is a function h such that, for all $x \in K$ and all n, $x(n) \leq h(n)$.*

In the study of Π^0_1 classes, the term "bounded" has the effective version given below, so that we use the modifier "topologically" to distinguish the two notions.

Definition III.2.5. *1. A Π^0_1 class P is* bounded *if there is a computable tree T such that $P = [T]$ and a function h (not necessarily computable) such that $\sigma(n) \leq h(n)$ for all $\sigma \in T$.*

2. A Π^0_1 class P is computably bounded (c. b.) *if $P = [T]$ for some highly computable tree T.*

3. A Π^0_1 class P is almost bounded *if $P = [T]$ for some a.b. tree T.*

4. A Π^0_1 class P is almost computably bounded (a. c. b.) *if $P = [T]$ for some a. c. b. tree T.*

In the next chapter, we will show that for any hyperarithmetical real $x \in \mathbb{N}^{\mathbb{N}}$, there exists $y \equiv_T x$ such that $\{y\}$ is a Π^0_1 class. Here we give the special case when x is a Σ^0_2 set.

Example III.2.6. *Here is an example of a Π^0_1 class P which is topologically bounded, but not bounded. That is, $P = \{g\}$ for a particular function g and hence P is bounded by g itself. At the same time, $P = [T]$ for some computable tree T. However, the tree T is not itself bounded by g, or even finite branching.*

Let A be a Σ^0_2 set which is not Π^0_2. Let R be a computable relation so that

$$e \in A \iff (\exists m)(\forall n) R(e, m, n).$$

Define the (non-computable) function f in two cases as follows.

(Case I): If $e \in A$, then $f(e, 0) = 1$, $f(e, 1)$ is the least m such that $(\forall n) R(e, m, n)$ and $f(e, 2) = \langle n_0, \ldots, n_{f(e,1)-1} \rangle$ where for each i, n_i is the least n such that $\neg R(e, i, n)$.

(Case II): If $e \notin A$, then $f(e, 0) = 0$ and $f(e, m+1)$ is the least n such that $\neg R(e, m, n)$.

Then define g by $g(2^e(2m+1)) = f(e, m)$. Observe that for each e, the values of $g(2^e(2m+1)$ tell us whether $e \in A$ and also verify the answer.

We claim that $\{g\}$ is a Π^0_1 class. That is, $\{g\} = [T]$ for the computable tree T defined as follows.

$\sigma \in T$ if and only if, for all e with $2^e < |\sigma|$, one of the following.

(0) $\sigma(2^e) = 0$ and, for all i with $2^e(2i+1) < |\sigma|$, $\neg R(e, i, \sigma(2^e(2i+1)))$, and for all $j < \sigma(2^e(2i+1))$, $R(e, i, j)$.

(1) $\sigma(2^e) = 1$ and, for all $n < |\sigma|$, $R(e, \sigma(3 \cdot 2^e), n)$. For all $i > 2$, $\sigma(2^e(2i+1)) = 0$. Finally, $\sigma(5 \cdot 2^e = \langle n_0, n_1, \ldots, n_{\sigma(3 \cdot 2^e)-1} \rangle$ where for all $i < \sigma(3 \cdot 2^e)$, n_i is the least n such that $\neg R(e, i, n)$.

Clearly $g \in [T]$. Now suppose there were some other function $h \in [T]$. Fix e and consider two cases as above.

(Case I): $h(2^e) = 0$. Then for each i, $\neg R(e, i, h(2^e(2i+1))$ and hence $e \notin A$. It follows that, for each m, $h(2^e(2m+1))$ is the least n such that $\neg R(e, m, n)$ and hence $h(2^e(2m+1)) = f(2^e(2m+1)$ for all m.

(Case II): $h(2^e) = 1$. Then for all n, $R(e, h(3 \cdot 2^e), n)$ and hence $e \in A$. Furthermore, for each $i < h(3 \cdot 2^e)$, $h(5 \cdot 2^e)$ witnesses that $\neg(\forall n)R(e, i, n)$, so that $h(3 \cdot 2^e)$ is the least m such that $(\forall n)R(e, m, n)$; that is, $h(3 \cdot 2^e) = f(3 \cdot 2^e)$. It follows that $h(2^e(2i+1) = h(3 \cdot 2^e) = f(3 \cdot 2^e) = f(2^e(2i+1)$ for all $i > 2$. Finally, $h(5 \cdot 2^3)$ must code the sequence of least witnesses n such that $\neg R(e, i, n)$ for each $i < h(3 \cdot 2^e)$ and hence $h(5 \cdot 2^e) = f(5 \cdot 2^e)$ as well.

It follows that $h = f$ and hence $[T] = \{f\}$ as desired.

Now we claim also that this class does not have a Δ^0_2 bounding function h. Suppose by way of contradiction that there were such a function h. Then

$$e \notin A \iff (\forall m < h(2^e 3))(\exists n)\neg R(e, m, n).$$

This would give a Π^0_2 definition of A. To see this, let

$$\phi(e) = (least\ n)(\forall m < h(2^e 3))(\exists n' < n)\neg R(e, m, n').$$

Then ϕ is computable in $\mathbf{0}'$ and $\mathbb{N} \setminus A = Dom(\phi)$, so that A is a Π^0_2 set.

Now the Π^0_1 class $\{g\}$ is certainly topologically bounded, but it follows from Lemma III.2.7 that it is not bounded.

Lemma III.2.7. *(a) A closed subset K of $\mathbb{N}^{\mathbb{N}}$ is (topologically) bounded if and only if there is a finite-branching tree T such that $K = [T]$;*

(b) A Π^0_1 class P is bounded if and only if there is a finite-branching computable tree T such that $P = [T]$. Furthermore, the bounding function may always be taken to be computable in $\mathbf{0}'$.

(c) A decidable Π^0_1 class P is effectively bounded if and only if it is bounded.

Proof. We leave the proof of part (a) to the reader. Let T be a computable tree such that $K = [T]$. Suppose first that K is bounded and let h be a function such that $\sigma(i) \leq h(i)$ for all $\sigma \in T$ and all $i < |\sigma|$. It is immediate that T is finite-branching, since for any $\sigma \in T$, if $\sigma^\frown j \in T$, then $j \leq h(|\sigma|)$. On the other hand, if T is finite-branching, then it is easy to see by induction that $T \cap \mathbb{N}^k$ is finite for all k and hence we may define the bounding function h by

$$h(k) = max\{i : (\exists \sigma \in T \cap \mathbb{N}^k)\sigma^\frown i \in T\}.$$

It is clear that h is computable in $\mathbf{0}'$.

(c) Let $P = [T]$ where T is computable and has no dead ends. If T is finite-branching, then P is bounded by (a). Now suppose that P is bounded by a function h. Since T has no dead ends, it follows that T is also bounded by h. Thus T must be finite-branching. □

Proposition III.2.8. *A Π^0_1 class P is computably bounded if and only if there is a computable function h such that $x(n) \leq h(n)$ for all $x \in P$.*

Proof. Let $P = [T]$ for the computable tree T. If P is computably bounded, then there is a computable function h such that $\sigma(n) \leq h(n)$ for all $\sigma \in T$ and all $n < |\sigma|$ and it follows that $x(n) \leq h(n)$ for all $x \in P$. For the other direction, let h be given as described. Then we can define a tree $S \subseteq T$ by having

$$\sigma \in S \iff \sigma \in T \ \& \ (\forall n < |\sigma|)\sigma(n) \leq h(n).$$

It is clear that $[S] = [T] = P$ so that P is computably bounded. □

Definition III.2.9. *(i) σ is an* extendible *node of T if $I(\sigma) \cap [T] \neq \emptyset$, that is, if σ has an infinite extension which belongs to $[T]$; $Ext(T)$ is the set of extendible nodes of T.*

(ii) A node σ such that $\sigma \notin T$, but all $\tau \prec \sigma$ are in T, is a dead end *of T.*

Observe that T_P has no dead ends and is the unique tree without dead ends such that $P = [T]$. The following lemma gives an alternate definition of the notion of a decidable Π^0_1 class. The proof is left as an exercise.

Lemma III.2.10. *For any Π^0_1 class P, the following are equivalent:*

(a) P is decidable;

(b) There is a computable tree T with $P = [T]$ such that $Ext(T)$ is computable.

(c) There is a computable tree T with no dead ends such that $P = [T]$. □

It is a fundamental property of the real line that a subset is compact if and only if it is closed and bounded.

Theorem III.2.11. *(a) A subset K of $\mathbb{N}^{\mathbb{N}}$ is compact if and only if it is closed and (topologically) bounded.*

(b) A Π^0_1 class P is bounded if and only if there exists a computable tree T with $P = [T]$ and a function h such that $\sigma(n) \leq h(n)$ for all $\sigma \in T$ and all $n < |\sigma|$. Furthermore, h may be taken to be computable in $\mathbf{0}'$.

Proof. (a) Suppose first that K is compact. Then K is certainly closed. For each n, $K \subset \bigcup_i \{x : x(n) = i\}$ and it follows from compactness that there exists some i_n such that $K \subseteq \{x : x(n) \leq i_n\}$. Then the function $h(n) = i_n$ satisfies the condition above. Suppose next that K is closed and bounded and let h be a bounding function for K. Then $K \subseteq \prod_{n \in \omega}\{0, 1, \ldots, h(n)\}$ and is therefore compact since it is a closed subset of a compact space.

(b) If $P = \emptyset$, then this is obvious. Thus we let P be a nonempty Π^0_1 class and let T be a computable tree such that $P = [T]$. Suppose first that T is finitely branching. Then we may define the bounding function h by letting $h(n)$ be the maximum of $\{\sigma(m) : \sigma \in T,\ m\||\sigma| \leq n\}$. It is clear that h is computable in $\mathbf{0}'$. Conversely, suppose that $P = [T]$ and that h is any bounding function such that $\sigma(n) \leq h(n)$ for all $\sigma \in T$. It is immediate that T must be finitely branching. □

A crucial result for bounded classes is König's Lemma, which follows from the compactness of bounded classes.

Lemma III.2.12 (König's Lemma)**.** *Any infinite, finite-branching tree has an infinite path.*

Proof. Let T be an infinite, finite-branching tree and let $P = [T]$. An infinite path x through T is defined as follows. Let $x(0)$ be the least i such that T contains infinitely many extensions of (i) and for each n, similarly let $x(n+1)$ be the least i such that T contains infinitely many extensions of $(x(0), x(1), \ldots, x(n), i)$. Since T is finite-branching, it follows by induction that such an i always exists. $\square$

Notice that the path defined in the proof of König's Lemma is not necessarily computable, despite the "recursive" definition. The complexity of this path will be considered further below in Chapter IV.

We can use König's Lemma to determine the complexity of $Ext(T)$ for a computable tree T.

Theorem III.2.13. *Let T be a computable tree in $\mathbb{N}^*$.*

(a) T, $Ext(T)$ is Σ^1_1.

(b) For a finite-branching tree T, $Ext(T)$ is Π^0_2.

(c) For a highly computable tree T, $Ext(T)$ is Π^0_1.

Proof. In general, we have

$$\sigma \in Ext(T) \iff (\exists x)[\sigma \prec x \ \&\ (\forall n) x \upharpoonright n \in T].$$

For a finite branching tree, König's Lemma implies that

$$\sigma \in Ext(T) \iff (\forall n)(\exists \tau \in \mathbb{N}^*)[|\tau| = n \ \&\ \sigma^\frown\tau \in T].$$

Part (c) is left as an exercise. $\square$

We now present the fundamental basis result is due to Kleene.

Theorem III.2.14 (Kleene)**.** *For any tree T such that the Π^0_1 class $P = [T]$ is nonempty, P contains a member which is computable in $Ext(T)$.*

Proof. The infinite path x through T can be defined recursively by letting $x(0)$ be the least n such that $(n) \in Ext(T)$ and, for each k, letting $x(k+1)$ be the least n such that $(x(0), \ldots, x(k), n) \in Ext(T)$. $\square$

Combining this with Theorem III.2.13, we get the following:

Theorem III.2.15. *For any nonempty Π^0_1 class $P \subseteq \mathbb{N}^{\mathbb{N}}$:*

(a) P has a member computable in some Σ^1_1 set;

(b) if P is bounded, then P has a member of Σ^0_2 degree (hence computable in $\mathbf{0}''$);

(c) if P is c. b., then P has a member of c.e. degree (hence computable in $\mathbf{0}'$);

(d) if P is decidable, then P has a computable member. □

Proof. Part (a) and the parenthetical consequences in parts (b) and (c) are immediate from Theorems III.2.13 and V.2.3. The existence of a member of c.e. Σ^0_2 degree in (b) and of a member of c.e. degree in (c) are left as an exercise. □

The following corollary is very useful.

Corollary III.2.16. *Any isolated element of a computably bounded Π^0_1 class is computable.*

Proof. Without loss of generality, Let $x \in \mathbb{N}^{\mathbb{N}}$ and let $P = \{x\}$ be a c.b. Π^0_1 class. Let $P = [T]$ where T is computable and computably bounded and let f be given so that $\sigma(m) < f(n)$ for all $\sigma \in \mathbb{N}^n$ and all $m < n$. Then $Ext(T)$ is Π^0_1 by Theorem III.2.13. But for any $\sigma \in Ext(T)$, $\sigma = x\lceil|\sigma|$ and is the unique member of $Ext(T)$ with length $|\sigma|$. Thus we also have

$$\sigma \in Ext(T) \iff (\forall \tau \in \{0, 1, \ldots, f(n)\}^{|}\sigma|)[\tau \in Ext(T) \implies \tau = \sigma]/$$

This shows that $Ext(T)$ is also c. e. and is therefore computable, so that x is computable by Theorem III.2.15. □

Part (a) is the Kleene basis theorem [108] and part (c) is the Kreisel basis theorem [115].

For two Π^0_1 classes $P = [S]$ and $Q = [T]$, define the *amalgamation* of P and Q, $P \otimes Q$, by $P \otimes Q = \{x \oplus y : x \in P \ \& \ y \in Q\}$. Then it is clear that $P \otimes Q = [S \otimes T]$. More generally, define the infinite amalgamation $\otimes_i S_i$ to be those strings σ such that for each i, $(\sigma([i, 0]), \sigma([i, 1]), \ldots, \sigma([i, j])) \in S_i$, where j is the maximum such that $[i, j] < |\sigma|$. Then $[\otimes_i S_i]$ is isomorphic to the direct product $\Pi_i[S_i]$.

We also wish to consider the disjoint union. For two Π^0_1 classes $P = [S]$ and $Q = [T]$, $P \oplus Q = \{0^\frown x : x \in P\} \cup \{1^\frown y : y \in Q\}$. It is easy to see that $P \oplus Q = [S \oplus T]$.

Another consequence of compactness is the dual reduction property of Π^0_1 classes in $\{0, 1\}^{\mathbb{N}}$. Some definitions are needed.

Definition III.2.17. *A family Γ of subsets of some set X has the* Reduction Property *if, for any A and B in Γ, there exist A_1 and B_1 in Γ such that*

(i) $A_1 \subseteq A$ and $B_1 \subseteq B$;

(ii) $A_1 \cup B_1 = A \cup B$;

(iii) $A_1 \cap B_1 = \emptyset$.

A standard result from classical computability theory is the following.

Proposition III.2.18. *The family of c.e. subsets of $\mathbb{N}$ satisfy the reduction property.*

Proof. Given c.e. sets A and B, let f and g be computable functions which enumerate A and B, respectively. Then define

$$A_1 = \{f(n) : (\forall m < n) f(n) \neq g(m)\}$$

and

$$B_1 = \{g(m) : (\forall n \leq m) g(m) \neq f(n)\}.$$

That is, $i = f(n)$ is enumerated into A_1 as long as it has not already appeared in B and similarly $j = g(m)$ is enumerated into B_1 as long as it does not come into A by stage m. □

We can modify this to show that the Σ^0_1 classes in $\mathbb{N}^{\mathbb{N}}$ also have the reduction property.

Proposition III.2.19. *The family of Σ^0_1 classes in $\mathbb{N}^{\mathbb{N}}$ has the reduction property.*

Proof. Since A and B are the complements of Π^0_1 classes, it follows from Theorem III.3.2 that there exist computable functions f and g such that $A = \bigcup_n I(\sigma_{f(n)})$ and $B = \bigcup_m I(\sigma_{g(m)})$ where $\sigma_{f(n_1)}$ and $\sigma_{f(n_2)}$ are incompatible for $n_1 \neq n_2$ and similarly for g. Now define computable sequences U_n and V_n of clopen sets by

$$U_n = I(\sigma_{f(n)}) \setminus \bigcup_{m<n} I(\sigma_{g(m)})$$

and

$$V_m = I(\sigma_{g(m)}) \setminus \bigcup_{n \leq m} I(\sigma_{f(n)}).$$

Then let $A_1 = \bigcup_n U_n$ and $B_1 = \cup_m V_m$. □

Definition III.2.20. *A family Γ of subsets of some set X has the* Dual Reduction Property *if, for any P and Q in Γ, there exist P_1 and Q_1 in Γ such that*

(i) $P \subseteq P_1$ *and* $Q \subseteq Q_1$;

(ii) $P_1 \cap Q_1 = P \cap Q$;

(iii) $P_1 \cup Q_1 = X$.

This is the dual of the usual reduction property and was first studied by Herrmann. It will be important in the study of the lattice $\mathcal{E}_\Pi$ in Chapter XVII. It is easy to see that a family Γ will satisfy the reduction property if and only if the family of complements of Γ satisfies the dual reduction property. (See the exercises.)

Corollary III.2.21. *The family of Π^0_1 classes in $\{0,1\}^{\mathbb{N}}$ has the dual reduction property.* □

Definition III.2.22. *A class Γ of subsets of some set X has the* Separation Property *if for any P and Q in Γ, if $P \cap Q = \emptyset$, then there exists R such that both R and $X \setminus R$ are in Γ and $P \subset R \subset X \setminus Q$. R is said to* separate *P and Q.*

If Γ is the family of Π^0_1 classes in $\{0,1\}^{\mathbb{N}}$ (or in general, the family of closed subsets of $\{0,1\}^{\mathbb{N}}$), then R and $\{0,1\}^{\mathbb{N}} \setminus R$ are both closed if and only if R is clopen.

The following result is left as an exercise.

Proposition III.2.23. *If Γ has the dual reduction property, then it also has the separation property.* □

Corollary III.2.24. *[Separation Property] The family of Π^0_1 classes has the separation property.* □

It is well-known that the c.e. sets do not satisfy the separation property. (See section III.5 below.) This also carries over to the Σ^0_1 classes.

Proposition III.2.25. *The family of Σ^0_1 classes does not have the separation property.*

Proof. Let A and B be disjoint, computably inseparable c.e. sets and let $U = \cup_{n\in A} I(0^n 1)$ and $V = \cup_{n\in B} I(0^n 1)$. Suppose by way of contradiction that G is a clopen set such that $U \subset G$ and $V \cap G = \emptyset$. Since G is closed, $0^\omega \in G$. Since G is open, there must be some finite m such that $I(0^m) \subset G$. But then there must be some $n \in B$ with $n > m$, so that $0^n 1^\omega \in V \cap G$. □

Exercises

III.2.1. Prove Lemma III.2.7(a).

III.2.2. Show that T_P is the least tree T such that $P = [T]$, that is, the intersection of all such trees.

III.2.3. Explain why the computable tree T in Example III.2.6 cannot be finite-branching. Where does the infinite branching occur?

III.2.4. Show that P as defined in Example III.2.3 is a Π^0_1 class and that T_P is not computable.

III.2.5. Let $P = \{0,1\}^{\mathbb{N}}$. Show that there are infinitely many computable trees in $\{0,1,2\}^{\mathbb{N}}$ such that $P = [T]$ and continuum many non-computable trees T in $\{0,1,2,3\}^{\mathbb{N}}$ with $P = [T]$.

III.2.6. Prove Lemma III.2.10.

III.2.7. Show that $Ext(T)$ is a Π^0_1 set for a highly computable tree T.

III.2.8. Complete the proof of Theorem III.2.15. Hint: If T is a computably bounded tree, then the leftmost infinite path of T is a c.e. real, as defined in Section II.4.

III.2.9. Verify that $[S \oplus T] = [S] \oplus [T]$ and that $[S \otimes T] = [S] \otimes [T]$.

III.2.10. Show that Γ has the reduction property if and only if $\tilde{\Gamma} = \{X \setminus S : S \in \Gamma\}$ has the dual reduction property.

III.2.11. Prove Proposition III.2.23

III.2.12. Show that the family of all open sets in $\mathbb{N}^{\mathbb{N}}$ has the reduction property and hence the class of closed sets has both the dual reduction property and the separation property.

III.2.13. Improve Proposition III.2.25 by showing that the Σ^0_1 classes defined in the proof cannot be separated by any *decidable* Π^0_1 class.

III.3 Effectively Closed Sets in the Arithmetic Hierarchy

The following lemma makes the connection between trees and quantified relations precise.

Proposition III.3.1. *For any class $P \subset \mathbb{N}^{\mathbb{N}}$, the following are equivalent:*

(a) $P = [T]$ for some computable tree $T \subset \mathbb{N}^*$;

(b) $P = [T]$ for some primitive recursive tree T;

(c) $P = \{x : (\forall n)R(n,x)\}$, for some computable relation R;

(d) $P = [T]$ for some Π^0_1 tree $T \subset \omega^{<\omega}$;

Proof. : [(a) $\to$ (b)]: Suppose that $P = [T]$, where T is a computable tree and let ϕ_e be a total $\{0,1\}$-valued computable function such that $\sigma \in T$ if and only if $\phi_e(\sigma) = 1$. Define the primitive recursive tree S by $\tau \in S \iff (\forall n < |\tau|)\neg\phi_{e,|\tau|}(\tau\lceil n) = 0$. Clearly $T \subset S$, so that $[T] \subset [S]$. Suppose now that $x \notin [T]$. Then for some n, $x\lceil n \notin T$. Thus we have some m such that $\phi_{e,m}(x\lceil n) = 0$. Then for any $k > max\{m,n\}$, we clearly have $x\lceil k \notin S$. It follows that $x \notin [S]$.

[(b) $\to$ (c)]: Suppose that $P = [T]$, where T is a primitive recursive tree. Define the relation R by $R(n,x) \iff x\lceil n \in T$. then we have $x \in [T] \iff (\forall n)x\lceil n \in T \iff (\forall n)R(n,x)$.

[(c) $\to$ (d)]: Suppose that $x \in P \iff (\forall n)R(n,x)$ where R is a computable relation, that is, there is a computable functional $\Phi = \Phi_e$ such that $R(n,x) \iff \Phi(n,x) = 1$ and $\neg R(n,x) \iff \Phi(n,x) = 0$. By the Master Enumeration Theorem II.II.2.5, we have $\Phi(n,x) = i$ if and only if $\Phi(n,x\lceil m) = i$ for some m. Define the tree T by

$$\sigma \in T \iff (\forall n < |\sigma|)\Phi(n,\sigma)\downarrow \to \Phi(n,\sigma) = 1.$$

It is clear that $P = [T]$.

[(d) $\to$ (a)]: Suppose that the tree T is a Π^0_1 subset of $\omega^{<\omega}$, so that there is a computable relation R such that $\sigma \in T \iff (\forall n)R(n,\sigma)$. Define the computable tree $S \supset T$ by $\sigma \in S \iff (\forall m, n \leq |\sigma|)R(m, \sigma\lceil n)$. It is easily verified that $[T] = [S]$. □

Standard topology tells us that a closed set may be defined as the complement of an open set, and that an open set in $\mathbb{N}^{\mathbb{N}}$ is a countable union of intervals $I(\sigma)$. Since $\mathbb{N}^{\mathbb{N}}$ is completely disconnected, this countable union can be made disjoint. The effective version of this fact is the following. Let $\sigma_0, \sigma_1, \ldots$ be an effective enumeration of $\mathbb{N}^*$. (This can be done in order first by the sum $\sigma(0) + \sigma(1) + \cdots + \sigma(|\sigma|)$ and then lexicographically.)

Theorem III.3.2. *(a) If $P \subseteq \mathbb{N}^{\mathbb{N}}$ is a Π^0_1 class, then there is a computable set W such that $\mathbb{N}^{\mathbb{N}} \setminus P = \bigcup_{n\in W} I(\sigma_n)$. Furthermore, the set W may be chosen so that $I(\sigma_m)$ and $I(\sigma_n)$ are disjoint for $m \neq n$.*

(b) For any c.e. set $W \subseteq \mathbb{N}^*$, $\mathbb{N}^{\mathbb{N}} \setminus \bigcup_{n\in W} I(\sigma)$ is a Π^0_1 class.

Proof. (a): Let P be a Π^0_1 class and let T be a computable tree such that $P = [T]$. Let $W = \{n : \sigma_n \text{ is a dead end of } T\}$. Then in fact W is a computable set and clearly $\mathbb{N}^{\mathbb{N}} \setminus P = \bigcup_{n\in W} I(\sigma_n)$. That is, if $x \in I(\sigma_n)$ for some $n \in W$ and $k = |\sigma_n|$, then $x \upharpoonright k = \sigma_n \notin T$, so that $x \notin P$. On the other hand, if $x \notin P$, then there is a least k such that $x \upharpoonright k \notin T$. Take n so that $x \upharpoonright k = \sigma_n$. Then $n \in W$ and $x \in I(\sigma_n)$. To check the disjointness condition, suppose that σ_m and σ_n are both dead ends of T. If they were comparable, then without loss of generality, $\sigma_m \prec \sigma_n$, which contradicts the assumption that both are dead ends.

(b) Given the c.e. set $W \subseteq \mathbb{N}^*$, let $P = \mathbb{N}^{\mathbb{N}} \setminus \bigcup_{n\in W} I(\sigma) = [T]$ and define the Π^0_1 tree T by

$$\tau \in T \iff (\forall \sigma \preceq \tau)\sigma \notin W.$$

Then $\mathbb{N}^{\mathbb{N}} \setminus \bigcup_{n\in W} I(\sigma) = [T]$. It follows from Proposition III.3.1 that P is a Π^0_1 class. □

Part (d) of Proposition III.3.1 may be used to define an enumeration of the Π^0_1 classes. That is, let we could let $P_e = \mathbb{N}^{\mathbb{N}} \setminus \bigcup_{n\in W_e} I(\sigma_n)$, where $\sigma_0, \sigma_1, \ldots$ is the standard enumeration of $\{0,1\}^*$ in length-lexicographic order. By applying Proposition III.3.1 uniformly, we obtain the following.

Theorem III.3.3. *There is a uniformly primitive recursive sequence $\langle T_e\rangle_{e\in\omega}$ of trees such that $\langle [T_e]\rangle_{e\in\omega}$ enumerates the Π^0_1 classes.*

Proof. Simply let

$$\sigma \in T_e \iff (\forall n < |\sigma|)[\sigma_n \preceq \sigma \to n \notin W_{e,|\sigma|}.$$

□

The official enumeration for the Π^0_1 classes will be defined below in Chapter VI and is based directly on primitive recursive trees.

There is another notion equivalent to being a computably bounded Π^0_1 class.

Proposition III.3.4. *Let $P \subset \mathbb{N}^{\mathbb{N}}$ and h a computable function such that $x(n) < h(n)$ for all $x \in P$. Then P is a Π^0_1 class if and only if T_P is a Π^0_1 set.*

Proof. If T_P is a Π^0_1 set, then P is a Π^0_1 class by Proposition III.3.1 since $P = [T_P]$. Now suppose that h is a computable function and $x(n) < h(n)$ for all $x \in P$ and all n. Then by König's Lemma,

$$\sigma \in T_P \iff (\forall k > |\sigma|)(\exists \tau \in h(0) \times h(1) \times \dots h(k))[\tau \in T \ \&\ \sigma \prec \tau].$$

□

We will also consider in general the families of Π^0_n classes and Σ^0_n classes. Analogous to the definition of Π^0_n sets, we have the following.

Definition III.3.5. *Let R be a relation on $\mathbb{N}^k \times \mathbb{N}^{\mathbb{N}}$ and let $n > 0$ be a natural number.*

1. *R is Π^0_0 if it is computable.*
2. *R is Σ^0_{n+1} if there is a Π^0_n relation $Q \subset \mathbb{N}^{k+1} \times \mathbb{N}$ such that*
$$R(a_1, \dots, a_k, x) \iff (\exists i)Q(i, a_1, \dots, a_k, x).$$
3. *R is Π^0_{n+1} if $\mathbb{N}^k \times \mathbb{N}^{\mathbb{N}} \setminus R$ is Σ^0_{n+1}.*
4. *R is Δ^0_{n+1} if R is both Σ^0_{n+1} and Π^0_{n+1}.*

Note that a computable class in $\mathbb{N}^{\mathbb{N}}$ is both open and closed. These definitions can also be relativized to a set oracle C, but the results are not quite analogous to those for sets.

Definition III.3.6. *(i) A subset of $\mathbb{N}^{\mathbb{N}}$ is a* **strong** *Π^0_{n+1} class if $P = [T]$ for some tree T computable in $\emptyset^{(n)}$.*

(ii) A strong Π^0_{n+1} class P is highly bounded *if $P = [T]$ for some tree T computable in $\emptyset^{(n)}$ and a bounding function f also computable in $\emptyset^{(n)}$ such that $\sigma(n) \leq f(n)$ for all $\sigma \in T$.*

Proposition III.3.7. *For any class $P \subset \mathbb{N}^{\mathbb{N}}$, the following are equivalent:*

(a) $P = [T]$ for some tree T computable in $\mathbf{0}^{(n)}$.

(b) $P = [T]$ for some Π^0_{n+1} tree T.

(c) $P = [T]$ for some Σ^0_n tree T.

Furthermore, if $P \subset \{0,1\}^{\mathbb{N}}$, then $T \subset \{0,1\}^*$.

Proof. The equivalence of (a) and (b) follows from a relativization of Proposition III.3.1. Clearly (c) implies (b). It remains to be shown that (b) implies (c). Let T be a Π^0_{n+1} tree such that $P = [T]$. Then there is a Σ^0_n relation R such that

$$\sigma \in T \iff (\forall i)R(i, \sigma),$$

so that

$$x \in P \iff (\forall m)(\forall i)R(i, x \restriction m).$$

Now define the Σ^0_n tree S by

$$\sigma \in S \iff (\forall m \leq |\sigma|)(\forall i \leq |\sigma|)R(i, \sigma \restriction m).$$

It is easy to check that S is a tree and that $P = [S]$. □

Exercises

III.3.1. Complete the proof of Proposition III.3.7 by showing that S is a tree and that $P = [S]$.

III.4 Graphs of Computable Functions

In classical computability theory, computable functions and computably enumerable sets are the two primary objects of study. In this context, the two are naturally related. For any (partial) computable function, the domain, the range and the graph are all c.e. sets. Furthermore, every c.e. set is the domain of a computable function and every nonempty c.e. set is the range of a total computable function. In addition, any partial function with a c.e. graph is necessarily computable. For our purposes, computably continuous functions and Π^0_1 classes are the primary objects of study. In this section, we explore possible analogues of the classical results.

Recall from the Master Enumeration Theorem II.II.2.5 that a (partial) computable function $\Phi : \mathbb{N}^\mathbb{N} \to \mathbb{N}^\mathbb{N}$ may be approximated by maps on sequences. Let $\Phi(\sigma, m) = n$ if Φ computes output n on input m using only oracle information from σ and in $|\sigma|$ or fewer steps; we may assume that $n < |\sigma|$. Let $\Phi(\sigma) = \tau$ denote the partial function on strings as before.

Before giving a characterization of a computably continuous function, we first consider arbitrary continuous functions.

Lemma III.4.1. *A function $F : \mathbb{N}^\mathbb{N} \to \mathbb{N}^\mathbb{N}$ (respectively, $F : \{0,1\}^\mathbb{N} \to \{0,1\}^\mathbb{N}$) is continuous if and only if there is a function $f : \mathbb{N}^* \to \mathbb{N}^*$ (resp. $f : \{0,1\}^* \to \{0,1\}^*$) such that*

1. *for all $\sigma \prec \tau$, $f(\sigma) \preceq f(\tau)$;*
2. *for all $x \in \mathbb{N}^\mathbb{N}$ ($\{0,1\}^\mathbb{N}$), $\lim_{n\to\infty} |f(x \restriction n)| = \infty$;*
3. *for all $x \in \mathbb{N}^\mathbb{N}$ ($\{0,1\}^\mathbb{N}$), $\cup_n f(x \restriction n) = F(x)$.*

Proof. Since F is continuous, it follows that $F^{-1}(I(\tau))$ is open for each $\tau \in \mathbb{N}^*$. Define $f(\sigma)$ to be the unique longest τ such that $I(\sigma) \subseteq F^{-1}(I(\tau))$ [equivalently, $F(I(\sigma)) \subseteq I(\tau)$].

Then f is certainly monotonic.

Fix $x \in \mathbb{N}^{\mathbb{N}}$ and let $Y = F(X)$. For each n, X belongs to the open set $F^{-1}(I(Y \upharpoonright n))$ and hence there is some basic open set $I(\sigma)$ such that $X \in I(\sigma) \subseteq F^{-1}(I(Y \upharpoonright n))$. It follows that $Y \upharpoonright n \preceq f(\sigma)$ and of course $\sigma = x \upharpoonright m$ for some m. Then for all $t > m$, $|f(x \upharpoonright t)| \geq n$. Thus $lim_{t\to\infty}|f(x \upharpoonright t)| = \infty$, as desired.

It now follows that $\cup_n f(x \upharpoonright n) = F(x)$. □

The fundamental notion of computable analysis is the computable version of Lemma III.4.1.

Lemma III.4.2. *A function $F : \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ (respectively, $F : \{0,1\}^{\mathbb{N}} \to \{0,1\}^{\mathbb{N}}$) is computably continuous if and only if there is a computable function $f : \mathbb{N}^* \to \mathbb{N}^*$ (resp. $f : \{0,1\}^* \to \{0,1\}^*$) such that*

1. *for all $\sigma \prec \tau$, $f(\sigma) \preceq f(\tau)$;*
2. *for all $x \in \mathbb{N}^{\mathbb{N}}$ ($\{0,1\}^{\mathbb{N}}$), $\lim_{n\to\infty} |f(x \upharpoonright n)| = \infty$;*
3. *for all $x \in \mathbb{N}^{\mathbb{N}}$ ($\{0,1\}^{\mathbb{N}}$), $\cup_n f(x \upharpoonright n) = F(x)$.*

Proof. Given such a representation f for F, compute $y(n)$ for $y = F(x)$ from x by computing $f(x \upharpoonright k)$ for sufficiently large k.

Given a computable function F, define the representation f as follows. On input σ of length n, compute the values of $\tau = f(\sigma)$ for each $i < n$ by applying the algorithm for F for n steps, using oracle σ. The length of τ will be the least $k < n$ such that $\tau(k)$ does not converge in n steps. □

Remark: The *modulus of convergence* function μ of $F : \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ as defined from the approximation map $f : \mathbb{N}^* \to \mathbb{N}^*$ may be defined by $\mu(x, n) = (least\ s)|f(x \upharpoonright n)| > s$. For a total computable function F, this modulus function is also computable. The following lemma will be useful. The proof is left as an exercise.

Lemma III.4.3. *For any computable function $F : \{0,1\}^{\mathbb{N}} \to \{0,1\}^{\mathbb{N}}$, there is a computable uniform modulus function $\mu : \mathbb{N} \to \mathbb{N}$ such that for all $x \in \{0,1\}^{\mathbb{N}}$, $|f(x\lceil\mu(n)| > n$.* □

Theorem III.4.4. *Let $\Phi : \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ be a (partial) computable function. Then the graph of Φ is a Π^0_2 class. Furthermore, if Φ is total, then the graph is a decidable Π^0_1 class.*

Proof. Let ϕ be a representing function for Φ. In general, we have,

$$\Phi(x) = y \iff (\forall m)(\exists k)[\phi(x \upharpoonright k)(m) = y(m)].$$

For a total function, define the computable tree T with $graph(\Phi) = [T]$ by putting $\sigma \oplus \tau \in T$ if and only if τ is consistent with $\Phi(\sigma)$, that is, for any m, if $\Phi(\sigma, m) = n$, then $\tau(m) = n$. Then $Ext(T)$ is Σ^0_1 and hence computable, since

$$\sigma \oplus \tau \in Ext(T) \iff (\exists \rho)[\sigma \prec \rho \ \& \ \tau \prec \Phi(\rho)].$$

To see this, note that if $\tau \preceq \Phi(\rho)$ and $\sigma \preceq \rho$, then for *any* $x \in I(\sigma)$, $\tau \prec F(x)$. □

Example III.4.5. *The graph of a partial computable function need not be closed. Define the partial function* $\Phi : \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ *by* $\Phi(x)(m) = 0\cdot[(least\ n)x(n) = 1]$. *For each* n, *let* $x_n = 0^n \frown 1 \frown 0^\infty$, *so that* $lim_n x_n = 0^\infty$. *Then, for each* n, $F(x_n) = 0^\infty$, *whereas* $F(0^\infty)$ *is undefined.*

For total functions on $\{0,1\}^{\mathbb{N}}$, there is a converse.

Theorem III.4.6. *A function* $F : \{0,1\}^{\mathbb{N}} \to \{0,1\}^{\mathbb{N}}$ *is computably continuous if and only if the graph is a* Π^0_1 *class.*

Proof. One direction follows from Theorem III.4.4. Next suppose that $F : \{0,1\}^{\mathbb{N}} \to \{0,1\}^{\mathbb{N}}$ and let T be a computable tree such that $graph(F) = [T]$. Define a computable function f on strings by letting $f(\sigma)$ be the common part of $\{\tau : \sigma \oplus \tau \in T\}$. □

Theorem III.4.7. *A subset* D *of* $\mathbb{N}^{\mathbb{N}}$ *is a* Π^0_2 *class if and only if* D *is the domain of some partial computable function* $\Phi : \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$.

Proof. Suppose first that D is the domain of Φ. Then

$$x \in Dom(\Phi) \iff (\forall n)(\exists k)[|\Phi(x \restriction k)| > k].$$

Next suppose that D is a Π^0_2 class and let R be a computable relation such that

$$x \in D \iff (\forall n)(\exists k)R(n, x\lceil k).$$

Then D is the domain of the partial computable function Φ defined by

$$\Phi(x)(n) = (leastk)R(n, x\lceil k).$$

□

We next examine the complexity of the image of a Π^0_1 class under a computably continuous function. The classical results is that the image of any compact set under a continous function is compact and that the image of a closed set is an analytic set.

Theorem III.4.8. *Let* F *be a computably continuous function on a* Π^0_1 *subclass* P *of* $\mathbb{N}^{\mathbb{N}}$ *and let* $F[P] = \{F(x) : x \in P\}$. *Then*

1. $F[P]$ *is a* Σ^1_1 *class;*

2. *if P is bounded, then $F[P]$ is a strong Π^0_2 class;*
3. *if P is computably bounded, then $F[P]$ is a computably bounded Π^0_1 class and, furthermore, if P is decidable, then $F[P]$ is decidable.*

Proof. (1) We have $y \in F[P] \iff (\exists x)(x \in P \;\&\; x \oplus y \in graph(F))$.

(2) Suppose that T is a finite branching, computable tree and let S be a computable tree such that $graph(F) = [S]$. Then it follows from König's Lemma that $F[P] = [R]$, for the finite branching Σ^0_1 tree R defined by

$$\tau \in R \iff (\exists \sigma)[\sigma \in T \text{ and } \sigma \oplus \tau \in S].$$

(3) Now suppose that T is computably bounded and let F be represented by the computable function $f : \mathbb{N}^* \to \mathbb{N}^*$. Then the definition of R above in (2) becomes computable since the $(\exists\sigma)$ quantifier becomes bounded.

To find a bound for the possible value of $\tau(n)$ for $\tau \in R$, compute the least m such that $|f(\sigma)| > n$ for all $\sigma \in T$ of length m. Then we compute the maximum value $h(r)$ of $f(\sigma(n))$ for all $\sigma \in T$ of length n. Thus R is seen to be highly computable. □

This result has a converse for $\{0,1\}^{\mathbb{N}}$.

Theorem III.4.9. *A Π^0_1 subclass P of $\{0,1\}^{\mathbb{N}}$ is the computably continuous image of $\{0,1\}^{\mathbb{N}}$ if and only if it is decidable.*

Proof. Let $P = T$, where T is a computable tree with no dead ends. We will define the computable map $f : \{0,1\}^* \to \{0,1\}^*$ which represents a map $F : \{0,1\}^{\mathbb{N}} \to \{0,1\}^{\mathbb{N}}$ such that $y = F(x)$ is some element of P which is nearest to x. Let $f(\sigma) = \sigma$ if $\sigma \in T$ and, if $\sigma \notin T$, let ρ be the longest initial segment of σ which is in T and let $f(\sigma)$ be the lexicograpical least extension of ρ which is in T and has length $|\sigma|$. □

Exercises

III.4.1. Prove Lemma III.4.3.

III.4.2. A mapping f from $\mathbb{N}^*$ (or $\{0,1\}^*$) into $\mathbb{N}^*$ is a *tree homomorphism* if $\sigma \prec \tau$ implies $f(\sigma) \prec f(\tau)$ for all σ and τ. Show that for any tree homomorphism f, $T = \{\tau : (\exists\sigma) : f(\sigma) \prec \tau\}$ is a tree and that if f is one-to-one and computable, then T is a computable tree and $[T]$ is a perfect Π^0_1 class.

III.5 Computably enumerable sets and Π^0_1 Classes

There are numerous connections between computable functions, Π^0_1 and c.e. subsets of $\mathbb{N}$, and Π^0_1 classes. We will consider in particular the class $S(A,B)$ of separating sets for a pair of c.e. sets A, B, and the class $I(C)$ of initial subsets of a Π^0_1 set C. Then we will take a first look at the notion of *forbidden words* and *sets* and their connection with Π^0_1 classes and with *subshifts*. In this section, the notion of retraceability

III.5.1 Separating classes

The most basic example here is that, for any Π^0_1 set C, the power set $\mathcal{P}(C)$ is a Π^0_1 class. That is, we have

$$x \subset C \iff (\forall n)[x(n) = 1 \to n \in C].$$

More generally, consider the notion of *separating sets.*

Definition III.5.1. *Let A and B be infinite disjoint c.e. sets and let $C \subset \mathbb{N}$.*

1. *C is a separating set for A and B if $A \subset C$ and $B \cap C = \emptyset$.*
2. *A and B are said to be computably (or recursively) inseparable if there is no computable separating set C for A and B.*
3. *The class of separating sets for A and B is denoted by $S(A, B)$.*
4. *P is a separating class if $P = S(A, B)$ for some c.e. sets A and B.*

Of course, $C \in S(A, B)$ if and only if $C \in \mathcal{P}(\mathbb{N} \setminus B)$ and $\mathbb{N} \setminus C \in \mathcal{P}(\mathbb{N} \setminus A)$. The notion of computably inseparable sets was introduced by Kleene in [109]. Shoenfield showed in [189] that every non-computable c. e. degree contains a pair of computably inseparable sets. Shoenfield observed in [189] that the class $S(A, B)$ of separating sets for A and B is a c. b. Π^0_1 class. We note that $S(A, B)$ is finite if and only if $A \cup B$ is cofinite, in which case A and B are both computable and every separating set is also computable. Otherwise, $S(A, B)$ is a perfect set and thus has the cardinality of the continuum. In either case, both the c. e. set A and the co-c. e. set $\mathbb{N} \setminus B$ are of course separating sets for A and B.

The classic example of a separating class comes from the notion of *diagonally non-computable* sets. Here a function $f \in \{0, 1\}^{\mathbb{N}}$ is diagonally non-computable if $f(e) \neq \phi_e(e)$ whenever $\phi_e(e)$ converges. Let $K_i = \{e : \phi_e(e) = i\}$. Then in particular, K_0 and K_1 are c.e. non-computable sets and any separating set for K_0 and K_1 has a diagonally non-computable characteristic function. (See the exercises.) Separating classes are important for the study of reverse mathematics and so will be examined further in Chapter VII. The complexity of the members of separating classes will be studied in Chapter IV.

For $\sigma, \tau \in \{0, 1\}^*$, let $\sigma \subseteq \tau$ mean that, for all $i \leq min\{|\sigma|, |\tau|\}$, $\sigma(i) = 1$ implies $\tau(i) = 1$. Define $\sigma \cup \tau$ to be the sequence ρ of length $max\{|\sigma|, |\tau|\}$ such that $\rho(i) = max\{\sigma(i), \tau(i)\}$ and similarly define $\sigma \cap \tau$ to be the sequence ρ of length $max\{|\sigma|, |\tau|\}$ such that $\rho(i) = min\{\sigma(i), \tau(i)\}$

Recall that for any closed subset P of $\mathbb{N}^{\mathbb{N}}$, $T_P = \{\sigma \in \{0, 1\}^* : I(\sigma) \cap P \neq \emptyset\}$.

Lemma III.5.2. *Suppose that P is a closed subset of $\{0, 1\}^{\mathbb{N}}$.*

1. *P is closed under subsets if and only if for every $\sigma \subset \tau$, if $\tau \in T_P$, then $\sigma \in T_P$.*
2. *P is closed under supersets if and only if for every $\sigma \subset \tau$, if $\sigma \in T_P$, then $\tau \in T_P$.*

3. *P is closed under union if and only if, for every σ and τ in T_P, $\sigma \cup \tau \in T_P$.*

4. *P is closed under intersection if and only if, for every σ and τ in T_P, $\sigma \cap \tau \in T_P$.*

Proof. We prove (1) and (3) and leave (2) and (4) to the reader.

(1) Suppose P is closed under subsets. Let $\sigma \subseteq \tau$ and $\tau \in T_P$. Then there exists $B \in P$ such that $\tau \preceq B$. Let $C = \{i \in B : \sigma(i) = 1\}$. Then $C \subseteq B$ so $C \in P$ by assumption and clearly $\sigma \preceq C$. On the other hand, suppose that T_P is closed under $\subseteq$. Let $B \in P$ and $C \subseteq B$. Then for any n, $C\lceil n \subseteq B\lceil n$ and $B\lceil n \in T_P$, so that $C \in P$.

(3) Suppose P is closed under union and let σ and τ be in T_P. Then there exist A and B in P such that $\sigma \prec A$ and $\tau \prec B$. It follows that $\sigma \cup \tau \prec A \cup B$ and $A \cup B \in P$ by assumption, so that $\sigma \cup \tau \in T_P$. Suppose next that T_P is closed under union and let $A, B \in P$. Fix n and let $\sigma = A \upharpoonright n$ and $\tau = B \upharpoonright n$. Then $(A \cup B) \upharpoonright n = \sigma \cup \tau \in T_P$ since both $\sigma, \tau \in T_P$. Hence $A \cup B \in P$. □

Lemma III.5.3. *Let P be a closed subset of $\{0,1\}^{\mathbb{N}}$. Then $P = \mathcal{P}(A)$ for some A if and only if P is closed under subsets and P is closed under union.*

Proof. If $P = \mathcal{P}(A)$, then P is certainly closed under subsets and union. Suppose that P is closed under subsets and union and let $A = \{i : (\exists \sigma \in T_P)\sigma(i) = 1\}$. Suppose first that $B \in P$. If $i \in B$, then $\sigma(i) = 1$ for $\sigma = B\lceil(i+1) \in T_P$, so that $i \in A$. Hence $B \subseteq A$. Next suppose that $B \subseteq A$. Then for each $i \in B$, there exists $\sigma_i \in T_P$ such that $\sigma_i(i) = 1$. Fix n and define $\sigma \in \{0,1\}^n$ by $\sigma = \cup\{\sigma_i : i < n \ \& \ i \in B\}$. Then $\sigma \in T_P$ by Lemma III.5.2 since P is closed under union. Also, $B\lceil n \subseteq \sigma$, so that $B\lceil n \in T_P$ again by Lemma III.5.2, since P is closed under subsets. It follows that $B \in P$. □

Observe that if P is actually a Π^0_1 class, then T_P is a Π^0_1 set and the set A defined in the proof of Lemma III.5.3 is in fact a Π^0_1 set. Thus we have the following.

Proposition III.5.4. *For any nonempty Π^0_1 class P of sets, the following are equivalent:*

1. *P is the class of subsets of a Π^0_1 set A;*

2. *P is the class of subsets of some set A;*

3. *P is closed under subsets and under union.* □

There is a similar result for supersets, which is left to the exercises.

Let us say that a class P of sets is closed under *between-ness* if, for any sets X, Y, Z, if $X \subset Y \subset Z$ and $X, Z \in P$, then $Y \in P$. It is clear that any separating class is closed under between-ness.

Proposition III.5.5. *For any Π^0_1 class P, the following are equivalent.*

1. *P is the class of separating sets of some pair A, B of r. e. sets.*

2. *P is the class of separating sets of some pair A, B*

3. *P is closed under union, intersection and between-ness.*

Proof. It is immediate that (1) implies (2) and (2) implies (3). Suppose therefore that P is closed under union, intersection and between-ness. Define the Π^0_1 class Q to be the family of subsets of sets in P. That is, for $\sigma \in \{0,1\}^n$,

$$\sigma \in T_Q \iff (\exists \tau \in \{0,1\}^n)\sigma \subseteq \tau \ \& \ \tau \in T_P.$$

It is clear that Q is closed under subsets and under union, so it follows from Proposition III.5.4 that $Q = \mathcal{P}(C)$ for some Π^0_1 set C. Let $B = \mathbb{N} \setminus C$.

Similarly define the Π^0_1 class R to be the family of supersets of sets in P. That is, for $\sigma \in \{0,1\}^n$,

$$\tau \in T_R \iff (\exists \sigma \in \{0,1\}^n)\sigma \subseteq \tau \ \& \ \tau \in T_P.$$

It follows that R is the class of supersets of some c.e. set A.

We claim that $P = Q \cap R = S[A, B]$. Suppose first that $X \in P$. Then certainly $X \in Q \cap R$ and therefore $A \subseteq X$ and $X \cap B = \emptyset$. Next suppose that $X \in S[A, B]$. Then $A \subseteq X$, so that $X \in R$ and therefore $Y \subseteq X$ for some $Y \in P$. Also, $X \cap B = \emptyset$, so that $X \subseteq C$ and $X \in Q$, which means that $X \subseteq Z$ for some $Z \in P$. It now follows by the between-ness property that $X \in P$. $\square$

The proof of the following corollary is left as an exercise 6.

Corollary III.5.6. *For any subset A of $\mathbb{N}$, if $\{A\}$ is a Π^0_1 class, then A is a computable set.* $\square$

III.5.2 Subsimilar classes

The notions of a *subsimilar set* (or *subshift*) and of *forbidden words* provides another link between c. e. sets and Π^0_1 classes and is also closely related to symbolic dynamics.

Definition III.5.7. 1. *A finite string $\sigma \in \mathbb{N}^*$ is a* factor *of another string τ if $\tau = \tau_0{}^\frown\sigma^\frown\tau_1$ for some strings τ_0, τ_1.*

2. *Similarly $\sigma \in \mathbb{N}^*$ is a* factor *of $x \in \mathbb{N}^{\mathbb{N}}$ if $x = \tau^\frown\sigma^\frown Y$ for some strings τ and some $Y \in \mathbb{N}^{\mathbb{N}}$.*

3. *$x \in \mathbb{N}^{\mathbb{N}}$* avoids *$\sigma$ if σ is not a factor of x and x avoids a set S of strings if x avoids each $\sigma \in S$.*

4. *$AV(S) \subseteq \mathbb{N}^{\mathbb{N}}$ is the set of reals which avoid S.*

Similar definitions apply for strings in $\{0,1\}^*$ and infinite words in $\{0,1\}^{\mathbb{N}}$ and also for finite and infinite sequences any other alphabet Σ. In symbolic dynamics, strings are referred to as *words* and infinite sequences as infinite words. The set S may be thought of a set of *forbidden words* for $AV(S)$.

The *shift* function $Shift$ is defined as follows.

Definition III.5.8. *1. For a finite string τ, $Shift(\tau) = (\tau(1), \tau(2), \dots, \tau(|\tau|-1)$.*

2. For an infinite sequence x, $Shift(x) = (x(1), x(2), \dots)$.

3. A set $Q \subseteq \mathbb{N}^{\mathbb{N}}$ is a subshift *(or* subsimilar*) if $Shift(x) \in Q$ for all $x \in Q$; that is, Q is closed under the subshift function.*

Informally, the shift function simply removes the first symbol of a finite or infinite sequence.

Lemma III.5.9. *For any set S, $AV(S)$ is closed; if S is c. e. , then $AV(S)$ is a Π_1^0 class.*

The proof is left as an exercise.

Theorem III.5.10 (Dashti [24]). *1. A closed set Q is a subshift if and only if $Q = AV(S)$ for some set S.*

2. A Π_1^0 class Q is a subshift if and only if $Q = AV(S)$ for some set c. e. set S.

Proof. For the first part, it is clear that $AV(S)$ is a subshift, so we will just sketch the reverse implication. Suppose that $P \subseteq \{0,1\}^{\mathbb{N}}$ is subsimilar and closed, and let $S = \{0,1\}^* - T_P$. If $x \notin P$, then for some n, $x \restriction n \in S$, so that $x \notin AV(S)$. On the other hand, suppose that $x \notin AV(S)$ and let $x = (x \restriction n)^\frown \tau^\frown y$ for some $n < \omega$ and some $\tau \in S$. Then $\tau \notin T_P$ and thus $\tau^\frown y \notin P$. Since P is subsimilar, it follows that $x \notin P$. A similar argument works for any alphabet Σ.

The proof of the effective version of this proposition is left as an exercise. □

Exercises

III.5.1. Show that the diagonally non-computable functions form a Π_1^0 class in $\{0,1\}^{\mathbb{N}}$.

III.5.2. Show that $S(K_0, K_1)$ is a Π_1^0 class of sets with no computable members, that is, K_0 and K_1 are computably inseparable.

III.5.3. Suppose that $P = [T]$ where T is a tree with no dead ends. Show the following.

(a) P is closed under supersets if and only if, for every $\sigma \in T$ and every τ such that $\sigma \subseteq \tau$, $\tau \in T$.

(b) P is closed under intersection if and only if, for every σ and τ in T, $\sigma \cap \tau \in T$.

III.5.4. Let P be a closed subset of $\{0,1\}^{\mathbb{N}}$. Then P is the class of supersets of some set A if and only if P is closed under supersets and P is closed under intersection.

III.5.5. Show that, for any nonempty Π^0_1 class P of sets, the following are equivalent:

(i) P is the class of supersets of a Σ^0_1 set A;

(ii) P is the class of supersets of some set A;

(iii) P is closed under supersets and under intersection.

III.5.6. Use Proposition III.5.5 to show that for any subset A of $\mathbb{N}$, if $\{A\}$ is a Π^0_1 class, then A is a computable set.

III.5.7. Show that a Π^0_1 class Q is subsimilar if and only if $Q = AV(S)$ for some c. e. set S.

III.6 Retraceability

For any infinite set A, recall that the *principal function* p_A enumerates the elements $a_0 < a_1 < \cdots$ in increasing order and that A is *hyperimmune* if, for any computable function f, there is an n such that $a_n > f(n)$. A c. e. set is *hypersimple* if its complement is hyperimmune. A is said to be *retraceable* if there is a partial computable function ϕ such that $\phi(a_{n+1}) = a_n$ for all n. Retraceable sets were introduced by Dekker and Myhill [56], who proved that any retraceable noncomputable Π^0_1 set A is hyperimmune. For Π^0_1 sets A, a stronger characterization can be given.

Theorem III.6.1. *The following are equivalent for any infinite* Π^0_1 *set* A*:*

(a) A *is retraceable*

(b) There is a total computable function Φ *such that, for all* n, $\Phi(a_{n+1}) = a_n$ *and, for all* y, $\{x : \Phi(x) = y\}$ *is finite.*

(c) There is a total computable function Ψ *such that, for all* n, $\Psi(a_n) = n$ *and* $\{x : \Psi(x) = n\}$ *is finite*

Proof. Let $A = \{a_0 < a_1 < \cdots\}$ be an infinite Π^0_1 set and let A be the decreasing intersection of uniformly computable sets A_s.

$(a) \to (b)$: Let ϕ be a partial computable retracing function for A. Assume, without loss of generality, that $\phi(a_0) = a_0$. Then for any x, we define $\Phi(x)$ as follows. Look for the least s such that either $x \notin A_s$, or such that $\phi_s(x) = y \leq x$ converges and, for all z with $y < z < x$, $z \notin A_s$. In the former case, we let $\Phi(x) = x$ and in the latter case, we let $\Phi(x) = \phi(x)$. Note that if $x \notin A$, then the former case will obtain and if $x \in A$, then the latter case will obtain, so that Φ is total and is a retracing function for A. It follows from the definition that for every x, $\Phi(x) \leq x$ and there are no elements of A between $\Phi(x)$ and x. Now for any y, let a be the least such that $a > y$ and $a \in A$. Then $\Phi(x) = y$ implies that $x \leq a$, so that $\{x : \Phi(x) = y\}$ is finite, as desired.

$(b) \to (c)$: Let Φ be given as described. Then we define $\Psi(x)$ to be length n of the chain $x > \Phi(x) > \Phi(\Phi(x)) > \cdots > \Phi^n(x) = a_0$, if there is such an

n-chain, and $\Psi(x) = x$ if $\Phi^{i+1}(x) = \Phi^i(x)$ for some i. Thus for $a_n \in A$, we obtain $\Psi(a_n) = n$. To complete the proof, we show by induction that, for each n, there are only finitely many n-chains $x > \Phi(x) > \cdots > \Phi^n(x) = a_0$ of length n. For $n = 1$, this follows from the assumption that $\Phi(x) = a_0$ for only finitely many x. Suppose now that there are only finitely many such n-chains of length n. Then any $n+1$-chain must extend one of these and, by our assumption, there are only finitely many ways to extend each chain. Thus there can be only finitely many $n+1$-chains.

$(c) \to (a)$: Let Ψ be given as described. Then for $a = a_{n+1} \in A$, $\Psi(a) = n+1$ and the retracing function $\phi(a_{n+1}) = a_n$ may now be computed by searching for the least s such that exactly $n+1$ elements of A_s are less than a and taking a_n to be the largest of those. □

We say that an infinite set $A = \{a_0 < a_1 < \cdots\}$ is *second-retraceable* if there is a (total) computable function Φ such that, for any $m < n$, $\Phi(a_m, a_n) = m$. In general, A is k-retraceable if there is a computable Φ such that $\Phi(a_{m_1}, a_{m_2}, \ldots, a_{m_k}) = m_1$ for any $m_1 < m_2 < \cdots < m_k$. Of course, any $k+1$-retraceable set is also k-retraceable.

A subset F of the set $\{a_0 < a_1 < \cdots\}$ is said to be an *initial subset* of A if $a_{n+1} \in F$ implies $a_n \in F$ for all n. Thus the initial subsets of A are A together with the finite sets $\{a_0, \ldots, a_{n-1}\}$ for each n. Let $I_1(A)$ denote the class of initial subsets of A. In general, the k-initial subsets $I_k(A)$ are the subsets F of A such that for any elements $a < b_1 < b_2 < \cdots b_k$ of A, if $b_1, \ldots, b_k \in F$, then $a \in F$.

Theorem III.6.2. *For each finite k, the set $A = \{a_0 < a_1 < \ldots\}$ is Π^0_1 and k-retraceable if and only if the class $I_k(A)$ of k-initial subsets of A is a Π^0_1 class.*

Proof. Suppose first that A is k-retraceable via the function Φ and that A is a Π^0_1 set. Let A_s denote the computable approximation to the set A at stage s, so that $A = \cap_s A_s$. Now define the computable tree T as follows.

$\lfloor b_0, b_1, \ldots, b_n, s\rfloor \in T \iff$

1. $(\forall i \leq n)(b_i \in A^s)$ and
2. if $n \geq k$, then $\Phi(b_{n-k+1}, b_{n-k+2}, \ldots, b_{n-1}, b_n) = n - k + 1$.

It is easy to check that $[T] = I_k(A)$, so that $I_k(A)$ is a Π^0_1 class.

Now suppose that $I_k(A)$ is a Π^0_1 class and let T be a computable tree so that $I_k(A) = [T]$. We will explain how to compute a k-retracing function Φ. Given $b_1 = a_{m_1} < b_2 = a_{m_2} < \ldots < b_k = a_{m_k}$, observe that there is only one possible string $\sigma = \lfloor a_0, a_1, \ldots, a_{m_1-1}, b_1, b_2, \ldots, b_k\rfloor^\frown 1$ of the form $\lfloor c_0, c_1, \ldots, c_r, b_1, b_2, \ldots, b_k\rfloor^\frown 1$ which has an extension in T; $\Phi(b_1, \ldots, b_k) = m_1$ is then easily computed from σ. To find σ, we just search through all strings of length $m > b_k$ until we find m large enough so that all strings τ in T of length m and with $\tau\lceil b_k + 1$ of the desired form, start with the same initial segment (σ) of length $b_k + 1$.

To see that A is a Π^0_1 set, recall that $Ext(T)$ is Π^0_1 and observe that

$a \in A \iff (\exists\sigma)[|\sigma| = a + 1 \& \sigma \in Ext(T) \& \sigma(a) = 1]$. □

We can now give a quick proof that any retraceable non-computable Π^0_1 set is hyperimmune.

Theorem III.6.3. *[Dekker-Myhill] If $A = \{a_0 < a_1 < \cdots\}$ is a retraceable non-computable Π^0_1 set, then A is hyperimmune.*

Proof. By Theorem III.6.2, $P(A)$ is a Π^0_1 class. Now suppose by way of contradiction that f were a computable function which dominated p_A, that is, $f(n) > p_A(n)$ for all n. Then the set $\{A\}$ would be the intersection of $I(A)$ with the following Π^0_1 class:

$\{B : (\forall n)(card(B \cap \{0, 1, \ldots, f(n)\}) \geq n\}$.

Thus $\{A\}$ would be a Π^0_1 class, so that A would be computable (this is seen below in Exercise V.1.11. This contradiction now demonstrates the result. □

For any k-retraceable Π^0_1 set A, the Π^0_1 class $I_k(A)$ is provides an example of a class with C-B rank k.

Theorem III.6.4. *For any set A, $D^k(I_k(A)) = \{A\}$.*

Proof. It is easy to see that $D(I(A)) = \{A\}$ and that, for each k, $D(I_{k+1}(A)) = I_k(A)$. □

It follows that if A is k-retraceable, then A has rank k in $I_k(P)$ and thus has rank $\leq k$.

We next give a result which shows how to define a retraceable Π^0_1 set by Π^0_1-recursion.

Theorem III.6.5. *Suppose that the set $A = \{a_0 < a_1 < \ldots\}$ is defined recursively by a Π^0_1 relation $Q(x, y)$ such that, for all n and x, $x = a_n \iff Q(x, < a_0, \ldots, a_{n-1} >)$. Then A is a Π^0_1 set and is retraceable.*

Proof. Define the Π^0_1 relation $R(n, x)$ by

$R(n, x) \iff (\exists x_0 < \cdots < x_{n-1} < x_n = x)(\forall i < n)Q(x_i, < x_0, \ldots, x_{i-1} >)$.

Then the set A is Π^0_1 since

$a \in A \iff (\exists n \leq a)R(n, a)$.

Define the uniformly computable relation $R_s(n, x)$ as in the definition of R above with Q_s in place of Q.

The counting function Ψ such that $\Psi(a_n) = n$ may of course be defined by the fact that n is the unique y such that $R(y, a_n)$. Since, given $a \in A$, there is just one $n \leq a$ such that $a = a_n$, $\Psi(a) = n$ may be computed by searching for an s large enough so that $R_s(n, a)$ for only one number $n \leq a$. □

This result can now be applied to give a quick proof of the following theorem of Dekker and Myhill [56] (Theorem T3).

Theorem III.6.6. *[Dekker-Myhill] Every r.e. set B is Turing equivalent to a retraceable Π^0_1 set A.*

Proof. Let the c.e. set B be the union of uniformly computable sets B_s and define the set A by Π^0_1 recursion as follows. There are two cases in the definition of a_n. If $n \notin B$, then $a_n = a_{n-1} + 1$ and if $n \in B$, then a_n is the least $s > a_{n-1}$ such that $n \in B_s$. It is clear that this is a Π^0_1-recursion, so that A is a Π^0_1 retraceable set. The definition also shows that A is computable in B. On the other hand, for any n, we have $n \in B \iff n \in B_{a_n}$, so that B is computable in A. □

Definition III.6.7. *The Cantor-Bendixson (C-B) rank of a set A is the least ordinal α such that A has rank α in some Π^0_1 class $P \subset \{0,1\}^{\mathbb{N}}$.*

It follows from Theorems III.6.4 and III.6.6 that every non-zero r.e. degree contains a set A of C-B rank one. A slightly better result was obtained in [25] by Cenzer, Downey, Jockusch and Shore, hereafter abbreviated as C-D-J-S.

Theorem III.6.8. *[C-D-J-S] Every c.e. non-computable set B is Turing equivalent to a hypersimple c.e. set E of rank one; furthermore there is a computable tree U with no dead ends such that $D([U]) = \{E\}$.*

Proof. Let $A = a_0 < a_1 < \cdots$ be the Π^0_1 retraceable set defined in Theorem III.6.6 and let A be the intersection of the uniformly computable, decreasing sequence A_s. Define the computable tree S to be a slight extension of the tree T defined in Theorem III.6.2. That is, we let Φ be the retracing function given by Theorem III.6.1 for A so that $\Phi(a_{n+1}) = a_n$ and so that $\{x : \Phi(x) = y\}$ is finite for each y, and define
$\lfloor c_0, \ldots, c_n, c_{n+1} \rfloor \in S \iff c_0 = a_0 \ \& \ (\forall i \leq n)[c_i \in A_{c_n} \ \& \ (i > 0 \to \Phi(c_i) = c_{i-1})]$.

S has no dead ends because for any string $\sigma \in S$, it is clear that $\sigma^\frown 0 \in S$.

We leave it to the reader to check that $D(P) = \{A\}$.

To obtain the c.e. set E, we note that the complement function $F(C) = \mathbb{N} \backslash C$ is a computable homeomorphism of $\{0,1\}^\omega$ to itself, so that the c.e. set $E = \mathbb{N} \backslash A$ has rank one in the Π^0_1 class of complements $\{F(C) : C \in P\}$. Finally, any retraceable noncomputable Π^0_1 set is hyperimmune by Theorem III.6.3, so that E is hypersimple. □

Exercises

III.6.1. Verify that $D(I(A)) = \{A\}$ and that, for each k, $D(I_{k+1}(A)) = I_k(A)$.

III.6.2. Show that $D(P) = \{A\}$, in the proof of Theorem III.6.8.

III.6.3. For any computable function f, the set $K(f) = \{x : (\forall n) x(n) \leq f(n)\}$ is a c. b. Π^0_1 class. Show that $K(p_A)$ is also a Π^0_1 class, where A is a infinite Π^0_1 set and p_A is the principal function defined above.

III.6.4. Show that for the total retracing function Φ of Theorem III.6.1, if $g(x) = card(\{y : \Phi(y) = x\})$ is computable, then A is a computable set. However, show that for the retraceing function from Theorem III.6.6, there is always a computable upper bound for the cardinality.

III.7 Reducibility

In this section, we consider some connections between subclasses of $\{0,1\}^{\mathbb{N}}$, computably bounded Π^0_1 classes, bounded classes and Π^0_n classes. Our goal is to reduce every class to a class of sets.

Computably bounded Π^0_1 classes play a fundamental role and occur frequently in the applications. A very useful result which simplifies the theory of c. b. Π^0_1 classes is that every such class is computably homeomorphic to a subclass of $\{0,1\}^{\mathbb{N}}$.

Definition III.7.1. *Classes P and Q are computably homeomorphic if there is a (total) computably continuous functional $F : \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ such that F maps P one-to-one and onto Q.*

Notice that we do not require that F be one-to-one onto $\mathbb{N}^{\mathbb{N}}$ or that it map $\mathbb{N}^{\mathbb{N}}$ onto $\mathbb{N}^{\mathbb{N}}$.

Lemma III.7.2. *If there is a partial computable function Φ mapping a subset of $\mathbb{N}^{\mathbb{N}}$ to a subset of $\mathbb{N}^{\mathbb{N}}$ such that Φ is total on P and maps P one-to-one and onto Q, then P and Q are computably homeomorphic.*

Proof. Let Φ have a representation $\phi : \mathbb{N}^* \to \mathbb{N}^*$; that is, let $\phi(\sigma)$ be the longest sequence of the form $(\Phi(0,\sigma), \Phi(1,\sigma), \ldots, \Phi(n-1,\sigma))$. Let $P = [T]$ and $Q = [S]$ for computable trees tree S and T, Define a new mapping $f : \mathbb{N}^* \to \mathbb{N}^*$ by letting $f(\sigma) = \phi(\sigma)$ for $\sigma \in T$ and letting $f(\sigma^\frown n) = f(\sigma)^\frown n$ if $\sigma^\frown n \notin T$. Then f represents a computably continuous function F such that $F(x) = \Phi(x)$ for $x \in P$ and $F(x) = \phi(x \upharpoonright k)^\frown(x(k), x(k+1), \ldots)$ if $x \notin P$ and k is the least such that $x \upharpoonright k+1 \notin T$. □

Theorem III.7.3. *Any c. b. Π^0_1 class P is computably homeomorphic to a Π^0_1 class Q of sets.*

Proof. Let T be a highly computable tree such that $P = [T]$ and let h be a computable function such that $\sigma(n) < h(n)$ for all $\sigma \in T$ and all $n < |\sigma|$.

The homeomorphism Φ will be defined by

$$\Phi(x) = 0^{x(0)}10^{x(1)}\ldots$$

and the class $Q \subset \{0,1\}^{\mathbb{N}}$ is simply $\{\Phi(x) : x \in P\}$.

The functional Φ is clearly one-to-one and maps P onto Q. Φ is computably continuous since it is represented by the computable map taking $\sigma = (\sigma(0), \ldots, \sigma(n-1))$ to $0^{\sigma(0)}1 \ldots 0^{\sigma(n-1)}$. $Q = \Phi[P]$ is a Π^0_1 class by Theorem III.4.8.

More specifically, $Q = [S]$, where

$$0^{x(0)}10^{x(1)} \ldots 0^{x(k-1)}10^i \in S \iff (x(0), \ldots, x(k-1)) \in T \;\&\; i < h(k)\}.$$

□

This result can be relativized to an oracle. Also, we can apply the same argument to Π^0_1 classes which are not computably bounded.

Theorem III.7.4. *(a) Any strong Π^0_2 class P which is highly computable in $\mathbf{0}'$ is computably homeomorphic to a strong Π^0_2 class Q of sets.*

(b) For any Π^0_1 class P, there exists a Π^0_1 class $Q \subset \{0,1\}^{\mathbb{N}}$ and a one-to-one degree-preserving correspondence between the non-computable members of P and the non-computable members of Q.

Proof. (a) This is just the relativization of Theorem III.7.3 to the oracle $\mathbf{0}'$.

(b) Consider the representation of the mapping from Theorem III.7.3. We can use the same mapping as above, but in this case Theorem III.4.8 only tells us that the image is a strong Σ^1_1 class. If we look at the definition of the tree S such that $Q = [S]$, we have to remove the condition $i < h(k)$. This potentially introduces computable elements $0^{x(0)}10^{x(1)} \dots 0^{x(k-1)}0^{\omega}$ into Q. However, any non-computable element has infinitely many 1's and hence will be the image of an element of P. □

If the same technique is applied to a bounded Π^0_1 class, we get one half of the following result from [93].

Theorem III.7.5. *[Jockusch-Lewis-Remmel]*

(a) Any bounded Π^0_1 class P is computably homeomorphic to a strong Π^0_2 class Q of sets.

(b) For any strong Π^0_2 class P which is highly computable in $\mathbf{0}'$, there is a bounded Π^0_1 class Q and an effective one-to-one degree-preserving correspondence between P and Q.

Proof. (a) This is left as an exercise.

(b) Let $P = [T]$, where S is highly computable in $0'$. By Theorem III.7.4, we may assume that T is a binary tree. It now follows from Proposition III.3.7 that T may be assumed to be a Σ^0_1 tree. Thus there is a computable relation $R \subset \mathbb{N} \times \{0,1\}^*$ such that

$$x \in P \iff (\forall m)(\exists n)R(n, x \upharpoonright m).$$

Now we may define Q by

$$z = x \oplus y \in Q \iff (\forall m)[R(y(m), x \upharpoonright m) \ \& \ \forall i < y(m) \neg R(i, x \upharpoonright m)].$$

Then for each $x \oplus y \in Q$, we have $x \in P$ and for each $x \in P$, there is a unique y such that $x \oplus y \in Q$ and that y is defined so that $y(m)$ is the least n such that $R(n, x \upharpoonright m)$. Thus y is computable in x and therefore $x \oplus y$ has the same degree as x. □

The proof of part (b) can be modified for an arbitrary Π^0_2 class to give a theorem from [94]. The proof is left as an exercise.

Theorem III.7.6 (Jockusch-McLaughlin). *For any Π^0_2 class P, there is a Π^0_1 class Q and an effective one-to-one degree-preserving correspondence between P and Q.* □

Jockusch and Soare showed in Theorem 1 of [98] that an arbitrary Π^0_1 class P with no computable members can be represented by a c. b. Π^0_1 class Q in the sense that the degrees of members of P are a subset of the degrees of members of Q. We give this result together with a relativized version. Let $\mathcal{D}(P)$ denote the set of degrees of members of the class P.

Theorem III.7.7. *(a) For any Π^0_1 class $P \subset \mathbb{N}^{\mathbb{N}}$, there is a Π^0_1 class R of sets such that (1) $\mathcal{D}(P) \subset \mathcal{D}(R)$ and (2) there is a one-to-one correspondence between the computable members of P and the computable members of R. (So that R has no computable members if P has no computable members.) Furthermore, there is a primitive recursive function k such that for $P = P_e$, we have $R = P_{k(e)}$.*

(b) For any Π^0_1 class $P \subset \mathbb{N}^{\mathbb{N}}$ with no members computable in $\mathbf{0}'$, there is a strong Π^0_2 class R of sets with no members computable in $\mathbf{0}'$ such that $\mathcal{D}(P) \subset \mathcal{D}(R)$. Furthermore, there is a primitive recursive function h such that for $P = P_e$, we have $R = P^2_{h(e)}$.

Proof. (a) Let Q be a Π^0_1 class of sets with no computable member. Let $P = [S]$, let $Q = [T]$ and assume without loss of generality that $S \subset (\mathbb{N} \setminus \{0,1\})^*$.

It suffices, by Theorem III.7.3 to obtain a class R which is computably bounded and otherwise meets the requirements of the conclusion. Define the tree U to be the set of strings

$$mu = \sigma_1 * \tau_1 * \sigma_2 * \tau_2 * \cdots * \sigma_n * \tau_n$$

such that $\sigma_i \neq \emptyset$ for $i > 1$ and $\tau_j \neq \emptyset$ for $j < n$, such that $s(\mu) = \sigma_1 * \cdots * \sigma_n \in S$ and $\tau_j \in T$ for all j, and such that $\mu(k) \leq k+1$ for all k.

We claim that $R = [U]$ satisfies the requirements of the theorem. Note first that the construction is uniformly computable in the tree T, so that there is a primitive recursive function k such that the for $T = T_e$, the tree $U = T_{k(e)}$.

The tree U is finite-branching by the restriction that $\mu(k) \leq k+1$. Thus R is a computably bounded Π^0_1 class. Now for any $x \in P$ we can define $z \in R$ with the same degree as x as follows. First of all, define a computable sequence $\emptyset = t_0, t_1, \ldots$ such that t_j is the lexicographically least string in T of length j. Now given x, let

$$z_x = t_{i_0} * (x(0)) * t_{i_1} * x(1)) \cdots ,$$

where for each n, i_n is the least such that

$$x(n) \leq |t_{i_0} * (x(0)) * \cdots * t_{i_n}| + 1.$$

Then z_x is computable in x by the definition, and x is computable in z_x, since it is the subsequence of z_x consisting of the entries $z_x(n) > 1$.

Now let z be any element of R which is not of the form z_x for any x. There are two cases.

(Case 1): Suppose that $z(i) > 1$ for infinitely many i and let $i_0, i_1, \ldots$ enumerate $\{i : z(i) > 1\}$. Define x by $x(n) = z(i_n)$. Then x is computable in z and that $x \in P$, so that x is not computable and therefore z is not computable.

(Case 2): Suppose that $z(i) > 1$ for only finitely many i and let m be the largest such that $z(m) > 1$. Define y by $y(n) = z(m + n)$. Then y is computable in z and that $y \in Q$, so that y is not computable and therefore z is not computable.

(b) The proof is just a modification of the proof of (a). Let $Q = [T]$ in this case be a strong Π^0_2 class of sets with no member computable in $\mathbf{0}'$. Then we define a tree U computable in $\mathbf{0}'$ with $\mu(k) \leq k + 1$ as above so that $R = [U]$ is a c. b. strong Π^0_2 class with the desired properties and apply Theorem III.7.5(a). □

We will consider members of Π^0_1 classes in detail in Chapter IV.

Exercises

III.7.1. Show that there exist Π^0_1 classes P and Q which are computably homeomorphic, but for which there can be no homeomorphism F of $\mathbb{N}^\mathbb{N}$ onto itself which maps P one-to-one and onto Q.

III.7.2. Show that if $P = [T]$ where T is a finite-branching, Σ^0_1 tree, then P is highly computable in $\mathbf{0}'$.

III.7.3. Show that there is a computably continuous map on $\mathbb{N}^\mathbb{N}$ such that the image is not even a closed set.

III.8 Thin and minimal classes

A Π^0_1 class P is said to be *thin* if, for every Π^0_1 subclass Q of P, there is a clopen set U such that $Q = U \cap P$. An infinite Π^0_1 class C is said to be *minimal* if every Π^0_1 subclass Q of C is either finite or cofinite in C. Thus the notion of a minimal Π^0_1 class is the analog of the notion of a co-maximal Π^0_1 subset of ω. In particular, if C is a co-maximal set, then the class of subsets of C containing either one or no elements is an example of a minimal Π^0_1 class which is not thin. (See the exercises.)

We have seen that any isolated element of a computably bounded Π^0_1 class must be computable. For a thin Π^0_1 class, the converse also holds. (Exercise 2). It follows that a perfect thin class has no computable members.

The first construction of a thin Π^0_1 class is due to Martin and Pour-El [144].

Theorem III.8.1. *(Martin–Pour-El) There exists a perfect thin Π^0_1 class with no computable member.*

Proof. Let $P_e = [T_e]$ be the e'th Π^0_1 class as in Theorem III.3.3 and let ϕ_e be the e'th partial recursive function from ω into $\{0,1\}$. We will construct a recursive tree S with corresponding Π^0_1 class $P = [S]$ and a homeomorphism F from $\{0,1\}^\omega$ onto P. F will be constructed by means of a map $f : \{0,1\}^{<\omega} \to S$ such that $\sigma \prec \tau \iff f(\sigma) \prec f(\tau)$; then for $x \in \{0,1\}^\omega$, $F(x) = \cup_n f(x\lceil n)$.

To ensure that P is thin, we construct f to satisfy the following requirement for each e.

R_e: For each $\sigma \in \{0,1\}^{e+1}$, if $f(\sigma) \in T_e$, then $(\forall \tau)(\sigma \prec \tau \to f(\tau) \in T_e)$.

To see that this makes P thin, let $U = \cup\{I(f(\sigma) : |\sigma| = e+1 \ \& \ f(\sigma) \in T_e\}$ and observe that if $P_e \subset P$, then $P_e = P \cap U$.

The map f is defined in uniformly computable stages f_s, beginning with f_0 as the identity function.

(Stage $s+1$): Look for $e < s$ and $\sigma \in \{0,1\}^{e+1}$ and $\tau \succ \sigma$ with $|\tau \leq s+1$ such that $f_s(\sigma) \in T_e$, but $f_s(\tau) \notin T_e$. If such e, σ and τ exist, then we take the least such e and the lexicographically least σ and τ for that e. Then we let $f_{s+1}(\sigma) = f_s(\tau)$ and in general, for any ρ we let

$f_{s+1}(\sigma^\frown\rho) = f_s(\tau^\frown\rho)$ and

$f_{s+1}(\rho) = f_s(\rho)$ for ρ incomparable with σ.

If no such e, σ and τ exist, then we just let $f_{s+1} = f_s$.

It is easy to see by induction on $|\sigma|$ that for each σ, $f_s(\sigma)$ converges to a limit $f(\sigma)$. Then we see by induction on e that the requirements R_e are satisfied. □

Countable thin classes were studied in [25].

The connection between thin and minimal classes is given by the following.

Theorem III.8.2. *(C-D-J-S) The following are equivalent for any Π^0_1 class P.*

(a) P is thin and $D(P)$ is a singleton.

(b) P is minimal and has a non-computable member.

Proof. (a) $\to$ (b): Suppose that P is thin and that $D(P) = \{A\}$. Then A is non-computable by Exercise 2. Let Q be a Π^0_1 class such that $Q \subset P$. Then $Q = U \cap P$ for some clopen U, so that $P \setminus Q$ is also a Π^0_1 class. If both Q and $P \setminus Q$ were infinite, then both would contain limit points, contradicting the assumption that P has only one limit point.

(b) $\to$ (a): Suppose that P is minimal and has a nonrecursive member A. Then $A \in D(P)$ by Corollary III.2.16. For any $B \neq A$ in P, let U be an interval such that $A \in U$ and $B \notin U$. Then $U \cap P$ is infinite, and therefore $P \setminus U$ must be finite, which implies that $B \notin D(P)$. Therefore $D(P) = \{A\}$. Now let Q be any Π^0_1 subclass of P. For any $B \neq A$ in P, let $U(B)$ be an interval such that $U(B) \cap P = \{B\}$. Since P is minimal, there are two cases.

Case 1: Q is finite. Then $Q = P \cap \cup_{B \in Q} U(B)$.

Case 2: $P \setminus Q$ is finite. Then $Q = P \cap [2^\omega \setminus \bigcup_{B \in P\setminus Q} U(B)]$. □

We next construct a minimal, thin class.

Theorem III.8.3 ([25]). *There exists a minimal, thin Π^0_1 class P; furthermore, P is decidable.*

Proof. Let $P_e = [T_e]$ be the e'th Π^0_1 class as above. We will construct a set A, a sequence $\tau_0 \prec \tau_1 \prec \dots$ of strings with $A = \cup_i \tau_i$ and a Π^0_1 class P such that

(1) $D(P) = \{A\}$.

(2) For any e and any extension $B \in P$ of τ_e, if $A \in [T_e]$, then $B \in [T_e]$.

Properties (1) and (2) imply that P is minimal, by the following argument. Note first that, for all $B \in P$, if $B \neq A$, then the set B is isolated in P by property (1), so that there exists a clopen set $U(B)$ such that $P \cap U(B) = \{B\}$. Suppose now that $[T_e]$ is a subset of P. Then there are two cases.

(Case 1) If $A \notin P_e$, then, since A is the only limit point of P and every infinite class has a limit point, it follows that Pe is finite.

(Case 2) If $A \in P_e$, then it follows from property (2) that every extension of τ_e is also in T_e. Now the set $P \setminus I(\tau_e)$ of paths through T which are not extensions of τ_e is a closed set and has no limit point (since A is the only limit point of P). Thus $P \setminus I(\tau_e)$ is finite and, since $P \setminus P_e \subset P \setminus I(\tau_e)$, $P \setminus [T_e]$ is also finite.

It also follows from properties (1) and (2) that A is not computable. To see this, suppose by way of contradiction that A were computable. Then $\{A\}$ would be a Π^0_1 class, so that $\{A\} = P_e$ for some e. Now by property (2), we have $P \cap I(\tau_e) \subset P_e$, which makes A isolated in P, contradicting property (1). It now follows from Theorem III.8.2 that P is thin.

It remains to construct the set P. The construction will proceed in stages. At stage s we will have, for $e \leq s$, finite sequences τ^s_e such that, for all $e < s$, $\tau^s_e{}^\frown 1 \prec \tau^s_{e+1}$. The construction will ensure the existence of the limits $\tau_e = \lim_s \tau^s_e$ for each e. The point A will the union of $\{\tau_e : e \in \omega\}$. At the same time we will be defining a sequence $k(0) < k(1) < \cdots$ so that $s \leq k(s)$ and constructing a computable tree T in stages T^s. At stage s, we will have decided whether each finite sequence of length $k(s)$ is in T. This will ensure that T is computable. We will always put $\sigma^\frown 0$ into T whenever σ is in T. This will imply that $x_e = \tau_e{}^\frown 0^\omega \in P$ for all e; since A is non-computable and therefore infinite, there are infinitely many distinct x_e, so that $A \in D(P)$. This also implies that P is decidable, that is, T has no dead ends. To obtain $D(P) = \{A\}$, we do the construction so that, whenever $\tau^{s+1}_e = \tau^s_e$, then there are no new branches added below τ^s_e. Thus once we have reached a stage s such that $\tau^s_e = \tau_e$ and counted the number n of distinct branches of T^s not passing through τ^s_e, then we know that all but n points of P will pass through τ_e. Now suppose that some path B is in $D(P)$ but is different from A. Just let k be the least number such that $A(k-1) \neq B(k-1)$ and let e be least such that $A\lceil k \subset \tau_e$. Then no extension of $B\lceil k$ passes through τ_e. It follows that the set of extensions of $B\lceil k$ in P is finite, so that B is isolated in P. This will take care of property (1).

In order to satisfy property (2), we want the construction to ensure the following requirements for each e.

(R_e): If $\tau_e \in T_e$, then every extension of τ_e which is in T is also in T_e.

We begin the construction by setting $k(0) = 1$, putting (0) and (1) in T^0 and setting $\tau^0_0 = \emptyset$.

Now suppose we have completed the construction as far as stage s. At stage $s+1$, we look for the least number $e \leq s$ such that $\tau^s_e \in T^s \cap T_e$ but τ^s_e has some extension $\tau \in T^s$ which is not in T_e. If such an e exists, then we act on requirement R_e at stage $s+1$, as follows. Let τ be the lexicographically least extension of τ_e^s of length $k(s)$ which is in $T^s \setminus T_e$. Then let $\tau_e^{s+1} = \tau$. For $i < e$, let $\tau_i^{s+1} = \tau_i^s$. For $i \leq s - e + 1$, let $\tau_{e+i}^{s+1} = \tau^\frown 1^i$. Now let $k(s+1) = k(s) + s - e + 1$ and define T^{s+1} to be the union of T^s with the set of the following strings. First, for any $\sigma \in T^s$ of length $k(s)$ and any $i \leq s - e + 1$, the extension $\sigma^\frown 0^i$. Next, for any $i \leq s - e + 1$, and any $j \leq s - e + 1 - i$, the extension $\tau^\frown(1^i)^\frown(0^j)$.

If there is no such e, just let $\tau_i^{s+1} = \tau_i^s$ for all $i \leq s$ and let $\tau_{s+1}^{s+1} = \tau_s^{s\frown}1$. Let $k(s+1) = k(s)+1$ and let T^{s+1} be the union of T^s with the set of all strings $\sigma^\frown 0$ where $\sigma \in T^s$ and the string τ_{s+1}^{s+1}.

Observe that in either case, we have extended all nodes in T^s by at least one node in T^{s+1}, so that T will have no dead ends.

Claim 1: For every e, the sequence τ_e^s converges to some limit τ_e.

Proof of Claim 1: This is by induction on e. Suppose therefore that Claim 1 is proved for all $i < e$ and that we have reached a stage s such that $\tau_i^s = \tau_i$ for all $i < e$. There are two cases. If $\tau_e^r = \tau_e^s$ for all $r > s$, then the limit $\tau_e = \tau_e^s$ and we are done. Otherwise, let $r > s$ be least such that $\tau_e^r \neq \tau_e^s$. It follows from the construction that we must have $\tau_e^r \notin T_e$. After stage r, there is no way that τ_e^t can be different from τ_e^r. Thus the limit $\tau_e = \tau_e^r$. □

Since $\tau_e^s \prec \tau_{e+1}^s$ for all s and e, it follows that $\tau_e \prec \tau_{e+1}$ for all e. Thus we can define the set A to have characteristic function $\cup_e \tau_e$.

Claim 2: For any e and any s, if $\tau_e^{s+1} = \tau_e^s$, then there are no new branches in $T^{s+1} \setminus T^s$ which do not pass through τ_e^s.

Claim 2 is immediate from the construction. □

It now follows that, for any e, all but finitely many points of P pass through τ_e. It follows from the discussion preceding the construction that $D(P) = \{A\}$.

Claim 3 If $\tau_e \in T_e$, then every extension of τ_e which is in T is also in T_e.

Proof of Claim 3: Suppose by way of contradiction that $\tau_e \in T_e$ but that τ_e has some extension $\tau \in T$ such that $\tau \notin T_e$. Consider a stage $s > lh(\tau)$ such that $\tau_i^s = \tau_i$ for all $i \leq e$ and $\tau \in T^s$. Then at stage $s+1$, we have $\tau \in T^s$ so the construction dictates that we act on requirement R_e and make $\tau_e^{s+1} = \tau$, contradicting the assumption that $\tau_e^s = \tau_e$. □

This establishes property (1) and (2) above and thus completes the proof □

We close the section with two important properties of thin classes.

Lemma III.8.4. *Let $Q \subseteq \{0,1\}^{\mathbb{N}}$ be a thin Π_1^0 class. Then*

1. *Every Π_1^0 $P \subseteq Q$ is thin;*
2. *For any computable $\Phi : \{0,1\}^{\mathbb{N}} \to \{0,1\}^{\mathbb{N}}$, $\Phi[Q]$ is thin.*

Proof. (1) is trivial. For (2), suppose that $R \subseteq \Phi[Q]$. Then $\Phi^{-1}[R]$ is a Π_1^0 subclass of Q and hence $\Phi^{-1}[R] = U \cap Q$ for some clopen U. Thus $R = \Phi[U] \cap Q$ and $\Phi[U]$ is clopen since $\{0,1\}^{\mathbb{N}}$ is compact. □

Exercises

III.8.1. Show that for any maximal c.e. set A, the class containing the empty set together with all singletons $\{m\}$ where $m \notin A$, is an example of a minimal Π^0_1 class which is not thin.

III.8.2. Show that any computable element of a thin Π^0_1 class must be isolated.

III.9 Mathematical Logic

In this section, we set up the framework for the representation and application of Π^0_1 classes, using the area of logical theories. There is a very close connection between Π^0_1 classes and logical theories and we will return to this topic in later sections as we develop the theory of Π^0_1 classes.

Recall the arbitrary first-order effective language $\mathcal{L}$ described in Chapter I. Let $Sent(\mathcal{L})$ be the set of sentences of $\mathcal{L}$. For any subset Γ of $Sent(\mathcal{L})$, the set $Con(\Gamma)$ of consequences of Γ is the closure of Γ under logical deduction and the set $Ref(\Gamma)$ of refutations of Γ is the set of negations of the consequences of Γ. A subset Γ of $Sent(\mathcal{L})$ is a first order logical theory if Γ is closed under logical deduction. Σ is said to be a set of axioms for Γ if $\Gamma = Con(\Sigma)$ and Γ is (computably) *axiomatizable* if Γ has a computable set of axioms; the modifier (computably) will normally be omitted. It is not hard to see that Γ is axiomatizable if and only if Γ is computably enumerable. A theory is said to be *decidable* if it is computable. It follows from Post's Theorem that a complete axiomatizable theory is decidable.

The usual idea for the application of Π^0_1 classes is that the set of solutions to some computable problem should correspond to a Π^0_1 class. The problem here is to find a complete consistent extension of a given computable or axiomatizable theory. The classical result here is the Completeness Theorem of Gödel that any consistent theory has an extension to a complete consistent theory and follows as usual from Zorn's Lemma. The other fundamental result is the Compactness Theorem, which states that if all finite subsets of a theory Γ have a model, then Γ has a model; this follows from König's Lemma.

Shoenfield observed in [189] that in general, the family of complete, consistent extensions of an axiomatizable first order theory can be represented by a Π^0_1 class. Now the undecidability of arithmetic was discovered by Turing and Church (independently) in 1936, following soon after Gödel's incompleteness theorem. This result stated that there is no decidable complete consistent extension of Peano Arithmetic and also showed, in our terminology, that there is a nonempty c. b. Π^0_1 class with no computable member. This led to the definition of an *essentially undecidable* theory as a theory with no decidable complete consistent extension. Now if Σ is any consistent complete extension of a theory Γ, then Σ separates the set T of consequences of Γ from the set R of refutations of Γ. A theory is said to be *separable* if the consequences and refutations can be separated by a computable set and is otherwise said to be *inseparable*. Rosser[180] observed also in 1936 that Peano arithmetic is an inseparable theory and that any inseparable theory is essentially undecidable. This also provided the first example of computably inseparable c. e. sets. Ehrenfeucht showed in 1961 [69] that there are separable theories which are essential undecidable. His construction, using theories of propositional calculus, also shows that every Π^0_1 class may be represented as the set of complete consistent extensions of a theory. A complete, consistent extension of Peano Arithmetic is of course just the the theory of some (possibly non-standard) model of Peano arithmetic. The theory of Peano arithmetic is of great interest in mathematical logic, due in part to the connection with Gödel's Incompleteness Theorem, and has been developed in the papers of Jockusch and Soare [99, 98], Knight [111], Marker [142] and many others.

Theorem III.9.1. *(Shoenfield [189] For any c. e. theory Γ of an effective language $\mathcal{L}$, both the class of consistent extensions of Γ and the class of complete consistent extensions of Γ can be represented as Π^0_1 classes. Furthermore, if Γ is a decidable theory, then these classes can be represented by computable trees with no dead ends.*

Proof. Let $\mathcal{L}$ be an effective first-order language and let $S = Sent(\mathcal{L})$ have an effective enumeration as $\gamma_0, \gamma_1, \ldots$. Then the sentence γ_i may be identified with the number i, so that a theory Γ is represented by the set $\{i : \gamma_i \in \Gamma\}$, and a class of theories is represented by a class in $\{0,1\}^\omega$. Let $\Gamma \vdash_s \gamma_i$ be the computable relation of which means that there is a proof of γ_i from Γ of length s. Then the class $P(\Gamma)$ of complete consistent extensions of Γ may be represented by the set of infinite paths through the computable tree T defined so that for any $\sigma = (\sigma(0), \ldots, \sigma(n-1))$, σ is in T if and only if the following conditions hold.

(1) For any $i < n$, if $\Gamma \vdash_n \gamma_i$, then $\sigma(i) = 1$.

(2) For any $i, j < n$, if $\Gamma \vdash_n \gamma_i \rightarrow \gamma_j$ and $\sigma(i) = 1$, then $\sigma(j) = 1$.

(3) For any $i, j, k < n$, if $\gamma_k = (\gamma_i \ \& \ \gamma_j)$, $\sigma(i) = 1$ and $\sigma(j) = 1$, then $\sigma(k) = 1$.

(4) For any $i, j < n$, if $\sigma(i) = 1$ and $\gamma_j = \neg\gamma_i$, then $\sigma(j) = 0$.

(5) For any $i, j < n$, if $\gamma_j = \neg\gamma_i$, then either $\sigma(i) = 1$ or $\sigma(j) = 1$.

Let x be an infinite path through T and let $\Delta = \{\gamma_i : x(i) = 1\}$. Condition (1) ensures that $\Gamma \subseteq \Delta$, while conditions (1), (2), and (3) ensure that Δ is a theory. Condition (4) ensures that Δ is consistent and condition (5) ensures that Δ is complete. To represent the class of consistent extensions of Γ, simply omit the final clause (5).

If Γ is decidable, then in each case we can modify the clauses given above as follows to get a tree S with no dead ends which has the same class of infinite paths. First, combine the first three clauses into the statement:

(1′) : For any $k < n$, if $\Gamma \vdash \wedge\{\gamma_i : i < n \ \& \ \sigma(i) = 1\} \rightarrow \gamma_k$, then $\sigma(k) = 1$.

Next, replace clause (4) with

(4′) It is not the case that $\Gamma \vdash [\wedge\{\gamma_i : i < n \;\&\; \sigma(i) = 1\} \to (\gamma_0 \& \neg\gamma_0)]$.

It follows that for any $\sigma \in S$, $\Gamma \cup \{\gamma_i : i < |\sigma| \;\&\; \sigma(i) = 1\} \cup \{\neg\gamma_i : i < n \;\&\; \sigma(i) = 0\}$ is consistent and therefore has an extension to a complete consistent theory $\Gamma(\sigma)$ which will be represented by an extension of σ. Thus S has no dead ends. □

We can now apply Theorem III.2.15 to logical theories.

Theorem III.9.2. *For any consistent, axiomatizable first-order theory* Γ*:*

(i) Γ *has a complete consistent extension which is computable in* $\mathbf{0}'$*.*

(ii) If Γ *is decidable, then* Γ *has a complete, consistent, decidable extension.*

Next we turn to the other direction of our correspondence, that is, representing an arbitrary Π^0_1 class by the set of complete consistent extensions of some axiomatizable theory.

Theorem III.9.3. *Any c. b.* Π^0_1 *class* P *may be represented by the set of complete, consistent extensions of an axiomatizable theory* Γ *in propositional logic. Furthermore, if* P *is a decidable* Π^0_1 *class, then* Γ *may be taken to be a decidable theory.*

Proof. We give the proof due to Ehrenfeucht [69]. Let the language $\mathcal{L}$ consist of a countable sequence $A_0, A_1, \ldots$ of propositional variables. For any $x \in \{0,1\}^{\mathbb{N}}$, we can define a complete consistent theory $\Delta(x)$ for $\mathcal{L}$ to be $Con(\{C_i : i \in \omega\})$, where $C_i = A_i$ if $x(i) = 1$ and $C_i = \neg A_i$ if $x(i) = 0$. It is clear that every complete consistent theory of $\mathcal{L}$ is one of these. Thus for any Π^0_1 class $P \subseteq \{0,1\}^{\mathbb{N}}$, we want a theory Γ such that $\Delta(P) = \{\Delta(x) : x \in P\}$ is the set of complete, consistent extensions of Γ.

For each finite sequence $\sigma = (\sigma(0), \ldots, \sigma(n-1))$, let $P_\sigma = C_0 \wedge C_1 \wedge \cdots \wedge C_{n-1}$, where $C_i = A_i$ if $\sigma(i) = 1$ and $C_i = \neg A_i$ if $\sigma(i) = 0$. Let the binary tree T be given such that $P = [T]$ and define the theory $\Gamma(T)$ to consist of all $P_\sigma \to A_n$ such that $\sigma \in T$ and $\sigma^\frown 0 \notin T$ and all $P_\sigma \to \neg A_n$ such that $\sigma \in T$ and $\sigma^\frown 1 \notin T$, where $|\sigma| = n$. We claim that $\Delta(P)$ is in fact equal to the set of complete consistent extensions of $\Gamma(T)$. Suppose first that $x \in P$ and let $Con(\{C_i : i \in \omega\}) = \Delta(x)$. Now any $\gamma \in \Gamma(T)$ is of the form $P_\sigma \to \pm A_i$ for some $\sigma \in T$; say that $|\sigma| = n$. There are several cases. If $\sigma \neq x\lceil n$, then $\Delta(x) \vdash \neg P_\sigma$, so that we always have $\Delta(x) \vdash P_\sigma \to \pm A_n$. Thus we may suppose that $\sigma = x\lceil n$. If $\sigma^\frown 0 \notin T$, then of course $x(n) = 1$, so that $C_n = A_n \in \Delta(x)$ and therefore $\Delta(x) \vdash P_\sigma \to A_n$. Similarly, if $\sigma^\frown 1 \notin T$, then $\Delta(x) \vdash P_\sigma \to \neg A_n$. Thus $\Delta(x)$ is a complete consistent extension of $\Gamma(T)$. On the other hand, let Δ be a complete consistent extension of $\Gamma(T)$. Then, for each i, we have either $\Delta \vdash A_i$ or $\Delta \vdash \neg A_i$; let $C_i = A_i$ if $A_i \in \Delta$ and $C_i = \neg A_i$ otherwise. Define $x \in \{0,1\}^\omega$ so that $x(i) = 1$ if and only if $\Delta \vdash A_i$. Then clearly $\Delta = \Delta(x)$. It remains to be shown that $x \in P$. Now if $x \notin P$, then there is some n such that $\sigma = x\lceil n+1 \notin T$ and $x\lceil n \in T$. Then $P_\sigma = C_0 \wedge \cdots \wedge C_{n-1}$, so that $\Delta \vdash P_\sigma$, and

$P_\sigma \to \neg C_i \in \Gamma(T)$, so that Δ is not consistent with $\Gamma(T)$. This contradiction proves that $\Delta = \Delta(x)$.

Now suppose that P is a decidable class, so that the tree T has no dead ends. Let a sentence $\gamma = \gamma(A_0, \ldots, A_{n-1})$ of the language $\mathcal{L}$ be given. We claim that $\Gamma(T) \vdash \gamma$ if and only if $\bigwedge\{P_\sigma \vdash \gamma : \sigma \in T \ \& \ |\sigma| = n\}$, that is, if and only if $P_\sigma \vdash \gamma$ for all $\sigma \in T$ with $|\sigma| = n$. This claim clearly implies that $\Gamma(T)$ is decidable.

We argue by the contrapositive. Suppose first that $\Gamma(T) \not\vdash \gamma$. Then there is some $x \in [T]$ such that $\Delta(x) \vdash \neg\gamma$. Since γ only depends on $A_0, \ldots, A_{n-1}$, it follows that $P_\tau \vdash \neg\gamma$, where $\tau = x\lceil n \in T$. Thus $P_\tau \vdash \gamma$ is clearly false, making it also false that $\bigwedge\{P_\sigma \vdash \gamma : \sigma \in T \ \& \ |\sigma| = n\}$. Suppose next that $\bigwedge\{P_\sigma \vdash \gamma : \sigma \in T \ \& \ |\sigma| = n\}$ is false. Then $P_\tau \vdash \gamma$ is false for some fixed $\tau \in T$, which means that $P_\tau \vdash \neg\gamma$ (since γ depends only on $A_0, \ldots, A_{n-1}$). Since T has no dead ends, there is some $x \in P$ such that $\tau \prec x$ and therefore $\Delta(x) \vdash \neg\gamma$ and therefore $\Gamma(T) \not\vdash \gamma$. □

This proof can be adapted to first order logic; see Exercise 1 below.

This representation theorem has the following corollary.

Theorem III.9.4. *There is a consistent axiomatizable first-order theory Gamma which has no computable consistent complete extension.*

The perfect thin class constructed by Martin and Pour-El (see Theorem III.8.1) was designed to produce a certain type of axiomatizable theory. A theory Γ is said to be *Martin–Pour-El* if every axiomatizable extension of Γ is generated by a single proposition. It follows from the proof of the next theorem that an axiomatizable theory Γ is a Martin–Pour-El theory if and only if the class of complete consistent extensions of Γ is a thin Π^0_1 class.

Theorem III.9.5. *There exists an axiomatizable, essentially undecidable theory T such that each axiomatizable extension of T is a finite extension of T.*

Proof. Let P be a perfect thin class with no computable members and let the axiomatizable theory T be given by Theorem III.9.3 such that P represents the family of complete consistent extensions of T. Now suppose that Γ is an axiomatizable extension of T. Then the family of complete consistent extensions of Γ is represented by a Π^0_1 subclass Q of P. Since P is thin, there is a clopen set U such that $Q = U \cap P$. Let $U = I(\sigma_1) \cup I(\sigma_2) \cup \cdots \cup I(\sigma_n)$ for some distinct finite sequences σ_i all having the same length k and let $\phi_i = P_{\sigma_i}$ as in the proof of Theorem III.9.3. Then the complete consistent extensions of Γ are exactly those complete consistent extensions of Δ which satisfy $P_1 \vee \cdots \vee P_n$. □

Exercises

III.9.1. Show that any c. b. Π^0_1 class may be represented as the set of complete consistent extensions of a first order logical theory in the language of one binary relation R Jockusch and Soare in ([99], p. 54). Hint: the underlying axioms assert that R is an equivalence relation and that, for any n, there are either one or two equivalence classes consisting of exactly n members. The propositional statement A_n in the proof above is replaced by the statement that there is exactly one equivalence class with n elements.

III.9.2. Let the propositional language $\mathcal{L}$ have variables $A_0, A_1, \ldots$. Variables and their negations $\neg A_i$ are said to be *literals*. Show that a consistent theory Γ for $\mathcal{L}$ is c. e. if and only if the set $C(\Gamma)$ of conjunctions of literals, consistent with Γ, is co-c. e. and that Γ is decidable if and only if $C(\Gamma)$ is computable. Show that this is not true for the set of *literals* consistent with Γ.

Chapter IV

Members of Π^0_1 Classes

In this chapter, we study the complexity of members of Π^0_1 classes. We present some "basis theorems" and "anti-basis theorems". The class $\Gamma \subset \mathbb{N}^{\mathbb{N}}$ is said to be a *basis* for a family Θ of subclasses of $\mathbb{N}^{\mathbb{N}}$ if every nonempty class from Θ has a member from Γ. For example, the class Δ^0_0 of computable reals is a basis for the family of open subclasses of $\mathbb{N}^{\mathbb{N}}$. This is an example of a "basis theorem". We have already given the simple positive result V.2.3 that the class $P = [T]$ of infinite paths through the tree T contains a member computable from $Ext(T)$, the set of nodes of T which have an infinite extension in P. Recall in particular, that if T is computably bounded, then P has a member computable in $\mathbf{0}'$ and if P is decidable, then P has a computable element. We will give several more basis results, including the Low Basis Theorem of Jockusch and Soare [99] is given.

On the other hand, the class of computable reals is not a basis for the family of closed subclasses of $\mathbb{N}^{\mathbb{N}}$ since every singleton is a closed class. This is an example of an "anti-basis theorem". One result given is that the set of Boolean combinations of c. e. sets is not a basis for the c. b. Π^0_1 classes. Any c. b. Π^0_1 class with no computable members is perfect and has a set of continuum many mutually Turing incomparable elements [99]. There is a c. b. Π^0_1 class of positive measure which has no computable element.

IV.1 Basis theorems

One of the most cited results in the theory of Π^0_1 classes is the Low Basis Theorem of Jockusch and Soare [99]. We will introduce the method of *forcing with* Π^0_1 *classes* in connection with this theorem. First we consider the notion a *generic* real.

Definition IV.1.1. *An element $x \in \mathbb{N}^{\mathbb{N}}$ is said to be* 1-generic *if, for every Π^0_1 class P, there exists n such that either $I(x \restriction n) \subset P$ or $I(x \restriction n) \cap P = \emptyset$.*

The existence of a generic real is obtained by forcing. Let $P = [T]$. The idea here is that the finite sequence $\sigma = x\lceil n$ "forces" $x \notin P$ if $\sigma \notin Ext(T)$, that is, $\sigma \Vdash x \notin P$ if *no* extension of σ is in P, and similarly $\sigma \Vdash x \in P$ if $I(\sigma) \subseteq P$, that is, if *every* extension of σ is in P. Then we write $\Vdash x \in P$ ($\Vdash x \notin P$) if there is some σ such that $\sigma \Vdash x \in P$ ($\sigma \Vdash x \notin P$). With this notation, x is 1-generic if, for every Π^0_1 class P, either $\Vdash x \in P$ or $\Vdash x \notin P$, or equivalently $x \in P$ implies $\Vdash x \in P$. (See Section III.6 of Hinman [87] for a presentation of arithmetical forcing.)

A subset D of $\{0,1\}^*$ is said to be *dense* if, for any σ, there exists a $\tau \succ \sigma$ such that $\tau \in D$. Now let $\mathcal{D} = \{D_i : i \in I\}$ be a family of dense sets. The element x of $\{0,1\}^{\mathbb{N}}$ is said to be *$\mathcal{D}$-generic* if, for each i, $x \upharpoonright n \in D_i$ for some n.

The standard forcing theorem shows that any countable family of dense sets possesses a generic set. We observe that this is an effective version of the Baire Category Theorem.

Lemma IV.1.2. *If $\mathcal{D} = \{D_i : i < \omega\}$ is a sequence of subsets of $\mathbb{N}^*$ uniformly computable in $\mathbf{0}'$, there exists a $\mathcal{D}$-generic $x \leq_T \mathbf{0}'$.*

Proof. Let m be the least such that $\sigma_m \in D_0$ and let $\tau_0 = \sigma_m$. Then for each n, find the least m such that σ_m is a proper extension of τ_n and let $\tau_{n+1} = \sigma_m$. Then $x = \cup_n \tau_n$ will be the desired generic real. This construction is computable using an oracle for the sequence $\mathcal{D}$. □

Recall that by Theorem III.3.3, there is a uniformly primitive recursive enumeration of trees T_e such that $P_e = [T_e]$ is the eth Π^0_1 class. Then the standard family of dense sets is now $D_i = \{\sigma : \sigma \notin T_i \ \vee \ (\forall \tau \succeq \sigma)\tau \in T_i\}$. Observe each D_i is dense and that the sequence of sets is uniformly Π^0_1. The element $x \in \{0,1\}^{\mathbb{N}}$ is 1-generic if it is generic for this sequence of dense sets. Then the remarks above imply the existence of a 1-generic real. The crucial property of a 1-generic real is given by the following well-known fact.

Theorem IV.1.3. *For any 1-generic $x \in \{0,1\}^{\mathbb{N}}$, if $x \leq_T \mathbf{0}'$, then $x' \leq_T x \oplus \mathbf{0}'$.*

Proof. Let x be 1-generic. For each e, let the Π^0_1 class $P_e = \{y : \phi^y_e(e)\uparrow\}$, so that $P_e = [U_e]$, where $\sigma \in U_e \iff \phi^\sigma_e(e)\uparrow$. Thus $e \in x' \iff x \notin P_e$. If $e \in x'$, then of course there is some n such that $x\lceil n \notin U_e$. Since x is 1-generic, if $e \notin x'$, then there is some n such that $(\forall \tau \succeq x\lceil n)\tau \in U_e$. Let $f(e)$ be the least n such that either $x\lceil n \notin U_e$ or $(\forall \tau \succeq x\lceil n)\tau \in U_e$. Then f is computable in $x \oplus \mathbf{0}'$. But then we have $e \in x'$ if and only if $x\lceil f(n) \notin U_e$, so that x' is also computable in $x \oplus \mathbf{0}'$. □

It follows that if a 1-generic real x is computable in $\mathbf{0}'$, then $x' = \mathbf{0}'$, that is, x is low. For the low basis theorem of Jockusch and Soare, a modification of this argument is used.

Theorem IV.1.4 (Low Basis Theorem). *(a) Every nonempty c. b. Π^0_1 class P contains a member of low degree.*

(b) There is a low degree $\mathbf{a}$ such that every nonempty r. b. Π^0_1 class contains a member of degree $\leq \mathbf{a}$.

Proof. (a) We may assume that $P \subset \{0,1\}^\omega$. Let $P = [T]$ and, as above, let $\sigma \in U_e \iff \phi_e^\sigma(e)\uparrow$. We will define, computably in $\mathbf{0}'$, a sequence $T = S_0 \supset S_1 \supset S_2 \supset \cdots$ of infinite subtrees of T and show that any member x of $\cap_e[S_e]$ has low degree. There are two cases in the definition of S_{e+1}.

1. If $S_e \cap U_e$ is finite, then $S_{e+1} = S_e$.

2. If $S_e \cap U_e$ is infinite, then $S_{e+1} = S_e \cap U_e$.

Observe that this construction is computable in $\mathbf{0}'$, in that there is a function $f \leq_T \mathbf{0}'$ such that $S_e = U_{f(e)}$ for each e. This is because the determination of whether $S_e \cap U_e$ is finite can be made using a $\mathbf{0}'$ oracle. Since each S_e is infinite by the construction, it follows that each $[S_e]$ is nonempty, so that $\cap_e[S_e]$ is nonempty. Now suppose that $x \in \cap_e[S_e]$. Then, for any e, we have $e \in x'$ if and only if $x \notin [U_e]$, and it follows from the construction that $x \notin [U_e]$ if and only if $S_e \cap U_e$ is finite. It follows by the observation above that $x' \leq_T 0'$.

(b) It suffices to prove the result for classes in $\{0,1\}^{\mathbb{N}}$. Let $P_e = [T_e]$ be an effective enumeration of the Π^0_1 in $\{0,1\}^{\mathbb{N}}$ and let p_e be the e'th prime number. Let the tree T be the amalgamation of the nonempty Π^0_1 classes, in the following sense. Let $\sigma \in T$ if, for each e such that P_e is nonempty and each k such that $p_e^k < |\sigma|$, $(\sigma(p_e), \sigma(p_e^2), \dots, \sigma(p_e^k)) \in T_e$. Since we can test computably in $\mathbf{0}'$ whether P_e is nonempty, the tree T is computable in $\mathbf{0}'$. Thus the construction above can be carried out to produce a member x of $P = [T]$ of low degree. Then any nonempty class P_e has a member $(x(p_e), x(p_e^2), \dots)$ computable in x. $\square$

The same technique can be used to prove other basis results. For example, a generalization shows that if the c. b. class P contains no computable member, then for any degree $\mathbf{b}$, P has a member A of degree $\mathbf{a}$ such that $\mathbf{a} \oplus \mathbf{0}' = \mathbf{a}' = \mathbf{b}'$. It follows (for $\mathbf{b} = \mathbf{0}'$) that any Π^0_1 class P has a member A of degree $\mathbf{a}$ such that $\mathbf{a} \oplus \mathbf{0}' = \mathbf{a}' = \mathbf{0}''$. Now a degree $\mathbf{a}$ is said to be *high* if $\mathbf{a} \leq \mathbf{0}'$ and $\mathbf{a}' = \mathbf{0}''$. It will be shown later that not every r. b. Π^0_1 class contains a member of high degree.

The following result is from Jockusch and Soare [98].

Theorem IV.1.5 (Jockusch and Soare)**.** *Every nonempty c. b. Π^0_1 class P contains a member of hyperimmune-free degree, that is, contains an almost computable member.*

Proof. We sketch the proof indicated in Soare [198] (p. 109). Let $P = [T]$, where T is an infinite computable binary tree. Recall that A is hyperimmune-free if every function f computable in A is majorized by some computable function. Thus we want to find $A \in P$ such that ϕ_e^A whenever total, is majorized by a computable function. We define the decreasing sequence S_e of computable subtrees of T beginning with $S_0 = T$. Then for each e and i, let $U_e^i = \{\sigma \in S_e : \phi_e^\sigma(i)\uparrow\}$. There are again two cases in the definition of S_{e+1}.

1. If U_e^i is finite for every i, let $S_{e+1} = S_e$.

2. If U^i_e is infinite for some i, choose such an i and let $S_{e+1} = S_e \cap U^i_e$.

Suppose now that $A \in \cap_e S_e$ and that $f = \phi^A_e$ is total. If the second case applied in the definition of S_{e+1}, then $\phi^A_e(i) \uparrow$, so that ϕ^A_e is not total. If the first case applied, then Φ^A_e is defined for all $A \in [S_{e+1}]$. Since $[S_e] \subseteq P$, $[S_e]$ is computably bounded and it follows from Theorem III.4.8 that $\Phi_e[S_e]$ is also bounded by some computable function h, which thus majorizes Φ^A_e. □

We will show in the next section that a nonempty Π^0_1 class $P \subseteq \{0,1\}^{\mathbb{N}}$ need not contain any sets which are c. e. or co-c. e.. However, it always contains elements which corresponds to c. e. and co-c. e. Dedekind cuts.

Theorem IV.1.6. *Any nonempty Π^0_1 class $P \subseteq \{0,1\}^{\mathbb{N}}$ contains elements x and y such that the Dedekind cut $L(r_x)$ is c. e. and the Dedekind cut $L(r_y)$ is co-c. e..*

Proof. Let $P = [T]$ where T is a computable tree. Let x be the "leftmost" element of P under the lexicographic order. For each n, let σ_n be the leftmost node in $T \cap \{0,1\}^n$ and let $q_n = r_\sigma = \sum_{i=0}^n \sigma(i)2^{-i-1}$. It is clear that $\{q_n\}_{n<\omega}$ is an increasing sequence and hence has limit r such that $L(r)$ is c. e. set by Proposition 2.II.4.3. We claim that $x = lim_n \sigma_n$ converges to a path in P and that $r = r_x$. For each n, $\sigma_n(0) \leq \sigma_{n+1}(0)$, since otherwise $\sigma_{n+1}\lceil n <_{lex} \sigma_n$. Thus the sequence $\sigma_n(0)$ converges to some $x(0)$. Once $\sigma_n(0), \ldots, \sigma_n(k-1)$ have all converged, then $\sigma_n(k)$ becomes increasing and thus also converges to some limit which we call $x(k)$. It remains to show that $x \in P$. Fix n and choose $s > n$ such that $\sigma_m(i) = x(i)$ for all $i < n$ and all $m \geq s$. Then $x\lceil n \prec \sigma_s$ and hence $x\lceil n \in T$.

A similar argument shows that the "rightmost" element Y of P corresponds to a real with a Π^0_1 Dedekind cut. □

Some further basis results are given below in Chapter V.

Exercises

IV.1.1. The Baire Category Theorem for $\mathbb{N}^{\mathbb{N}}$ states that the countable intersection of a sequence of dense open sets is nonempty. Use forcing to prove this theorem.

IV.1.2. Show that x is 1-generic if and only if it belongs to every Σ^0_1 co-meager set (equivalently, every non-meager Σ^0_1 set).

IV.1.3. Show that if the c. b. class P contains no computable member, then for any degree $\mathbf{b}$, P has a member A of degree $\mathbf{a}$ such that $\mathbf{a} \oplus \mathbf{0}' = \mathbf{a}' = \mathbf{b}'$. Then show that any Π^0_1 class P has a member of degree $\mathbf{a}$ such that $\mathbf{a} \oplus \mathbf{0}' = \mathbf{a}' = \mathbf{0}''$.

IV.1.4. Show that if x is an isolated member of P, then x is computable. (Define a decidable subclass of P and use Theorem III.2.15.)

IV.1.5. Show that any nonempty Π^0_1 class which is almost recursively bounded must contain an element computable in $\mathbf{0}'$.

IV.1.6. Show that for any x and y with $x <_{lex} y$ such that $L(r_X)$ is Σ^0_1 and $L(r_y)$ is Π^0_1, there is a Π^0_1 class with leftmost element x and rightmost element y.

IV.2 Special Π^0_1 classes

A c. b. Π^0_1 class is said to be *special* if it has no computable members.

We have seen in Section 3.III.5(Exercise (3)) that the diagonally non-computable sets form a Π^0_1 class with no computable element. We give an improvement of this result due to Jockusch [91]. Recall that a set A is *immune* if it has no infinite recursive subset; A is said to be *bi-immune* if both A and $\mathbb{N}\setminus A$ are immune. It is elementary that any infinite c. e. set has an infinite recursive subset, and it follows that the difference of two c. e. sets cannot be bi-immune and then by induction that no Boolean combination of c. e. sets can be bi-immune.

Theorem IV.2.1 (Jockusch). *There is a nonempty Π^0_1 class of sets containing only bi-immune sets.*

Proof. Let W_e be the e'th c.e. set and let D_n be the n'th finite set. Let ψ be a partial recursive function such that, whenever $|W_e| \geq e+3$, then $\psi(e)$ is defined and $D_{\psi(e)} \subset W_e$ and $|D_{\psi(e)}| = e+3$. Define the Π^0_1 class $P = \cap_e P_e$, where $A \in P_e$ if and only if, if $\psi(e)$ is defined, then $A \cap D_{\psi(e)} \neq \emptyset$ and $(\mathbb{N}\setminus A) \cap D_{\psi(e)} \neq \emptyset$. Any element A of P is clearly bi-immune. To see that P is nonempty, note that for each e, $\{0,1\}^{\mathbb{N}} \setminus P_e$ has measure $\leq 2^{-e-2}$. (For $A \notin P_e$, either all $e+3$ elements of $D_{\psi(e)}$ are in A or all $e+3$ elements are not in A, which allows only 2 of the 2^{e+3} possibilities.) It follows that $\{0,1\}^{\mathbb{N}} \setminus P$ has measure $\leq \sum_e 2^{-e-2} = \frac{1}{2}$, so that $P \neq 0$. □

This immediately implies the following.

Theorem IV.2.2. *(Jockusch) There is a nonempty c. b. Π^0_1 class with no member a Boolean combination of r.e. sets.* □

A set A is said to be *effectively immune* if there is a computable function g such that for any e, if $W_e \subset A$, then $|W_e| \leq g(e)$. For the Π^0_1 class P constructed in the proof of Theorem IV.2.1, it is clear that any set $A \in P$ is effectively bi-immune via the function $g(e) = e+3$. This yields the following corollary, which we note is essentially exercise 4.2 on page 87 of [198].

Theorem IV.2.3. *There is a nonempty c. b. Π^0_1 class such that if* $\mathbf{a}$ *is the degree of a member of P and* $\mathbf{b}$ *is a c.e. degree and* $\mathbf{a} \leq \mathbf{b}$ *then* $\mathbf{b} = \mathbf{0}'$.

Proof. Let P be the Π^0_1 class defined in the proof of Theorem IV.2.1. Then every member of P is effectively immune. Now suppose that P had a member C of c. e. degree. We claim that C must have degree $\mathbf{0}'$. By the Modulus

Lemma (Soare [198], $C = \lim_s C_s$ with a *modulus function* $m(i)$ computable in C such that $s \geq m(i)$ implies that $i \in C \iff i \in C_s$. We may assume without loss of generality that each C_s is infinite and let $c_{0,s}, c_{1,s}, \ldots$ enumerate in increasing order the elements of C_s. Let $C = \{c_0, c_1, \ldots\}$ enumerate C in increasing order, so that for each n, $c_n = \lim_s c_{n,s}$. Now let the complete c. e. set K have enumeration K_s and let the partial recursive function θ be defined so that $\theta(i) = s$ if and only if s is the least such that $x \in K_s$.

By the recursion theorem, define the computable function h so that $W_{h(i)} = \emptyset$, if $i \notin K$ and otherwise $W_{h(i)} = \{c_{0,\theta(i)}, c_{1,\theta(i)}, \ldots, c_{g(h(i)),\theta(i)}\}$. Let $r(i)$ be the least s such that, for all $j \leq g(h(i))$ and all $t \geq s$, $c_{j,t} = c_j$. Then the function r can be computed from C using the modulus function m. If $r(i) \leq \theta(i)$, then $W_{h(i)} \subset C$, so that C has $g(h(i)) + 1$ elements, contradicting the hypothesis on g. It follows that $\theta(i) < r(i)$ for all i, so that K is computable from C, as desired. □

It follows that the only possible c. e. degree of a member of P is $\mathbf{0}'$. There is a more general version of this result, Theorem 5 of [98].

Theorem IV.2.4. *For any c. e. degree* $\mathbf{c}$*, there is a nonempty c. b.* Π^0_1 *class* P *such that the c. e. degrees of member of* P *are precisely the c. e. degrees above* $\mathbf{c}$.

Proof. For $\mathbf{c} = 0$, this is trivial. For $\mathbf{c} \neq 0$, let A be the simple but not hypersimple c. e. set of degree $\mathbf{c}$ from Theorem II.7.8. Let $D_{f(n)}$ be a disjoint strong array such that $D_{f(n)} \setminus A \neq \emptyset$ for all n and assume without loss of generality that $Card(D_{f(n)} \setminus A) \geq 2$. Now define the Π^0_1 class P by

$$D \in P \iff D \cap A = \emptyset \ \& \ (\forall n)(D \cap D_{f(n)} \neq \emptyset).$$

For any $D \in P$ of c. e. degree, it follows from Theorem II.7.8 that $A \leq_T D$.

Now let E be a set of any degree $\mathbf{d} \geq c$ and find $D \in P$ such that $D \leq_T E$ and

$$Card(D_{f(n)} \setminus D) = 1 \iff n \in E.$$

Clearly $E \leq_T D$. Since $A \leq_T D$, we can compute from D a sequence of pairs $\langle a_n, b_n \rangle$ with a_n, b_n both in $D_{f(n)} \setminus A$. Let $E = \{e_0 < e_1 < \ldots$. Then we can code D into E by letting $D_{f(e_i)} \setminus D = \{a_{e_i}\}$ if $i \in D$ and $D_{f(e_i)} \setminus D = \{b_{e_i}\}$ otherwise. Thus D will be a member of P with degree $\mathbf{d}$. □

We next show that every special c. b. Π^0_1 class satisfies a weakened form of Theorem IV.2.3.

Theorem IV.2.5. *(Jockusch-Soare [98]) For any special* Π^0_1 *class, there exists a non-zero c. e. degree* $\mathbf{a}$ *such that* P *has no members of degree* $\leq \mathbf{a}$.

Proof. We give the proof from [98]. Let $P = [T]$ where T is a computable tree. We will construct a simple (and hence noncomputable) c. e. set A in stage A_s, such that $\phi^A_e \notin P$ for any e.

Two binary computable functions will be used in the construction.

$p(e,s) = max\{n \leq s : \Phi_e(A_s\lceil s)\lceil n) \in T\}$;

$q(e,s) = (least\ j) : |\Phi_e(A_s\lceil j)| \geq p(e,s)\}$.

Initially, $A_0 = \emptyset$. At stage $s+1$, let e_{s+1} be the least $e \leq s$ such that $W_{e,s} \cap A_s = \emptyset$ and $W_{e,s}$ contains $u \geq max\{q(e',s) : e' \leq e\} \cup \{2e\}$. Let $A_{s+1} = A_s \cup \{u\}$, where u is the least such number for $e = e_{s+1}$. (If no such e exists, let $A^{s+1} = A_s$.)

Claim 1 No element of P is computable in A.

Proof of Claim 1: Suppose by way of contradiction that $y = \phi_e^A \in P$ (and is thus a total function). It follows that $lim_s p(e,s) = \infty$. For the contradiction, we will show that y is computable. Let t be a stage such that for all $e' < e$, if $W_e \cap A \neq \emptyset$, then $W_{e,t} \cap A_t \neq \emptyset$. Then for all $s \geq t$, either e_s is undefined or $e_s \geq e$. To compute $y(n)$, find $s_n \geq t$ such that $n < p(e,s_n)$, so that $\Phi_e(A\lceil s_n, n)$ is defined and in T. We claim that $y(n) = \Phi_e(A\lceil s_n, n)$. To show this, it suffices to prove that no number $u < q(e,s_n)$ enters A after stage s_n. But if $u \in A^{s+1} - A_s$ and $s+1 > s_n$, then $e_{s+1} \geq e$ and so $u \geq q(e,s_n)$ by the construction.

Claim 2 For fixed e, $q(e,s)$ is bounded over all s.

Proof of Claim 2:
Let n be the least such that $\Phi_e^A\lceil n \notin T$ and let t be as in the proof of Claim 1. It follows that $p(e,s) \leq n$ for all $s \geq t$. Choose j such that $\Phi_e^A\lceil n \preceq \Phi_e(A\lceil j)$. It follows that $q(e,s) \leq j$ for all sufficiently large s.

Claim 3 A is simple.

Proof of Claim 3: For each c. e. set W_e, A contains at most one member $u \geq 2e$ from W_e, so that A contains at most e elements $\leq 2e$ and the complement of A is infinite. Fix e and let $j = max\{q(e',s) : e' \leq e, s \in \mathbb{N}\}$. Let t be given from Claim 1. Then if $s \geq t$ and $W_{e,s}$ contains a member $\geq max\{j, 2e\}$, then $W_{e,s+1} \cap A \neq \emptyset$ by the construction. Hence if W_e is infinite, then $W_e \cap A \neq \emptyset$.
This completes the proof of the theorem. □

As we saw in Exercise 4, any isolated member of a c. b. Π_1^0 class is computable. Hence, if P is a special c. b. Π_1^0 class, then it is perfect and hence has cardinality $2^{\aleph_0}$. It follows that there are $2^{\aleph_0}$ different degrees of members of P. Several results deal with the comparability of these degrees.

It is easy to see that if F is a computable function from $\{0,1\}^{\mathbb{N}}$ to $\{0,1\}^{\mathbb{N}}$, $r \in \{0,1\}^{\mathbb{N}}$, and P is a Π_1^0 class such that $F(x) = r$ for all $x \in P$, then r is computable. (See the exercises.) The next lemma, from Jockusch-Soare [99], improves this observation.

Lemma IV.2.6. *Let P be a nonempty c. b. Π_1^0 class P and let $\Phi : \mathbb{N} \times \mathbb{N}^{\mathbb{N}} \to \mathbb{N}$ be a partial computable functional. Suppose that for any $n \in \mathbb{N}$ and any $x_1, x_2 \in P$, $\Phi(n,x_1) = \Phi(n,x_2)$ whenever they are both defined. Then there is a nonempty Π_1^0 class $Q \subseteq P$ such that, for all $y \in Q$, if $\lambda n \Phi(n,y)$ is total then it is computable.*

Proof. Let $P = [T]$ where T is a computable tree and suppose that P and Φ satisfy the hypothesis. There are two possibilities. Suppose first that for some n, $\Phi_e(n,\sigma)$ is undefined for infinitely many $\sigma \in T$. Then we may let S be the subtree $\{\sigma \in T : \Phi_e(n,\sigma) \uparrow\}$ and let $Q = [S]$, since $y \in Q$ then implies that $\Phi_e(n,y) \uparrow$. Thus we may suppose that for any given n, $\Phi(n,\sigma) \downarrow$ for all but finitely many $\sigma \in T$. Thus $\Phi(n,y) \downarrow$ for all $y \in P$ and by the hypothesis there is a unique k_n such that $\Phi(n,y) = k_n$ for all $y \in P$. Let $r(n) = k_n$ for each n. It now follows from the remark above that r is computable. Thus in fact $\Phi(n,y) = r(n)$ for all $y \in P$ and all n. □

When F is the identity function, we obtain the following corollary.

Lemma IV.2.7. *If P is a nonempty c. b. Π^0_1 class with no computable elements, then any $\sigma \in Ext(T)$ has incompatible extensions $\tau_1, \tau_2 \in Ext(T)$.* □

Theorem IV.2.8. *(Jockusch-Soare [99]) For any c. b. Π^0_1 class P with no computable members and any countable set $\{\mathbf{a}_i : i < \omega\}$ of noncomputable degrees, P has continuum many mutually incomparable members x such that the degree of x is incomparable with each $\mathbf{a}_i$.*

Proof. We may assume without loss of generality that $P \subseteq \{0,1\}^{\mathbb{N}}$ and let $P = [T]$ for some computable tree $T \subseteq \{0,1\}^*$. For simplicity, we construct the elements of P to be incomparable to a single noncomputable degree $\mathbf{a}$ and let z have degree $\mathbf{a}$; for the general argument simply work on $\mathbf{a_0}, \dots, \mathbf{a_n}$ at level n.

We will define a function $\psi : \{0,1\}^* \to Ext(T)$ such that $\sigma_1 \prec \sigma_2$ implies $\psi(\sigma_1) \prec \psi(\sigma_2)$ and infinite trees $S_\sigma \subseteq T \cap I(\psi(\sigma)$ such that, for all n

(1) If $|\sigma| = n+1$ and $x \in [S_\sigma]$, then $\Phi^x_n \neq z$.

(2) If $|\sigma = |\tau| = n+1$ and $\sigma \neq \tau$, then for any $x \in S_\sigma$ and $y \in S_\tau$, $\Phi^x_n \neq y$.

Initially, we let $F(\emptyset) = \emptyset$. Now suppose that we have defined $\psi(\sigma)$ and S_σ for $|\sigma| = n$ and let $\Phi = \Phi_n$. By Lemma IV.2.6, there are two possible cases. First suppose that there is a nonempty Π^0_1 subclass Q of S_σ such that Φ^y_n is either computable or not total for all $y \in Q$. Then apply Lemma IV.2.7 to obtain incompatible extensions τ_0 and τ_1 of $F(\sigma)$ from $Ext(Q)$, by Lemma IV.2.7 and let $F(\sigma^\frown i) = \tau_i$ and $S_{\sigma^\frown i} = I(\tau_i) \cap Q$ for $i = 0, 1$.

In the second case, by Lemma IV.2.6, $F(\sigma)$ has extensions $\rho_0, \rho_1 \in Ext(S_\sigma)$ such that $\Phi_n(m,\rho_1) \neq \Phi_n(m,\rho_2)$ for some m. Without loss of generality, $\Phi_n(m,\rho_1) \neq z(m)$ and thus we first extend $F(\sigma)$ to ρ_1 to satisfy condition (1) and then take extensions τ_0 and τ_1 as above. In this case, $F(\sigma^\frown i)$ will be an extension of the provisional value τ_i for $i = 0, 1$ after we take care of condition (2). The provisional values for $S_{\sigma^\frown i}$ are $S_\sigma \cap I(\tau_i)$.

Now we will indicate how to satisfy condition (2). Let σ_1, σ_2 have length $n+1$ with provisional values τ_1 and τ_2 for $F(\sigma_1)$ and $F(\sigma_2)$ and provisional values Q_1 and Q_2 for S_{σ_1} and S_{σ_2} with $Q_i \subseteq P \cap I(\tau_i)$. Our goal is to ensure that $\Phi^x \neq y$ for any $x \in S_{\sigma_1}$ and $y \in S_{\sigma_2}$. If σ_1 came under case 1 above, then this is already satisfied. Otherwise, find m and extensions $\rho_1, \rho_2 \in S_{\sigma_1}$ such that

$\Phi(m, \rho_1) \neq \Phi(m, \rho_2)$ as above and find an extension τ of τ_2 such that, without loss of generality, $\tau(m) \neq \Phi(m, \rho_1)$. Then the new provisional value of $F(\sigma_1)$ is ρ_1, the new provisional value of $F(\sigma_2)$ is τ; the new provisional value of S_{σ_1} is $Q_1 \cap I(\rho_1)$ and the new provisional value of S_{σ_2} is $Q_2 \cap I(\tau)$. This satisfies one of the finitely many subconditions of (2).

We repeat this procedure for each pair $\sigma_1, \sigma_2 \in \{0,1\}^{n+1}$ to obtain final provisional values which will satisfy all of the necessary conditions. □

This leads to the following. Two noncomputable reals x, y are said to be a *minimal pair* if any real computable both in x and in y is itself computable.

Corollary IV.2.9. *(Jockusch-Soare [99] Let $P \subset \{0,1\}^{\mathbb{N}}$ be a Π^0_1 class with no computable members. Then P contains a minimal pair.*

Proof. Let x be any member of P. By Theorem IV.2.8 P contains an element y which is incomparable with each of the (countably many) noncomputable reals which are computable in x. □

Theorem IV.2.10. *(Jockusch-Soare [98]) There is an infinite c. b. Π^0_1 class P such that any two different members of P are Turing incomparable.*

Proof. We will use a priority argument to define a uniformly computable sequence $f_s : \{0,1\}^* \to \{0,1\}^*$ of one-to-one, computable tree homomorphisms and corresponding perfect Π^0_1 classes $P_s = [T_s]$, where

$$T_s = \{\tau : (\exists \sigma)[f_s(\sigma) \preceq \tau]\}.$$

Then the desired (perfect) Π^0_1 class P is defined to be $P = \cap_s P_s$. In particular, each f_s satisfies the following for all $\sigma \in \{0,1\}^*$.

(1) $f_s(\sigma^\frown 0)$ and $f_s(\sigma^\frown 1)$ are incompatible extensions of $f_s(\sigma)$ for all s;

(2) $range(f_{s+1}) \subseteq range(f_s)$;

(3) $\lim_s f_s(\sigma) = f(\sigma)$ exists.

It follows that f induces a homeomorphism F from $\{0,1\}^{\mathbb{N}}$ onto P, defined by $F(x) = \cup_n f(x \lceil n)$. It follows that that P is a perfect set, hence uncountable and therefore certainly nonempty.

The construction of P will ensure that for each $y \in P$ and each partial function Φ_e, $\Phi_e(y) \neq y$ implies that $\Phi_e(y) \notin P$. To accomplish this, we will ensure that for each e and each $x \in \{0,1\}^{\mathbb{N}}$, if $z = \Phi_e(F(x))$ converges and $z \neq F(x)$, then $\Phi_e(f(x \lceil e))$ is incompatible with $F(x)$. That is, for each e and each $\sigma \in \{0,1\}^{e+1}$, we have the following requirement.

$$\mathbf{R}_{e,\sigma} : (\forall y)[y \in I(f(\sigma)) \ \&\ \Phi_e(y) \text{ total } \&\ \Phi_e(y) \neq y \ \rightarrow\ \Phi_e(y) \notin P].$$

The priority order on the requirements is lexicographic, first on e and then on σ.

Requirement $R_{e,\sigma}$ is said to be *satisfied at stage s* if $\Phi_e(f(\sigma)) \notin T_s$. If this requirement is satisfied at stage s and $y \in I(f(\sigma))$, then of course requirement $R_{e,\sigma}$ is actually satisfied, since $\Phi_e(f(\sigma)) \preceq \Phi_e(y)$.

Initially we set $f_0(\sigma) = \sigma$ for all $\sigma \in \{0,1\}^*$.

Stage $s+1$. Requirement $R_{e,\sigma}$ needs attention at stage $s+1$ if it is not satisfied at stage s and there exists $\sigma' \succ \sigma$ such that $\Phi_e(f_s(\sigma'))$ is incompatible with $f_s(\sigma')$ and extends $f_s(\rho^\frown i)$ for some $\rho \in \{0,1\}^{e+1}$ and some $i \in \{0,1\}$.

If no requirement needs attention at stage $s+1$, then $f_{s+1} = f_\sigma$. Otherwise, let $R_{e,\sigma}$ be the requirement of highest priority which needs attention and let σ', ρ and i be given as above. Define f_{s+1} as follows, for any $\nu \in \{0,1\}^*$.

(4) $f_{s+1}(\sigma^\frown\nu) = f_s(\sigma'^\frown\nu)$;

(5) If $\rho \neq \sigma$, then $f_{s+1}(\rho^\frown\nu) = f_s(\rho^\frown(1-i)^\frown\nu)$;

(6) If ν does not extend either ρ or σ, then $f_{s+1}(\nu) = f_s(\nu)$.

Now suppose that $f_{s+1}(\sigma) \prec y$. Then $\Phi_e(f_{s+1}(\sigma)) = \Phi_e(f_s(\sigma')) \succ f_s(\rho^\frown i) =\prec \Phi_e(y)$ and is incompatible with $f_{s+1}(\sigma)$. There are two cases. If $\sigma \neq \rho$, then $f_{s+1}(\rho) = f_s(\rho)^\frown(1-i)$ so that $f_s(\rho^\frown i) \notin T_{s+1}$. If $\sigma = \rho$, then $f_s(\rho^\frown i)$ extends $f_s(\sigma)$ but is incompatible with $f_{s+1}(\sigma)$ and therefore again is not in T_{s+1}.

The functions f_s clearly satisfy (1) and (2). To verify (3), we show simultaneously that, for each e and σ, action is taken on each requirement $R_{e,\sigma}$ at most finitely often. Fix e and σ and choose s by induction so that for all $t \geq s$ and all requirements $R_{d,\tau}$ of higher priority

(7) $f_s(\tau) = f_t(\tau)$ and

(8) no action is ever taken on requirement $R_{d,\tau}$ after stage s.

Then $f_s(\tau) = f_t(\tau)$ for all $\tau \prec \sigma$ so that $f_s(\sigma) \preceq f_t(\sigma)$. Thus once requirement $R_{e,\sigma}$ is satisfied at stage t, it will remain satisfied. Hence it requires attention at most one more time after stage s and therefore $f_t(\sigma)$ converges.

Finally, suppose that $y \in P$, $\Phi_e(y)$ is total and $\Phi_e(y) \neq y$. Let $y = F(x)$, $\sigma = x\lceil n$ and $\tau = f(\sigma) \prec y$. Since $\Phi_e(y) \neq y$, there must exist $\sigma' \succeq \sigma$ such that $\Phi_e(f(\sigma'))$ is incompatible with y. It follows that in fact $\Phi_e(\tau) \notin T$, since otherwise $R_{e,\sigma}$ would require attention at arbitrarily large stages s. □

There is a version of this theorem for classes of separating sets.

Theorem IV.2.11. *(Jockusch-Soare [98]) There exist disjoint c. e. sets A and B such that $A \cup B$ is cofinite, but any two members of Sep(A, B) either have finite difference or are Turing incomparable.*

Proof. The disjoint c. e. sets A and B will be defined by a priority argument as the effective, increasing unions $A = \cup_s A_s$ and $B = \cup B_s$. The construction will ensure that for each partial function Φ_e and any pair of C, D of separating sets for A, B, the following requirement is satisifed.

$\mathbf{R}_e$: If $C \neq D$ (modulo finite difference), then $D \neq \Phi_e(C)$.

As in the construction of a maximal set ([198], p. 188), an increasing sequence $\{m_i^s : i < \omega\}$ will be defined at stage s so that for each i, $m_i^0 \leq m_i^1 \leq \dots$ and $m_i = \lim_s m_i^s$ exists. At any stage s, $A_s \cup B_s = \omega \setminus \{m_i^s : i \in \omega\}$, so that $\{m_i : i < \omega\}$ is the complement of $A \cup B$.

For fixed e, let

$$D_e = \{m_i : i < e\}.$$

Let C and D be two subsets of D_e. The subrequirement $R_e^{C,D}$ associated with C and D asserts that for any separating sets C_1 and D_1 for A and B such that $C_1 \cap D_e = C$ and $D_1 \cap D_e = D$, $D_1 \neq \Phi_e(C_1)$.

This requirement is said to be *satisfied at stage s* if there exists a string σ such that, for any n,

(1) if $n \in D_e$, then $n \in C \iff \sigma(n) = 1$;

(2) if $n \in Dom(\sigma)$, then $n \in A_s \to \sigma(n) = 1$ and $n \in B_s \to \sigma(n) = 0$.

(3) $Dom(\sigma) \subseteq D_e \cup A_s \cup B_s$

(4) $\Phi_e^s(m, \sigma)$ is defined and $\neq D(m)$ for some m.

That is, let (1) through (4) hold and suppose that C_1 and D_1 are separating sets for A and B such that $C_1 \cap D_e = C$ and $D_1 \cap D_e = D$. Then $\sigma \prec C_1$ so that $\Phi(m, C) \neq D(m)$.

$R_e^{C,D}$ requires attention at stage s if it is not satisfied at stage s and if there exists a string σ satisfying (1) and (2) and such that

(5) $\Phi_e^s(m_e^s, \sigma)$ is defined.

Then we also say that R_e requires attention at stage s.

The construction proceeds as follows. Initially $A_0 = B_0 = \emptyset$ and $m_e^0 = e$ for all e.

Stage $s+1$: Choose the requirement R_e of highest priority which requires attention at stage s. (If none exists, let $A_{s+1} = A_s$, $B_{s+1} = B_s$ and $m_i^{s+1} = m_i^s$ for all i.) Then choose C and D such that $R_e^{C,D}$ requires attention at stage s and let σ be given as above. For $m = m_e^s$, put $m \in A_{s+1}$ if $\Phi_e^s(m, \sigma) = 0$ and otherwise put $m \in B_{s+1}$. For $m < |\sigma|$ such that $m \notin D_e$ and $m \neq m_e^s$, put $m \in A_{s+1}$ if $\sigma(m) = 1$ and put $m \in B_{s+1}$ otherwise.

Then the sequence m_i^{s+1} is defined so that m_i^{s+1} is the ith element not in $A_s \cup B_s$.

Subrequirement $R_e^{C,D}$ is now satisfied at stage $s+1$, since σ' satisfies the conditions (1) through (4) where $\sigma' = \sigma$ except possibly for $\sigma(m_e^s)$.

It is easy to see that each requirement R_e requires attention only finitely often and that m_e^s converges for each e. To verify that all requirements are satisfied, let C' and D' be separating classes for A and B which have infinite difference and suppose by way of contradiction that $C' = \Phi_e(D')$. Choose s_0 so that every requirement of priority R_e or higher has ceased to require attention by stage s_0. Then for $i \leq e$, $m_i^s = m_i$ for all $s \geq s_0$. Let $C = C' \cap D_e$ and $D = D' \cap D_e$. Now at some stage $s > s_0$, $\Phi_e^s(m_e, D)$ must converge so that there is a $\sigma \prec D$ with $\Phi^s_e(m_e, D)$ defined. But then $R^{C,D}_e$ requires attention at stage s, contrary to the assumption on s_0. This completes the proof. □

The proof of the following theorem is omitted.

Theorem IV.2.12. *(Jockusch-Soare [96]) There exist disjoint pairs* (A_0, B_0) *and* (A_1, B_1) *of disjoint, recursively inseparable c. e. sets such that if* $C \in S(A_0, B_0)$ *and* $D \in S(A_1, B_1)$*, then* C *and* D *form a minimal pair, that is, any set computable in both* C *and* D *is computable.* □

Corollary IV.2.13. *For any degree* $\mathbf{a}$*, there is a special c. b.* Π^0_1 *class which has no members of degree* $\geq \mathbf{a}$*.*

Proof. Let (A_i, B_i) be given for $i = 0, 1$ by Theorem IV.2.12. Each class $S(A_i, B_i)$ contains no computable elements. Suppose that $S(A_0, B_0)$ has a member of degree $\geq a$. Then no member of $S(A_1, B_1)$ can have degree $\geq a$. □

Exercises

IV.2.1. Theorem IV.2.3 can be improved to say that for any degree $\mathbf{c}$ of a member of P and any c. e. degree $\mathbf{a} \geq \mathbf{c}$, $\mathbf{a} = \mathbf{0}'$. Show this using the full Modulus Lemma, which gives a modulus $m \leq_T A$ for C whenever A is a c.e. set such that $C \leq_T A$.

IV.2.2. Show that any nonempty c. b. Π^0_1 class P contains members x and y whose degrees have greatest lower bound $\mathbf{0}$. If P is special, then this gives a minimal pair of members.

IV.2.3. Show that if F is a computable function from $\{0,1\}^{\mathbb{N}}$ to $\{0,1\}^{\mathbb{N}}$, $r \in \{0,1\}^{\mathbb{N}}$, and P is a Π^0_1 class such that $F(x) = r$ for all $x \in P$, then r is computable.

IV.3 Measure, Category and Randomness

that is, $\{q \in \mathbb{Q} : q \leq r\}$ is a Π^0_1 set, since r can be approximated from above as the limit of a decreasing computable sequence. That is, given a computable tree $T \subset 2^{<\omega}$ with $P = [T]$, let $P_n = \bigcup\{I(\sigma) : \sigma \in T \cap 2^n\}$. Then $m(P_n)$ is just $k/2^n$, where $k = card\{\sigma \in T \cap 2^n\}$ and $m(P) = \lim_n m(P_n)$. The measure is not necessarily computable, which will follows from the next two results.

The measure of a Π^0_1 class of reals will be discussed in detail in Chapter XV.

It is easy to modify the class of diagonally non-computable reals to obtain a Π^0_1 class which has positive measure but has no computable elements.

Theorem IV.3.1. *There is a* Π^0_1 *class* $P \subset \{0,1\}^{\mathbb{N}}$ *with positive measure which has no computable members.*

The proof is left as an exercise.

On the other hand, there is a computable basis result.

Theorem IV.3.2. *Any Π^0_1 class $P \subseteq \{0,1\}^{\mathbb{N}}$ with positive, computable measure has a computable member.*

Proof. Let P have computable measure $r > 0$. It follows that the measure of $P \cap I(\sigma)$ is computable uniformly in σ, since we can approximate the measure from below by subtracting the measure of the complement $P \cap (\{0,1\}^{\mathbb{N}} - I(\sigma))$ from r. Thus we can recursively select paths $\sigma = (x(0), \ldots, x(n))$ of length n such that $P \cap I((x(0), \ldots, x(n)))$ always has measure $\geq r/2^n$. The infinite path x will be a computable member of P. □

Proposition IV.3.3. *For any thin Π^0_1 class $Q \subseteq \{0,1\}^{\mathbb{N}}$, $\mu(Q) = 0$.*

Proof. Let Q be a thin class and let T be a computable tree such that $Q = [T]$. Suppose that $\mu(Q) > 0$ and choose a rational $p > 0$ so that $\mu(Q) \geq p$. Uniformly define the class P_x to be $\{y \in Q : (\forall n) y \upharpoonright n \leq_{lex} y \upharpoonright n\}$. Then let $P = \{x \in Q : \mu(P_x) \geq p$. This is a Π^0_1 class since

$$x \in P \iff (\forall n) card(\{\sigma \in T : \sigma \leq_{lex} x \upharpoonright n) \geq 2^n p.$$

P has the property that, for $x, y \in Q$, if $x \in P$ and $x \leq_{lex} y$, then $y \in Q$. Since Q is thin, the complement $Q - P$ is also a Π^0_1 class and $\mu(Q - P) = p$. But as a closed set Q has a greatest element z. It follows that $Q - P = P_z$ so that $\mu(P_z) = p$, which would mean that $z \in P$, a contradiction. □

Definition IV.3.4.

1. *The real $x \in \{0,1\}^{\mathbb{N}}$ is said to be* random *if x does not belong to any Π^0_1 class of measure 0.*
2. *The real $x \in \{0,1\}^{\mathbb{N}}$ is said to be* 1-random *if for any computable function f such that $\mu(P_{f(n)}) > 1 - 2^{-n}$ for all n, $x \in P_{f(n)}$ for some n.*

It follows from this definition that the class of random reals is simply the intersection of all Σ^0_1 classes of measure 1 and therefore has measure 1 itself. Thus *any* set of positive measure must contain a random real.

We will also consider degrees of members of Π^0_1 classes. For any class $P \subset \mathbb{N}^{\mathbb{N}}$, $\mathcal{D}(P)$ is the collection of all sets Turing equivalent to a member of P and $\mathcal{U}(P)$ is the collection of all sets A such that some member of P is Turing reducible to A. Note that $\mathcal{D}(P) \subseteq \mathcal{U}(P)$

P. Martin-Lof [146] introduced the notion of 1-randomness and showed that there is a universal, increasing sequence P_{e_n} of Π^0_1 classes such that $\mu(P_{e_n}) > 1 - 2^{-n}$ and such that $\cup_n P_{e_n}$ is precisedly the class of 1-random reals. Thus the class of 1-random reals also has measure 1. The degree of a 1-random real is called a *1-random degree*. Recall that a function $f : \omega \to \omega$ is *fixed-point-free* if there is no e such that $\phi_e = \phi_{f(e)}$.

Theorem IV.3.5. *(Martin-Lof) There is a computable function g such that if $Q_n = P_{g(n)}$, then*

1. *$Q_0 \subseteq Q_1 \subseteq \ldots$*
2. *For each n, $\mu(Q_n) > 1 - 2^{-n}$*

3. $\cup_n Q_n$ *is the set of 1-random reals.*

Proof. For each n, we construct a c. e. set $U_n \subset \{0,1\}^*$ and let $Q_n = \{0,1\}^{\mathbb{N}} \setminus \cup_{\sigma \in U_n} I(\sigma)$. First enumerate all c. e. Martin-Lof tests as follows. Let W_e denote the eth c. e. subset of $\{0,1\}^*$ and let $W_{e,s} \subseteq \{0,1\}^s$ denote the finite subset of $W_{e,s}$ enumerated into W_e by stage s. Then let $W_{e,j,s} = \bigcap_{i=1}^{j} W_{\langle e,j\rangle,s}$ as long as $\mu(\bigcup\{I(\sigma) : \sigma \in W_{e,j,s}\}) \leq 2^{-(j+1)}$ and otherwise $W_{e,j,s} = W_{e,j,s-1}$. Then for $\mathcal{W}_{e,j} = \bigcup\{I(\sigma) : s \in \mathbb{N}, \sigma \in W_{e,j,s}\}$. the c. e. Martin-Lof tests consist exactly of the sequences $\mathcal{W}_{e,0}, W_{e,1}, \dots$ for $e \in \mathbb{N}$.

Now define

$$U_n = \cup_{e \in N} W_{e,n+e+1},$$

and let

$$\mathcal{U}_n = \cup\{I(\sigma) : \sigma \in U_n\}.$$

Note that $\mu(\mathcal{U}_n) \leq \sum_{e \in N} \mu(\mathcal{W}_{e,n+e+1}) \leq \sum_e 2^{-(n+e+1)} \leq 2^{-n}$. Then $\{\mathcal{U}\}_{n<\omega}$ is a c. e. Martin-Lof test. Let $Q_n = \{0,1\}^{\mathbb{N}} \setminus \mathcal{U}_n$.

If x is 1-random, then $x \in \cup_n Q_n$ since it passes all c. e. Martin-Lof tests. If x is not 1-random, suppose that it fails the test $W_{e,n_{e<\omega}}$, Then $x \in \cap_{n \geq e+1} \mathcal{W}_{e,n}$ and hence $x \in \cap_n \mathcal{U}_n$. □

Theorem IV.3.6. *(Kucera). For any Π^0_1 class P of positive measure, $\mathcal{D}(P)$ contains every 1-random degree.*

Proof. Suppose $\mu(P) > 0$. Let W be a c. e. set such that $\{0,1\}^{\mathbb{N}} \setminus P = \cup_{\sigma \in W} I(\sigma)$. For each n, define a sequence of uniformly c. e. sets U^k by $U^0 = W$ and for each k,

$$U^{k+1} = \{\tau^\frown\sigma : \tau \in U^k \ \&\ \sigma \in W\}.$$

Then for each k, let $V^k = \bigcup_{\sigma \in U^k} I(\sigma)$. It is easy to see that, for all k, $V^{k+1} \subseteq V^k$ and that $\mu(V^k) = \mu(V)^k$. Choose m such that $\mu(V^k) < \frac{1}{2}$. Then for each n, $\mu(V^{nk}) < 2^{-n}$. Thus $\cap_m V^m$ contains no 1-random elements. Suppose now that A is 1-random. It follows that $A \notin V^j$ for some j; let k be the least such. If $k = 0$, then $A \in P$. If $k > 0$, then $A \in V^{k-1} \setminus V^k$, so that $A = \tau^\frown B$ for some $\tau \in U^{k-1}$ and some $B \in P$. □

It follows from this proof that for any Π^0_1 class P of positive measure and any random real A, the iterated shift $\sigma^n(A) \in P$ for some n.

Corollary IV.3.7. *For each n, $\mathcal{D}(Q_n)$ equals the set of 1-random degrees.*

Proof. By Theorem IV.3.5, every element of Q_n is 1-random and by Theorem IV.3.6, $\mathcal{D}(Q_n)$ contains all 1-random degrees. □

This also proves the following result from [99], which significantly improves Theorem IV.3.1, since now we know that $\mathcal{D}(Q_n)$ includes the 1-random degrees and thus has measure one.

Corollary IV.3.8. *(Jockusch-Soare) There is a c. b. Π^0_1 class P with no computable members such that $\mu(\mathcal{D}(P)) = 1$.*

The following is due to Kucera [117].

Theorem IV.3.9. *Every 1-random set is bi-effectively immune.*

Proof. Let Q_n be defined as in Theorem IV.3.5 above. Following Kucera's proof, we will construct computable functions f and g such that for all $k, x \in \mathbb{N}$:

$$A \in Q_n \ \& \ W_x \subseteq A \implies card(W_x) \leq f(n,x);$$
$$A \in Q_n \ \& \ W_x \subseteq \mathbb{N} - A \implies card(W_x) \leq g(n,x).$$

Now define the uniformly Π^0_1 classes $P_{x,n} \subseteq \{0,1\}^{\mathbb{N}}$ by

$$A \in P_{x,m} \iff [A \text{ contains the first } m+1 \text{ elements of } W_x \rightarrow card(W_x) < m.$$

Then $\mu(P_{x,m}) > 1 - 2^{-m}$. Now let $e = f(n,x)$ be defined so that $\phi_e(j)$ is an index of $P_{x,j}$ for all $j \in \mathbb{N}$. It follows from the definition of Q_e that $Q_e \subseteq P_{x,e}$, so that $Q_k \subseteq P_{x,e}$. Thus if $A \in Q_k$ & $W_x \subseteq A$, then $card(W_x) \leq e$. The argument for $\mathbb{N} - A$ and g is similar. □

The next two results are taken from [99].

Theorem IV.3.10. *For any closed subset P of $\mathbb{N}^{\mathbb{N}}$ with no computable member, $\mathcal{U}(P)$ (and hence $\mathcal{D}(P)$) is meager.*

Proof. Let $P = [T]$ where T is not necessarily computable and assume by way of contradiction that $\mathcal{U}(P)$ is not meager. Then for some e, $S_e = \{A : \phi_e^A \in P\}$ is not meager. Now S_e is not nowhere dense, so there exists σ such that every extension τ of σ can be extended to some $A \in S_e$. Note that S_e is not necessarily a closed set. We will use e and σ to construct a computable member of P. By assumption there is an extension A of σ in S_e so that $\phi_e^A \in P$. Thus we can find $\tau_0 \succ \sigma$ such that $\phi_e(\tau_0, 0)$ is defined. Proceeding recursively, we can find $\tau_{n+1} \succ \tau_n$ such that $\phi_e(\tau_n) \in T$ and $|\phi_e(\tau_n)| > n$. It follows that the computable function $f(n) = \phi_e(\tau_n, n) \in P$. (Although $x \cup_n \tau_n$ is not necessarily an element of S_e.) □

This theorem implies the result of Sacks [181] that $\mathcal{U}(\{A\}$ is meager for any noncomputable set A and the following theorem from [99] will imply the result from [181] that $\mu(\mathcal{U}(\{A\})) = 0$ for noncomputable A.

Theorem IV.3.11. *(Jockusch-Soare) For any Π^0_1 class $P = S[A,B]$ where A and B are computably inseparable, $\mu(\mathcal{U}(P)) = 0$.*

Proof. Suppose by way of contradiction that $\mu(\mathcal{U}(P)) > 0$. Then for some e and some rational m, $Q_e = \{C : \phi_e^C \in P\}$ has positive measure p with $m < p < 3m/2$ for some rational m. Let $Q_e = [T]$ for some computable tree T. For $i = 0, 1$, let

$$C_i = \{n : \mu(\{C : \phi_e^C(n) = i\}) > m/2\}.$$

Then each C_i is a c. e. set, since to test whether $n \in C_i$, simply find k such that $card(\{\sigma \in \{0,1\}^k : \phi_e^\sigma(n) = i\}) > m2^{k-1}$. Since ϕ_e^C is total for a set of measure

$> m$, it follows that $C_0 \cup C_1 = \mathbb{N}$. Hence by the reduction principle, there are disjoint c. e. sets $E_0 \subseteq C_0$ and $E_1 \subseteq C_1$ such that $E_0 \cup E_1 = \mathbb{N}$ and therefore E_0 and E_1 are computable. We claim that E_1 is a separating set for A and B. If $n \in A$, then $\mu(\{C : \phi_e^C \in P\}) > m$, so that $\mu(\{C : \phi_e^C(n) = 1\}) > m$ and hence $\mu(\{C : \phi_e^C(n) = 0\}) < m/2$, so that $n \notin C_0$ and therefore $n \in E_1$. Similarly if $n \in B$, then $n \in E_0$. But this contradicts the assumption that A and B are computably inseparable. □

Exercises

IV.3.1. Construct a Π^0_1 class which has positive measure m but has no computable members. Show that for any real number $\epsilon > 0$, we can make $m > 1 - \epsilon$. (Hint: we can ensure that $\phi_e \notin P$ by making $\phi_e \lceil n \notin T$ for some large value of n.)

IV.3.2. Say that a real r is Σ^0_1 if $\{q \in \mathbb{Q} : q < r\}$ is a Σ^0_1 set. Show that r is Π^0_1 (respectively, Σ^0_1) if and only if r is the limit of a computable, decreasing (resp. increasing) sequence of rationals. Thus r is computable if and only if r is both an increasing and a decreasing limit of rationals.

IV.3.3. Show that the class of 1-generic reals has measure 0 and is not a basis for the family of Π^0_1 classes of positive measure. (Hint: x is 1-generic if it never belongs to the boundary of any P_e.)

IV.3.4. Show that 1-generic reals and 1-random reals are also random.

IV.3.5. Show that if a Π^0_1 class P contains a random element, then $\mu(P) > 0$.

IV.4 Mathematical Logic: Peano Arithmetic

In this section, we consider further the connection between Π^0_1 classes and mathematical logic and, in particular, Peano Arithmetic. We begin with some applications of the present chapter together with sections III.III.9.

Theorem IV.4.1. *Any axiomatizable theory Γ has a complete consistent extension of low c. e. degree.*

Proof. Let Γ be an axiomatizable theory and let the Π^0_1 class P represent the family of complete consistent extensions of Γ, by Theorem III.III.9.1. Then P has a member of low c. e. degree by Theorem IV.1.4. □

On the other hand, we have the following.

Theorem IV.4.2. *There is a (propositional) axiomatizable theory Γ which has no c. e. complete consistent extension.*

Proof. By Theorem IV.2.1, there is a Π^0_1 class P with no c. e. member. Now, by Theorem III.III.9.3, there is an axiomatizable theory Γ such that P represents the set of complete consistent extensions of Γ. □

Theorem IV.4.3. *For any c. e. degree* $\mathbf{c}$*, there is an axiomatizable theory* Γ *such that the c. e. degrees of complete consistent extensions of* Γ *are exactly the c. e. degrees above* $\mathbf{c}$*.*

Proof. This is an immediate consequence of Theorems III.III.9.3 and IV.2.4. □

Theorem IV.4.4. *For any essentially undecidable, axiomatizable theory* Γ*, there exists a c. e. degree* $\mathbf{a}$ *such that* Γ *has no complete consistent extensions of degree* $\leq \mathbf{a}$*.*

Proof. This follows from Theorems III.III.9.1 and IV.2.5. □

Theorem IV.4.5. *For any essentially undecidable, axiomatizable theory* Γ*, there exists two complete consistent extensions,* Δ_1 *and* Δ_2*, of* Γ *such that any set computable from* Δ_1 *and computable from* Δ_2 *is in fact computable.*

Proof. This follows from Theorems III.III.9.1 and IV.2.12. □

Theorem IV.4.6. *There is an axiomatizable theory* Γ *such that any two complete consistent extensions of* Γ *are Turing incomparable.*

Proof. This follows from Theorems III.III.9.3 and IV.2.10. □

IV.4.1 Peano Arithmetic

The standard model $\mathbb{N} = (\mathbb{N}, S, +, \cdot, <)$ of arithmetic is fundamental in mathematics. The language of arithmetic is often defined to consist of a one-place function symbol S, representing the successor function, and binary function symbols $+$ for addition and $\cdot$ for multiplication. The usual linear ordering $\leq$ may then be defined by

$$x \leq y \iff (\exists z)(x + z = y);$$
$$x < y \iff x \leq y \ \&\ x \neq y.$$

Each natural number n may be represented by the term $S^n 0$. We will generally identify $S^n 0$ with n for simplicity of expression. Peano Arithmetic is a first order theory for $\mathbb{N}$ the consisting of nine axioms needed to define the functions and also the axiom scheme of induction. The eight axioms of *Robinson* arithmetic are the following.

S1 $Sx = Sy \implies x = y$

S2 $(\forall x) Sx \neq 0$

L1 $(\forall x) \neg x < 0$

L2 $(\forall x)(\forall y)[x < Sy \iff (x < y \ \vee\ x = y)]$

A1 $(\forall x) x + 0 = x$

A2 $(\forall x)(\forall y) x + Sy = S(x + y)$

M1 $(\forall x) x \cdot 0 = 0$

M2 $(\forall x)(\forall y) x \cdot Sy = x \cdot y = y$

The axiom scheme of induction provides for each formula ϕ with one free variable:

IP $[\phi(0) \;\&\; (\forall x)(\phi(x) \;\rightarrow\; \phi(Sx))] \rightarrow (\forall y)\phi(y)$

The fundamental results of Gödel and others [100, 209] connecting Peano Arithmetic and computability theory is the following.

Theorem IV.4.7. *For any c. e. set A, there is a formula ϕ (which represents A) such that, for all m, the following are equivalent:*

(1) $m \in A$;

(2) $\mathbb{N} \models \phi(m)$ *and*

(3) $PA \vdash \phi(S^m 0)$.

A partial converse to this result is the result of Gödel that *any* axiomatizable theory Γ is computably enumerable (in the sense that the set of Gödel numbers of members of Γ is a c. e. subset of $\mathbb{N}$). This means that in particular PA itself is a c. e. set. The Incompleteness Theorem of Gödel follows from these results and tells us in particular, that PA can have no axiomatizable, complete consistent extension.

We shall now consider the connection between Peano Arithmetic and Π^0_1 classes. It follows from Theorem III.9.1 that the set of complete consistent extensions of PA as well as the set of consistent extensions of PA may be represented as Π^0_1 classes. It then follows that there is a complete consistent extension of PA with c. e. degree. Now any axiomatizable extension Γ of PA is necessarily incomplete by Gödel's theorem, hence the family of complete consistent extensions of Γ will be an uncountable Π^0_1 class. Certainly not every Π^0_1 class may be represented as the set of complete extensions of such a Γ. However, the Scott Basis Theorem shows that these classes are as complicated as an arbitrary Π^0_1 class in a certain sense.

Theorem IV.4.8. *(Scott [186]) For any consistent extension Γ of PA, the sets computable in Γ form a basis for the c. b. Π^0_1 classes.*

Proof. Let Γ be a consistent extension of PA, let P be a c. b. Π^0_1 class and let $T_P \subseteq \mathbb{N}^*$ be the tree of nodes which have an extension in P. Then T_P is a Π^0_1 set and it follows from Theorem IV.4.7 that there is a formula ψ such that, for all strings σ,

$$\sigma \notin T_P \implies PA \vdash \phi(\langle\sigma\rangle) \implies \phi(\langle\sigma\rangle \in \Gamma.$$

Now we can define a subtree T of T_P which is computable in Γ by

$$\sigma \in T \iff \phi(\langle\sigma\rangle \notin \Gamma.$$

The tree T has no dead ends and thus there P contains an infinite path which is computable in T and hence computable in Γ. □

Tennenbaum [210] observed that no nonstandard model of PA can be computable and the following improvement of that result was noted by Jockusch and Soare [98].

Theorem IV.4.9. *If $\mathcal{M} = (\mathbb{N}, +^M, \cdot^M)$ is any nonstandard model of Peano arithmetic, then the sets computable in M form a basis for the c. b. Π^0_1 classes.*

Proof. We follow the proof of Cohen [54]. Since PA and the set of negations of sentences in PA form a pair of computabley inseparable c. e. sets, it suffices by Theorem IV.4.8 to show that $\mathcal{M}$ can compute a separating set for any pair of disjoint c. e. sets A and B. First observe that we can recursively compute the standard numbers in $\mathcal{M}$ by starting with a_0 and a_1 which represent 0 and 1 and letting $a_{n+1} = a_n +^M a_1$ for all n. Similarly we may compute from $\mathcal{M}$ the sequence p_n of (finite) prime numbers of $\mathcal{M}$. Now by the Chinese Remainder Theorem, there must exist an infinite element a of $\mathcal{M}$ such that, in $\mathcal{M}$, $a = 0(modp_n)$ if and only if $a \in A$ and $a = 1(modp_n)$ if and only if $n \in B$. A separating set for A and B may now be computed from $\mathcal{M}$ as follows. Given n, compute the unique q and unique $r < p_n$ such that $a = q \cdot^M p_n +^M r$ and put $n \in C$ if and only if $r \neq 1$. It is immediate that $A \subset C$ and $B \cap C = \emptyset$. □

We have the following corollary to Theorem IV.2.3.

Corollary IV.4.10. *If* **a** *is the degree of any consistent extension or nonstandard model of Peano Arithmetic and* **b** *is a c. e. degree such that* $\mathbf{a} \leq b$*, then* $\mathbf{b} = \mathbf{0}'$.

It now follows that no consistent extension of PA can have c. e. degree $< \mathbf{0}'$. However, can get degree exactly $\mathbf{0}'$, as announced by Scott and Tennenbaum in [187].

Corollary IV.4.11. *There is a complete consistent extension of Peano Arithmetic of degree* $\mathbf{0}'$.

Proof. Let P be the c. b. Π^0_1 class of all complete consistent extensions of PA. Then P has a member of c. e. degree **b** and it follows from Corollary IV.4.10 that $\mathbf{b} = \mathbf{0}'$. □

Here is a nice corollary to Theorem IV.2.8.

Corollary IV.4.12. *(Jockusch-Soare [99]) There is a complete consistent extension Γ of Peano Arithmetic such that every set definable in Γ is either computable or not arithmetical.*

Proof. By Theorem IV.2.8, there is a theory Γ whose degree is incomparable with $\mathbf{0}^{(n)}$ for each $n > 0$. The corollary now follows from the fact that each set definable in Γ is computable from T. □

Chapter V

The Cantor-Bendixson Derivative

The perfect set theorem states that any closed subset of $\{0,1\}^{\mathbb{N}}$ is the union of a perfect closed set (the perfect kernel) and a countable set. The perfect kernel results from iterating the Cantor Bendixson derivative $D^\alpha(P)$ until a fixed point (an analytic set) is reached. The effective version of this theorem is that the perfect kernel of a Π^0_1 class is a Σ^1_1 and that the iteration must stop by the first non-computable ordinal ω_1^{C-K}.

For a countable Π^0_1 class P, the perfect kernel is the empty set and the iteration must stop at some computable ordinal (the C-B rank of P. The Kreisel basis theorem [116] showed that isolated members of Π^0_1 classes must be hyperarithmetic in general, must be computable in $\mathbf{0}'$ if P is bounded, and must be computable if P is computably bounded. Furthermore, any countable closed set must have an isolated element.

A finer analysis was given in [23] for Π^0_1 classes in $\{0,1\}^{\mathbb{N}}$. The C-B rank of a real x in a class P is the least ordinal α such that $x \notin D^\alpha(P)$. In particular, if x has C-B rank $\lambda + n$ in P for some limit ordinal λ and finite n, then x is computable in $\mathbf{0}^{\lambda+2n}$. It follows that every element of a countable class is hyperarithmetic. The C-B rank of a real x is defined to be the minimum of the ranks of x in any Π^0_1 class.

V.1 Cantor-Bendixson derivative and rank

The Cantor-Bendixson derivative $D(P)$ of a compact subset P of $\mathbb{N}^{\mathbb{N}}$ is the set of nonisolated points of P. Thus a point $x \in P$ is not in $D(P)$ if and only if there is some open set U containing x which contains no other point of P. Equivalently, $x \notin D(P)$ if and only if there is some clopen set U such that $U \cap P = \{x\}$. Another useful observation is that, for any compact set P, $D(P)$ is empty if and only if P is finite.

The iterated Cantor-Bendixson derivative $D^\alpha(P)$ of a closed set P is defined for all ordinals α by the following transfinite induction.

Definition V.1.1. *1. $D^0(P) = P$; $D^{\alpha+1}(P) = D(D^\alpha(P))$ for any α;*

2. $D^\lambda(P) = \bigcap_{\alpha<\lambda} D^\alpha(P)$ for any limit ordinal λ.

There is a related derivative which we can define for a tree $T \subseteq \{0,1\}^*$.

Definition V.1.2. *1. $d(T) = \{\sigma \in T : (\exists\tau)[\sigma \prec \tau \;\&\; \tau^\frown 0 \in Ext(T) \;\&\; \tau^\frown 1 \in Ext(T)]\}$.*

2. $d^0(T) = T$, $d^{\alpha+1}(T) = d(d^\alpha(T))$ and $d^\lambda(T) = \cap_{\alpha<\lambda} d^\alpha(T)$ for limit λ.

Then $d(T) \leq_T T^{(2)}$ and is in fact Σ^0_2 in T. We iterate this derivative by

Lemma V.1.3. *For any countable ordinal α and any tree $T \subseteq \{0,1\}^*$, $D^\alpha([T]) = [d^\alpha(T)]$.*

Proof. The proof is by transfinite induction. The equality is clear for $\alpha = 0$.

The interesting case is when $\alpha = 1$. Suppose that $x \in [d(T)]$ and fix $n \in \omega$. Then $x\lceil n \in d(T)$, so that there exists $\tau \succ x\lceil n$ such that both $\tau^\frown 0$ and $\tau^\frown 1$ are in $Ext(T)$. Thus $I(x\lceil n) \cap [T]$ contains at least two elements. It follows that there is some path $y_n \in [T]$ such that $x\lceil n \prec y_n$ and $x \neq y_n$. Hence $x \in D([T])$.

Next suppose that $x \in D([T])$ and again fix $n \in \omega$. Since x is not isolated in $[T]$, there must exist $y \in [T]$ such that $y \neq x$ and $x\lceil n \prec y$. Let $m > n$ be the least such that $y\lceil m+1 \neq x\lceil m+1$ and let $\tau = x\lceil m$. Then $x\lceil n \prec \tau$ and $\tau^\frown 0$ and $\tau^\frown 1$ are both in $Ext(T)$, which demonstrates that $x\lceil n \in d(T)$. Hence $x \in [d(T)]$. v Now suppose that $D^\alpha([T]) = [d^\alpha(T)]$. Then

$$D^{\alpha+1}([T]) = D(D^\alpha([T]) = D([d^\alpha(T)]) = [d([d^\alpha(T))] = [d^{\alpha+1}(T)].$$

For a limit ordinal λ,

$$D^\lambda([T]) = \cap_{\alpha<\lambda} D^\alpha([T]) = \cap_{\alpha<\lambda}[d^\alpha(T)] = [d^\lambda(T)].$$

This completes the proof. □

The Cantor-Bendixson (CB) rank of a closed set P is the least ordinal α such that $D^{\alpha+1}(P) = D^\alpha(P)$; then $D^\alpha(P)$ is a perfect closed set, denoted $K(P)$, called the *perfect kernel* of P and $P \setminus K(P)$ is a countable set. For $A \in P \setminus K(P)$, the C-B rank $rk_P(A)$ of A in P is the least ordinal α such that $A \in D^\alpha(P) \setminus D^{\alpha+1}(P)$; the C-B rank $rk(P)$ is the least α such that $D^\alpha(P) = K(P)$. The set A is *ranked* if there is a Π^0_1 class P such that $A \in P \setminus K(P)$, and the C-B rank $rk(A)$ is the least α such that $rk_P(A) = \alpha$ for some Π^0_1 class P. Kreisel [116] used the Boundedness Principle of Spector [201] to show that $rk(P) \leq \omega_1^{\mathrm{C-K}}$ for any Π^0_1 class P, so that any ranked point A has $rk(A) < \omega_1^{\mathrm{C-K}}$.

Theorem V.1.4 (Kreisel). *For any Π^0_1 class P, $K(P)$ is a Σ^1_1 class and $rk(P) \leq \omega_1^{C-K}$.*

Proof. Fix a computable tree T such that $P = [T]$ and define the Σ^0_2 monotone inductive operator Γ on $\{0,1\}^*$ as follows.

$$\sigma \in \Gamma(A) \iff \sigma \notin T \;\vee\; \sigma \notin d(\{0,1\}^* - A).$$

It follows that $\Gamma^1 = \{0,1\}^* - T$, $\Gamma^1 = \{0,1\}^* - d(T)$, and so on, so that $Cl(\Gamma) = \{0,1\}^* - T_{K(P)}$. It follows from Theorem II.II.9.8 that $T_{K(P)}$ is Σ^1_1, so that $K(P) = [T_{K(P)}]$ is also Σ^1_1. It follows from Corollary II.9.13 that $rk(P) \leq \omega_1$. □

We will show in Section VI.VI.7 that there is a Π^0_1 class P such that $rk(P) = \omega_1^{C-K}$ and $K(P)$ is not Δ^1_1.

For the sake of completeness, let $rk_P(A) = rk(P)$ if $A \in K(P)$. This means that the rank function will define a prewellordering on P.

A fundamental idea here is that the complexity of an element x of a Π^0_1 class P is related to the Cantor-Bendixson rank of x in P. Kreisel [116] first noticed that the Turing degree of a member x of a Π^0_1 class is related to the CB rank; he used this to show that every member of a countable Π^0_1 class is hyperarithmetic. In particular, it is easy to see that a real has rank 0 if and only if it is computable.

We need a series of lemmas from [46]. Note first that for two closed sets P and Q, $D(P \cup Q) = D(P) \cup D(Q)$ but $D(P \cap Q)$ is in general only a subset of $D(P) \cap D(Q)$.

More generally, we have

Lemma V.1.5. *For any closed sets P and Q and any ordinal α:*

(a) $D^\alpha(P \cup Q) = D^\alpha(P) \cup D^\alpha(Q)$ *and*

(b) $D^\alpha(P \cap Q) \subset D^\alpha(P) \cap D^\alpha(Q)$.

Proof. The proofs are by induction on α. For $\alpha = 0$, these are trivial.

(a) For $\alpha = 1$, $D(P)$ and $D(Q)$ are subsets of $D(P \cup Q)$ since P and Q are subsets of $P \cup Q$. For the reverse inclusion, suppose that $x \notin D(P) \cup D(Q)$. Then there are clopen sets U and V with $U \cap P = \{x\}$ and $V \cap Q = \{x\}$. Thus $U \cap V \cap (P \cup Q) = \{x\}$, so that $x \notin D(P \cup Q)$. Now assume that $D^\alpha(P \cup Q) = D^\alpha(P) \cup D^\alpha(Q)$. Then using the case $\alpha = 1$, we have

$$\begin{aligned} D^{\alpha+1}(P \cup Q) &= D(D^\alpha(P \cup Q)) = D(D^\alpha(P) \;\cup\; D^\alpha(Q)) \\ &= D(D^\alpha(P)) \;\cup\; D(D^\alpha(Q)) = D^{\alpha+1}(P) \;\cup\; D^{\alpha+1}(Q). \end{aligned}$$

(b) This is left as an exercise. □

Note that equality holds in (b) if one of the sets is clopen.

Lemma V.1.6. *For any compact subset Q of $\{0,1\}^\omega$, any clopen set K and any ordinal α,*

(a) $D^\alpha(K \cap Q) = K \cap D^\alpha(Q)$.

(b) $rk_{K\cap Q}(A) = rk_Q(A)$ *for any* A.

The proof is left as an exercise.

Lemma V.1.7. *For any set A and any computable ordinal α, $rk(A) \leq \alpha$ if and only if there is some Π^0_1 class P such that $D^\alpha(P) = \{A\}$.*

Proof. The "if" direction is immediate from the definition of rank. Suppose now that $rk(A) \leq \alpha$, so that $A \in D^\alpha(Q) \setminus D^{\alpha+1}(Q)$ for some Π^0_1 class Q. Thus A is isolated in $D^{\alpha(Q)}$, so that for some clopen K, $K \cap D^\alpha(Q) = \{A\}$. Let $P = K \cap Q$. It follows from Lemma V.1.6 that
$D^\alpha(P) = D^\alpha(K \cap Q) = K \cap D^\alpha(Q) = \{A\}$. □

Lemma V.1.8. *(Cenzer-Smith [46])*

(a) Let Φ be a continuous map from $\{0,1\}^\omega$ into $\{0,1\}^\omega$ and let P, Q be compact sets such that $\Phi(P) = Q$. Then for any $y \in Q$, $rk_Q(y) \leq max\{rk_P(x) : x \in P \ \&\ \Phi(x) = y\}$.

(b) For any sets A, B, if $A \leq_{tt} B$ and B is ranked, then A is ranked and $rk(A) \leq rk(B)$.

(c) For any sets A, B, if $A \equiv_{tt} B$ and B is ranked, then A is ranked and $rk(A) = rk(B)$.

Proof. Let $rk_Q(y) = \beta$, so that $y \in D^\beta(Q) \setminus D^{\beta+1}(Q)$. It is shown in Lemma 1.2 of [46] that $D^\alpha(\Phi(P)) \subset \Phi(D^\alpha(P))$ for any α. Since $y \in D^\beta(Q)$, it follows that $y = \Phi(x)$ for some $x \in D^\beta(P)$, so that $rk_Q(y) = \beta \leq rk_P(x)$, where it is possible that x is not ranked in P.

(b) By Theorem II.5.10, if $A \leq_{tt} B$, then there is a computable function $\Phi : \{0,1\}^\omega \to \{0,1\}^\omega$ such that $\Phi(B) = A$. Now let $rk(B) = \alpha$ and, by Lemma 4.2, let P be a Π^0_1 class such that $D^\alpha(P) = \{B\}$, so that $rk_P(B) = max\{rk_P(x) : x \in P\}$, and let $Q = \Phi(P)$. It follows from (a) that $rk(A) \leq rk_Q(A) \leq rk_P(B) = rk(B)$.

(c) This is immediate from (b). □

The following improvement of Lemma 1.4 of [46] is due to J. Owings and C. Laskowski [166]. Recall that $\alpha \oplus \beta$ is the Hessenberg sum of two ordinals, that $A \oplus B$ is the disjoint union of two sets and that $P \oplus Q = \{A \oplus B : A \in P \ \&\ B \in Q\}$ for two classes P and Q of sets.

Theorem V.1.9. *(Owings, [166]) For any sets $A, B \in \{0,1\}^{\mathbb{N}}$ and any compact $P, Q \subset \{0,1\}^{\mathbb{N}}$, $rk_{P\oplus Q}(A \oplus B) = rk_P(A) \oplus rk_Q(B)$.*

Proof. We first show by induction on $rk_P(A) \oplus rk_Q(B)$ that $rk_{P\oplus Q}(x \oplus y) \leq rk_P(A) \oplus rk_Q(B)$. If $rk_P(A) \oplus rk_Q(B) = 0$, then A is isolated in P and B is isolated in Q, so that there are open intervals I and J such that $P \cap I = \{A\}$ and $Q \cap J = \{B\}$. It follows that $(P \oplus Q) \cap (I \oplus J) = \{A \oplus B\}$, so that $rk_{P\oplus Q}(A \oplus B) = 0$. Now let $rk_P(A) = \alpha$ and $rk_Q(B) = \beta$ and suppose the inequality holds for all x, y such that $rk_P(x) \oplus rk_Q(y) < \alpha \oplus \beta$. By intersecting with open intervals, as above, we may assume that $D^\alpha(P) = \{A\}$ and that $D^\beta(Q) = \{B\}$. It suffices to show that $rk_{P\oplus Q}(x \oplus y) < \alpha \oplus \beta$ for all $x \oplus y \neq A \oplus B$ in $P \oplus Q$. But if $x \oplus y \neq A \oplus B$, then either $x \neq A$ or $y \neq B$, so that either $rk_P(x) < \alpha$ or $rk_Q(y) < \beta$. In either case, $rk_P(x) \oplus rk_Q(y) < \alpha \oplus \beta$, so that $rk_{P\oplus Q}(x \oplus y) < \alpha + \beta$.

For the reverse inequality, we prove by induction on $\alpha \oplus \beta$ that

$D^\alpha(P) \oplus D^\beta(Q) \subset D^{\alpha\oplus\beta}(P \oplus Q)$.

For $\alpha \oplus \beta = 0$, this is obvious. We also need the case where $\alpha \oplus \beta = 1$. Suppose without loss of generality that $\alpha = 1$ and $\beta = 0$ and suppose that $A \in D(P)$ and $B \in Q$. Then for any interval $I \subset \{0,1\}^\omega$, there is some $A' \neq A$ in $P \cap I$. Then for any basic open set $I \oplus J \in \{0,1\}^\omega \oplus \{0,1\}^\omega$, there is an element $A' \oplus B \neq A \oplus B$ in $(P \oplus Q) \cap (I \oplus J)$. Thus $A \oplus B \in D(P \oplus Q)$. This shows that

$D(P) \oplus Q \subset D(P \oplus Q)$.

Now suppose the inclusion holds for all ordinals σ, τ with $\sigma \oplus \tau < \alpha \oplus \beta$. There are two cases.

(Case 1) If $\alpha \oplus \beta$ is a limit ordinal, then α and β are both limit ordinals and $\alpha \oplus \beta = sup\{\sigma \oplus \tau : \sigma < \alpha \ \& \ \tau < \beta\}$. Thus

$$\begin{aligned} D^{\alpha\oplus\beta}(P \oplus Q) &= \cap_{\gamma<\alpha\oplus\beta} D^\gamma(P \oplus Q) = \cap_{\sigma<\alpha,\tau<\beta} D^{\sigma\oplus\tau}(P \oplus Q) \\ &= \cap_{\sigma<\alpha,\tau<\beta} D^\sigma(P) \oplus D^\tau(Q) = D^\alpha(P) \oplus D^\beta(Q). \end{aligned}$$

(Case 2) If $\alpha \oplus \beta$ is a successor, then either α is a successor or β is a successor– without loss of generality say that $\alpha = \gamma + 1$, so that $\alpha \oplus \beta = (\gamma \oplus \beta) + 1$. Then

$$\begin{aligned} D^\alpha(P) \oplus D^\beta(Q) &= D(D^\gamma(P)) \oplus D^\beta(Q) \subset D(D^\gamma(P) \oplus D^\beta(Q)) \\ &\subset D(D^{\gamma\oplus\beta}(P \oplus Q)) = D^{\alpha\oplus\beta}(P \oplus Q). \end{aligned}$$

This completes the proof. □

Thus we have the following corollary.

Theorem V.1.10. *For any sets A and B, $max\{rk(A), rk(B)\} \leq rk(A \oplus B) \leq rk(A) \oplus rk(B)$.*

Proof. The first inequality follows from Lemma V.1.8, since both A and B are $\leq_{tt} A \oplus B$. The second inequality follows from Theorem V.1.9. □

The basic result for rank 0 is the following.

Lemma V.1.11. *For any $x \in \{0,1\}^\omega$, the following are equivalent:*

(a) x is computable;

(b) $\{x\}$ is a Π^0_1 class;

(c) x has Cantor-Bendixson rank 0.

Proof. Suppose first that x is computable. Then $\{x\} = [T]$, where $\sigma \in T \iff (\forall i < |\sigma|)(\sigma(i) = x(i))$. Next suppose that $\{x\}$ is a Π^0_1 class. Then the rank of x in $\{x\}$ is 0, so that the C-B rank of x is 0. Next suppose that x has C-B rank 0 and let $P = [T]$ be a Π^0_1 class such that x is isolated in P, where T is a computable tree. Then for each sufficiently large n, $x\lceil n+1$ is the unique path of length n which has an extension in P. Thus we may compute $x\lceil n+1$ (and therefore compute $x(n)$) by searching for the least m such that all strings $\sigma \in T$ of length m have the same initial segment $\sigma\lceil n$. □

We remark that Lemma V.1.11 and its proof can be relativized to computability in B for any set B.

Exercises

V.1.1. Give a proof of Lemma V.1.5(b).

V.1.2. Give a proof for Lemma V.1.6.

V.1.3. Prove a relativized version of Lemma V.1.11.

V.1.4. Show by induction on ordinals α that for any continuous map $\Phi : \{0,1\}^{\mathbb{N}} \to \{0,1\}^{\mathbb{N}}$, and any compact set Q, $D^\alpha(\Phi(P)) \subseteq \Phi(D^\alpha(P))$.

V.1.5. Show that if $rk(B) = rk(A \oplus B)$, then A is computable in B.

V.1.6. Owings [166] defined a Cantor singleton as being the unique noncomputable element of some Π^0_1 class. By Theorem III.6.2, every noncomputable Π^0_1 retraceable set is a Cantor singleton. Show that any noncomputable set which is the union of a computable set with a Π^0_1 retraceable set is also a Cantor singleton.

V.1.7. Show that if B is a Cantor singleton and $A \leq_{tt} B$, then A is a Cantor singleton.

V.1.8. Show that if $A \oplus B$ is a Cantor singleton, then either B is computable or A is computable in B.

V.1.9. A set A is said to be *autoreducible* if there is a computablee functional F such that, for all n, $A(n) = F(n, A \setminus \{n\})$. Show that every ranked set A is autoreducible (Owings). (Hint: note that, for any n, $A \setminus \{n\}$ and $A \cup \{n\}$ are both $\equiv_{tt} A$ and thus have rank α.)

V.2 Basis results

In this section, we consider basis results for countable Π^0_1 classes. The key observation here is the following.

Theorem V.2.1. *Any countable closed $P \subset \{0,1\}^{\mathbb{N}}$ has countable rank and has an isolated point.*

Proof. If P has no isolated points, then P is perfect and hence uncountable. For the rank, observe that for $\alpha < rk(P)$, $D^\alpha(P) \setminus D^{\alpha+1}(P)$ is nonempty. $\square$

Basis results for countable Π^0_1 classes can thus be obtained from the following.

Theorem V.2.2. *(Kreisel [116]) Let P be a Π^0_1 class.*

(a) Any isolated member of P is hyperarithmetic; if P is finite, then every member of P is hyperarithmetic.

(b) Suppose that P is bounded. Then any isolated member of P is computable in $0'$; if P is finite, then every member of P is computable in $0'$.

(c) Suppose P is computably bounded. Then any isolated member of P is computable; if P is finite, then every member of P is computable.

Proof. Let x be isolated in $P = [T]$ and take n large enough so that $P \cap I(x\lceil n) = \{x\}$. Now define the Π^0_1 class Q to be $P \cap I(x\lceil n)$. That is, $Q = [S]$, where S is the computable tree consisting of all strings in T which are compatible with $x\lceil n$. It follows that $Ext(S) = \{x\lceil n : n < \omega\}$, so that x is computable in $Ext(S)$. Now consider the three cases.

(a) For an (unbounded) tree S, $Ext(S)$ is Σ^1_1 by Theorem III.2.13 and since $[S]$ is a singleton, we have

$\sigma \in Ext(S) \iff (\forall \tau \in \omega^{|\sigma|}(\tau \neq \sigma \rightarrow \tau \notin Ext(S)),$

so that $Ext(T)$ is also Π^1_1. It follows that $Ext(T)$ and hence x are hyperarithmetic.

(b) For a finitely branching tree T, $Ext(T)$ is Π^0_2 by Theorem III.2.13. Now it follows from König's Lemma that for each n, there is some $k \geq n$ such that every sequence in T of length k is an extension of $x\lceil n$. Thus we have

$\sigma \in Ext(S) \iff (\exists k \geq |\sigma|)(\forall \tau \in \omega^k)(\tau \in S \rightarrow \sigma \prec \tau),$

so that $Ext(S)$ is also Σ^0_2. It follows that $Ext(S)$ and hence x are computable in $0'$.

(c) In this case, $Ext(S)$ is Π^0_1 by Theorem III.2.13 and is also Σ^0_1 since

$\sigma \in Ext(S) \iff (\exists n \geq |\sigma|)(\forall \tau \in \{0,1,\ldots,f(n)\}^n)(\tau \in S \rightarrow \sigma \prec \tau),$

where f is a computable bounding function for S. Thus $Ext(S)$ and x are both computable.

Suppose now that P is finite. The conclusion in each case follows from the fact that every member of P will be isolated. $\square$

Combining this with the previous result, we have

Theorem V.2.3. *(Kreisel) Let P be a countable Π^0_1 class.*

(a) P has a hyperarithmetic member.

(b) If P is bounded, then P has a member computable in $0'$.

(c) If P is computably bounded, then P has a computable member.

As a corollary to the proof of Theorem V.2.2, we also have the following.

Theorem V.2.4. *Let $P = [T]$ be a Π^0_1 class.*

(a) Suppose that T is finite branching. Then any isolated member of P is computable in T'; if P is finite, then every member of P is computable in T'. Thus if P is countable, then P has a member computable in T'.

(b) Suppose T is computably bounded. Then any isolated member of P is computable in T; if P is finite, then every member of P is computable in T. Thus if P is countable, then P has a member computable in T.

V.3 Ranked Points and Rank-Faithful Classes

The following notions were introduced by G. Martin [145] and J. Owings [166]. A Π^0_1 class is said to be *rank-faithful* if $rk_P(x) = rk(x)$ for all $x \in P$. The Π^0_1 classes constructed in Theorems III.6.8, V.4.5 and V.4.8 are clearly rank-faithful. A real x is said to be a *Cantor Singleton* if it is the unique non-computable member of some Π^0_1 class. Theorem III.ref2.2 implies that every non-computable Π^0_1 retraceable set is a Cantor singleton. G. Martin improved this result by showing that any noncomputable set which is the union of a computable set with a Π^0_1 retraceable set is a Cantor singleton. (See the exercises.)

Theorem III.III.6.2 was improved to the following.

Theorem V.3.1. *(G. Martin) For each computable ordinal α and every non-zero r.e. degree $\mathbf{a}$, there are c. e. sets A and B of degree $\mathbf{a}$ and rank α such that A is a Cantor singleton and B belongs to a rank-faithful Π^0_1 class.*

Recall from Theorem III.8.2 that a minimal Π^0_1 class is a thin class of rank one such that all computable elements are isolated and the unique nonisolated element is noncomputable. It follows that any minimal Π^0_1 class is rank-faithful. This can be extended to arbitrary thin classes as follows.

Theorem V.3.2. *Any thin Π^0_1 class P is rank-faithful.*

Proof. Suppose that $A \in P$ and that $rk(A) = \alpha$. Let $D^\alpha(Q) = \{A\}$. Then $A \in P \cap Q$ and $P \cap Q = P \cap U$ for some clopen set U. It follows from Lemma V.1.6 that $rk_P(A) = rk_{P\cap Q}(A) \leq rk_Q(A)$. Equality follows since $rk_Q(A)$ is minimal by assumption. □

Here are two interesting results from Owings [166].

Theorem V.3.3. *(Owings) (a) If $A \oplus B$ is a Cantor singleton, then either B is computable or A is computable in B, so that the degree of $A \oplus B$ is either the degree of A or the degree of B.*

(b) If $rk(B) = rk(A \oplus B)$, then A is computable in B.

Proof. (a) Suppose that $A \oplus B$ is the unique noncomputable element of P and that B is noncomputable. Let $Q = P \cap (\{0,1\}^\omega \oplus \{B\}$ and observe that $B \leq_{tt} C$ for every $C \in Q$, so that any $C \in Q$ is noncomputable. Thus $A \oplus B$ is the unique element of Q and is therefore computable in B by the relativized version of Lemma V.1.11.

(b) Let $\alpha = rk(A \oplus B)$ and let P be a Π^0_1 class such that $D^\alpha(P) = \{A \oplus B\}$. As in (a), let $Q = P \cap (\{0,1\}^\omega \oplus \{B\})$. It follows from Lemma 4.3(b) that $rk(C) \geq \alpha$ for all $C \in Q$, so that in fact $Q = \{A \oplus B\}$. Then $A \oplus B$ is computable in B as above. □

Recall that a set A is *autoreducible* if there is a computable functional F such that, for all n, $A(n) = F(n, A \setminus \{n\})$.

Theorem V.3.4. *(Owings) Every ranked set A is autoreducible.*

Proof. Let $rk(A) = \alpha$ and let $P = [T]$ be a Π^0_1 class such that $D^\alpha(P) = \{A\}$. Note that for any n, $A \setminus \{n\}$ and $A \cup \{n\}$ are both $\equiv_{tt} A$ and thus also have rank α. It follows that only one of them can belong to P. Thus, given n and $A \setminus \{n\}$, we search for the least k such that every $\sigma \in T$ of length k consistent with $A \setminus \{n\}$, except possibly at n, has the same value $\sigma(n)$. Then $A(n) = \sigma(n)$. □

Exercises

V.3.1. Prove G. Martin's result that any noncomputable set which is the union of a computable set with a Π^0_1 retraceable set is a Cantor singleton.

V.3.2. For any sets A, B such that B is a Cantor singleton and $A \leq_{tt} B$, show that A is a Cantor singleton (Owings [166].) Hint: recall the proof of Lemma V.4.3.

V.4 Rank and Complexity

In this section, we examine the connection between the rank and the hyperarithmetic complexity of a set.

Our first theorem will be a generalization of Theorem V.2.2 (c) to arbitrary rank.

First we consider the complexity of the tree resulting from the iterated C-B derivative.

Lemma V.4.1. *For any computable tree $T \subseteq \{0,1\}^*$, any computable limit ordinal λ and any finite $n > 0$,* *(a)* $d^n(T) \leq_T \mathbf{0}^{(2n)}$ *and is* Σ^0_{2n}*;*

(b) $d^\lambda(T) \leq_T \mathbf{0}^{(\lambda+1)}$ *and is* $\Pi^0_{\lambda+1}$*;*

(c) $d^{\lambda+n}(T) \leq_T \mathbf{0}^{(\lambda+2n)}$ *and is* $\Sigma^0_{\lambda+2n+1}$*.*

Proof. Let T be given and define the inductive operator Γ on $\{0,1\}^*$ by

$$\sigma \in \Gamma(U) \iff \sigma \notin d(T) \ \vee \ \sigma \notin d(\{0,1\}^* - U).$$

Then Γ is Π^0_2 by Definition V.1.2 and it is easy to see that for all ordinals α,

$$d^\alpha(T) = \{0,1\}^* - \Gamma^\alpha.$$

The result now follows immediately from Theorem III.II.10.9. □

For decidable trees, there is a slight improvement.

Lemma V.4.2. *For any decidable tree* $T \subseteq \{0,1\}^*$*, any computable limit ordinal* λ *and any finite* $n > 0$*,* $d^n(T) \leq_T \mathbf{0}^{(2n-1)}$ *and is* Σ^0_{2n-1}*.*

Proof. Since T is decidable, $Ext(T)$ is a computable set and it follows that $d(T)$ is a Π^0_1 set. The result now follows from Exercise III.3. □

Theorem V.4.3. *For any* $A \in \{0,1\}^{\mathbb{N}}$*, any* Π^0_1 *class* P*, any finite* n *and any limit ordinal* λ*,*

(a) If $rk_P(A) = n$*, then* $A \leq_T 0^{(2n)}$*. Furthermore, if* $rk_P(A) = n$ *for some* **decidable** P*, then* $A \leq_T 0^{(2n-1)}$*.*

(b) If $rk_P(A) = \lambda + n$*, then* $A \leq_T 0^{(\lambda+2n+1)}$*.*

Proof. (a) Let T be a computable binary tree such that $[d^n(T)] = \{A\}$. It follows from Lemma V.4.1 that $d^n(T) \leq_T 0^{(2n)}$ and then by Theorem V.2.4 that $A \leq_T 0^{(2n)}$. For a decidable class $P = [T]$, $d^n(T) \leq_T 0^{(2n-1)}$ by Lemma V.4.2.

(b) Similarly $x \leq_T d^{\lambda+n}(T) \leq_T \mathbf{0}^{(\lambda+2n)}$. □

Theorem V.4.4. *For any countable* Π^0_1 *class* $P \subseteq \{0,1\}^{\mathbb{N}}$*,*

(a) $rk(P)$ *is a computable ordinal;*

(b) Every element of P *is hyperarithmetic.*

Proof. Since P is countable, it follows that the perfect kernel of P is empty. Thus the inductive definition Γ given in the proof of Lemma V.4.1 has closure $\{0,1\}^*$ which is a computable set. It follows from the Boundedness Theorem for inductive definability (Theorem II.II.9.12) that $|\Gamma| < \omega_1$ and therefore $rk(P) < \omega_1$. It now follows from Theorem V.4.3 that every element of P is hyperarithmetic. □

It follows from Theorem V.4.3 that any set of rank one must be computable in $0''$. We now analyze the rank one sets further. Recall from Theorem III.6.8 that every c. e. set is Turing equivalent to a hypersimple r.e. set of rank one. This result has the following improvement.

Theorem V.4.5. *(Cenzer-Smith [46] For any noncomputable degree* $\mathbf{b} \leq 0'$*, there is a hyperimmune set* B *with degree* $\mathbf{b}$ *of rank one; furthermore, there is a computable tree* T *with no dead ends such that* $D([T]) = \{B\}$.

Proof. Let A be a set of degree $\mathbf{b}$. By the limit lemma, there is a computable function f such that, for all e,
$A(e) = \lim_n f(n,e)$.

Let $n(0)$ be the least $n > 0$ such that $f(n,0) = A(0)$ and, for any e, let $n(e+1)$ be the least $n > n(e)$ such that, for all $i < e+2$, $f(n,i) = A(i)$. Then $n(0) < n(1) < \cdots$ is a *modulus* for the set A, so that $B = \{n(0), n(1), \cdots\}$ has degree $\mathbf{b}$.

We define a Π^0_1 class P with $rk_P(B) = 1$.

The (possibly finite) set $C = \{m(0) < m(1) < \cdots\}$ is in P if and only if $0 < m(0)$ and for all e, i, and m:

1. $(0 < m < m(0) \ \& \ C \neq 0) \rightarrow f(0,m) \neq f(0,m(0))$;
2. $e < i < card(C) \rightarrow f(m(i),e) = f(m(e),e)$;
3. $(e+1 < card(C) \ \& \ m(e) < m < m(e+1)) \rightarrow (\exists j < e+2)(f(m(e+1),j) \neq f(m,j))$.

It is clear that P may be defined by a tree T without dead ends and in fact closed under extension by 0. Also, P contains all initial subsets of B and in fact is closed under initial subsets, so that $rk_P(B) \geq 1$.

If $C = \{m(0) < m(1) < \cdots\}$ is any infinite set in P, then it follows from (ii) that $f(m(e),e) = f(n(e),e) = A(e)$ for all e. But it then follows from (i) or, by induction, from (iii) that $m(e) = n(e)$ for all e, so that $C = B$.

If $C = \{m(0) < m(1) < \cdots < m(k)\}$ is any finite set in P, then it follows from (i) and (iii) that there are at most two extensions $C \cup \{m\}$ of C in P (one with $f(m,k+1) = 0$ and one with $f(m,k+1) = 1$), so that C is isolated in P.

Thus B is the only non-isolated element of P and therefore $rk_P(B) = 1$.

To see that B is hyperimmune, suppose by way of contradiction that h were a computable function with $h(e) > n(e)$ for all e. Then we could define a Π^0_1 subclass Q of P by adding the restriction

(iv) $(\forall e)[card(C \cap \{0,1,\ldots,h(e)-1\}) > e]$.

□

We shall see later that not every Σ^0_2 degree contains a ranked set. Thus there is a more complicated result for degrees below $\mathbf{0}''$. The following theorem gives a partial inverse C-B derivative and shows that a large class of degrees contain ranked points.

Theorem V.4.6. *[23] For any real B and any tree $S \subset \{0,1\}^*$ such that S is computable in B'', there is a tree T computable in B and a homeomorphism Φ from $[S]$ onto $D([T])$ such that for all $x \in [S]$, $x \leq_T \Phi(x) \leq_T x \oplus B'$.*

Proof. Let F be a function, computable in B, such that, for all $\sigma \in \{0,1\}^*$,

$$\chi_S(\sigma) = \lim_{p\to\infty} \lim_{n\to\infty} F(p,n,\sigma).$$

For each p and σ, let $F_p(\sigma) = \lim_{n\to\infty} F(p,n,\sigma)$.

Let $x \in \{0,1\}^{\mathbb{N}}$ be given and assume that $x \in [S]$. We define "outer modulus" values $p = p_k(x)$ such that $F_p(x\lceil k) = 1$ and "inner modulus" values $n = n_k(x)$ such that $F(p_k, n, x\lceil k) = 1$.

Let $p_1 = p_1(x)$ be the least $p > 0$ such that $F_p(x\lceil 1) = 1$ and let $n_1 = n_1(x)$ be the least $n > p_1$ such that, for all $m \geq n$, $F(p,m,x\lceil 1) = 1$. Now inductively define $p_{e+1}(x)$ to the least $p > n_e$ such that $F_p(x\lceil i) = 1$ for all $i \leq e+1$. Since $x \in [S]$, p_{e+1} exists. Now let n_{e+1} be the least $n > p_{e+1}$ such that

(1) for all $i \leq e+1$ and all $m \geq n$, $F(p_{e+1}, m, x\lceil i) = 1$, and

(2) for all p with $n_e < p < p_{e+1}$, there is an $i \leq e+1$ such that, for all $m \geq n$, $F(p,m,x\lceil i) = 0$.

Note that our choice of n_{e+1} not only verifies that $F_{p_{e+1}}(i) = 1$ for all $i \leq e+1$, it also verifies that p_{e+1} is the least $p > n_e$ with this property.

Now for $x \in [S]$, let

(3) $H(x) = \{x(0)^\frown 1^\frown 0^{n_1\frown} 1^\frown 0^{p_1\frown} 1x(1)^\frown 1^\frown 0^{n_2\frown} 1^\frown 0^{p_2\frown} \ldots$

It is clear from the definition that x is computable from $H(x)$ and that $H(x)$ is computable in $0' \oplus B'$.

The tree T is to contain all initial segments of each image $H(x)$ together with certain strings which resemble those initial segments when only finitely many values of F are considered. Every element of $[T] - H([S])$ will be isolated in $[T]$ and each point $H(x)$ will be the limit of a distinct sequence of points from $[T] \setminus H([S])$.

Elements of the computable tree T with $D([T]) = \{B\}$ will be initial segments of strings σ of the form

(4) $\sigma = r(0)10^{n_1}10^{p_1}1r(1)10^{n_2}10^{p_2}\ldots r(k-1)10^{n_k}10^{p_k}1.$

The string σ as above is said to be *consistent* if it satisfies the following conditions:

(5) $0 < p_1 < n_1 < p_2 < n_2 < \cdots < p_k < n_k$.

(6) For all i, j, k and m, if $i \leq j \leq k$ and $n_j \leq m \leq n_k$, then $F(p_j, m, r\lceil i) = 1$.

(7) For all j, k and p, if $j \leq k$ and $n_{j-1} < p < p_j$, then there is an $i \leq j$ such that, for all m with $n_i \leq m \leq n_k$, $F(p,m,i) \neq F(p,m,r\lceil i) = 0$.

If $\tau = \sigma^\frown r$ or $\tau = \sigma^\frown r^\frown 1^\frown 0^n$, then τ is consistent if σ is consistent. The string $\rho = \sigma^\frown r^\frown 1^\frown 0^{n\frown} 1$ is consistent if σ is consistent and if conditions (ii) and (iii) hold when n_k is replaced by n and the string $\rho^\frown 1^\frown 0^p$ is consistent if ρ is consistent and $p < n$.

It is clear that for all $x \in [S]$, every initial segment of $H(x)$ is consistent. Now let $y \in \{0,1\}^{\mathbb{N}}$ be an extension of σ as in (4) above and suppose that every initial segment of y is consistent. There are two possibilities. Either y has the form in (3) above or $y = \tau^\frown 0^\omega$, where $\tau = \sigma^\frown r(k+1)^\frown \dots {}^\frown r(\ell)^\frown 1$. In either case, the values of n_j are unbounded in the initial segments of y. It now follows from (5) that:

(8) For all $i \leq j \leq k$, $F_{p_j}(r\lceil i) = 1$.

It also follows from (v) that:

(9) For all $j \leq k$ and all p with $n_{j-1} < p < p_j$, there exists $i \leq j$ such that $F_p(r\lceil i) = 0$.

It is now clear that if y has form (3), then for all j, p_j is the least p greater than n_{j-1} (or > 0 for $j = 1$) such that $F_p(r\lceil i) = 1$ for all $i \leq j$. It follows that $x \in [S]$.

A consistent string τ with $\sigma \preceq \tau \preceq \sigma^\frown r^\frown 1^\frown 0^n_{k+1}{}^\frown 1^\frown 0^p_{k+1}$ is said to be *exact* if each n_i is minimal, that is,

(10) For all i, k, n, if $i \leq k$ and $p_i < n < n_i$, then $r(0)10^{n_1}10^{p_1}1 \dots 1r(i)10^n 10^{p_i}1$ is not consistent.

The desired tree T is defined to be the set of all exact strings. It is clear that for each $x \in [S]$, $H(x)$ is in $[T]$; furthermore, for each k, $x(0)^\frown 1^\frown 0^{n_1(x)\frown} 1 \dots {}^\frown 1^\frown x(k)^\frown 1^\frown 0^\omega$ belongs to $[T]$. Thus $H(x) \in D([T])$ as desired. It remains to show that all other elements of $[T]$ are isolated.

First suppose that $y \in [T]$ has infinitely many 1's. Then we saw above that y has the form (3) where $x \in [S]$ with each p_j minimal. It now follows from (10) that each n_i is also minimal so that $y = H(x)$. Thus any element of y of $[T] = H([S])$ must have the form

$$y = r(0)10^{n_1}10^{p_1}1r(1)10^{n_2}10^{p_2} \dots r(k)10^\omega = \tau^\frown 0^\omega$$

Now suppose that $z \neq y$ and $z \in [T] \cap I(\tau)$. Then for some n_k and p_k, $\tau^\frown 1^\frown 0^{n_k\frown} 1^\frown 0^{p_k\frown} 1 \prec z$. It follows from (8) and (9) that p_k is uniquely determined and from (10) that n_k is uniquely determined. Thus y is the unique element of $[T]$ which extends $\tau^\frown 1^\frown 0^{n_k+1}$ and hence y is isolated in $[T]$ as desired.

We now have that H is an isomorphism of $[S]$ onto $[S]$ onto $D([T])$. Since H is computable in B, it must be continuous and since $[S]$ and $D([T])$ are compact, it follows that H^{-1} is also continuous. This completes the proof. $\square$

We note that this proof is uniform in S. That is, there is a primitive recursive function ϕ such that if e is an index for S as a Π^0_3-in-B set, then $\phi(e)$ is an index for T as a Π^0_1-in-B set.

Corollary V.4.7. *(a) For any finite n and any tree $S \leq_T \mathbf{0}^{(2n)}$, there is a computable tree T and a homeomorphism H from $[S]$ onto $D^n([T])$ such that, for all $x \in [S]$, $x \leq_T H(x) \leq_T x \oplus \mathbf{0}^{(2n-1)}$.*

(b) For any finite n and any real x such that $\mathbf{0}^{(2n-1)} \leq_T x \leq_T \mathbf{0}^{(2n)}$ or such that $\mathbf{0}^{(2n-2)} \leq_T x \leq_T \mathbf{0}^{(2n-1)}$, there is a computable tree T and a real $y \equiv_T x$ with $|y|_T = |y| = n$.

Proof. Part (a) follows easily from Theorem V.4.6 by induction on n. Part (b) now follows from (a) by letting $[S] = \{x\}$. □

This result can be extended to higher levels of the hyperarithmetic hierarchy as follows. The following is Theorem 42 of [39].

Theorem V.4.8. *For any computable ordinal α and any c. b. $\Pi^0_{2\alpha+1}$ class Q, there exists a Π^0_1 class P of sets and a homeomorphism from Q onto $D^\alpha(P)$ such that $x \leq_T H(x) \leq x \oplus 0^{2\alpha-1}$ for all $x \in Q$.*

Proof. The proof is by a uniform recursion up to a fixed computable ordinal κ with a set of notations as given by Lemma II.8.12. We may assume that Q is actually a class of sets and build a P which is computably bounded. In fact, we only need to assume that $Q = [S]$, where for all $\sigma \in S$ and all relevant notations a, $\sigma(i) \neq a$ for any i.

We will actually define computable functions f and ψ such that $\phi^{0^{o(a)-1}}_{\psi(e,a)}$ is a homeomorphism from $P_{e,2o(a)+1}$ onto $D^{o(a)}(P_{f(e,a)})$ and such that $x \leq \phi_{\psi(e,a)}(x)$ for each x.

The construction will be presented as a transfinite recursion on $o(a)$, but is actually obtained by the recursion theorem.

We need a series of lemmas. We shall write $A = \sqcup_n B_n$ if $A = \cup_n B_n$ and the elements of $\{B_n : n \in \omega\}$ are pairwise disjoint

Lemma V.4.9. *There is a primitive recursive function ρ such that, for each $\alpha = o(a)$, $P_{e,\alpha+3} = [T]$, with $T = \sqcup_n U_{\rho(e,a,n),\alpha+1}$.*

Proof. From the definition, $P_{e,\alpha+3} = [U]$, where $U = [U_{e,\alpha+3}]$ is uniformly $\Pi^0_{\alpha+3}$. It is then easy to define, as in Proposition III.3.7 a $\Sigma^0_{\alpha+2}$ tree S such that $[U] = [S]$. Thus there exists a Σ^0_α relation R such that

$$\sigma \in S \iff (\exists n)(\forall m) R(m,n,e,\sigma)$$

As usual, we may assume that $\neg R(m,n,e,\sigma) \to \neg R(m,n+1,e,\sigma)$ and that $(\forall m)R(m,n,e,\sigma) \to (\forall m)R(m,n+1,e,\sigma)$.

Now define the desired trees by

$\sigma \in U_{\rho(e,a,\langle m,n\rangle),\alpha+1}$ if and only if

1. $(\forall m')(\forall i \leq |\sigma|)R(m', n, e, \sigma\lceil i)$
2. $(\forall n' < n)(\exists m' < m)(\exists i \leq |\sigma|)\neg R(m', n', e, \sigma\lceil i)$
3. $(\forall m' < m)(\exists n' < n)(\forall m'' < m')(\forall i \leq |\sigma|)R(m'', n', e, \sigma\lceil i)$.

$\square$

We need a slightly modified notion from [23] of a *completely ranked* tree.

Definition V.4.10. *A tree T is completely ranked up to α if, for all $y \in [T]$ and all notations a with $o(a) < \alpha$,*

1. *$y \in D^{o(a)}([T])$ if and only if $y(n) = a$ for infinitely many n.*
2. *$y \notin D^{o(a)}([T])$ implies that for some n, any extension $\tau \in T$ of $y\lceil n$ never has $\tau(i) = a$ for $i \geq n$.*

T is completely ranked if there exists a recursive ordinal α such that T is completely ranked up to α and, for any $\tau \in T$, any notation a with $o(a) \geq \alpha$, and any i, $\tau(i) \neq a$.

Let Q be a r. b. strong $\Pi^0_{2\alpha+1}$ class of sets. Note that elements of Q are only allowed to contain notations for ordinals $> \alpha$. We will define a Π^0_1 class P, a recursively bounded, tree T, completely ranked up to $\alpha+1$, such that $P = [T]$ and a homeomorphism Φ from Q onto $D^\alpha(P)$ such that $x \leq_T \Phi(x) \leq_T \oplus 0^{2\alpha-1}$ for all $x \in Q$. In fact, x is always a subsequence of $\Phi(x)$. Furthermore, an element y of P will contain infinitely many notations for ordinals $> \alpha$ only if $y = \Phi(x)$ for some $x \in Q$.

For $\alpha = 0$, just let $Q = P$ and let Φ be the identity.

Successor Case: Suppose that $\beta = o(b) = o(a) + 1 = \alpha + 1$ and that $Q = P_{e,2\alpha+3} = [T]$. By Lemma V.4.9, we have

$T = \sqcup_n U_{\rho(e,a,n),2\alpha+1} = \sqcup_n T_n$.

Now for $x \in Q$, let n_i for each i be the unique n such that $x\lceil i+1 \in T_n$ and let

$$\Gamma(x) = (bx(0)0^{n_0+1}bx(1)0^{n_1+1}b\ldots).$$

Define the $\Pi^0_{2\alpha+1}$ tree U to contain $\tau = (bx(0)0^{n_0+1}bx(1)0^{n_1+1}\ldots x(k)0^n)$ and its initial segments provided that $x\lceil i+1 \in T_{n_i}$ for all $i < k$.

It is easy to see that $G = [U]$ contains exactly $\{\Gamma(x) : x \in Q\}$ together with paths $y = (bx(0)0^{n_0+1}bx(1)0^{n_1+1}bx(k))^\frown 0^\omega$ for all $x \in Q$ and all k. Each of the latter will thus be isolated in G and each of the former will have rank 1, so that Q is homeomorphic to $D(G)$. Furthermore, for $x \in Q$, x a subsequence of $\Gamma(x)$ and therefore is recursive in $\Gamma(x)$, since the values of x are just those following immediately after b's. $\Gamma(x)$ is recursive in $x \oplus \mathbf{0}^{2\alpha+1}$, since the computation of the sequence of witnesses n_i may be performed with a $\Pi^0_{2\alpha+1}$ complete oracle.

In addition, for any $y \in G$, $y \in D(G)$ if and only if y has infinitely many occurrences of b, and y has no occurrences of any notation $\leq \alpha$. Let f be a recursive function so that $\sigma(i) \leq f(i)$ for all $\sigma \in T$ and all i. Then for any $\tau \in U$ and any i, $\tau(i) \leq max\{b, f(i)\}$, so that G is recursively bounded. In addition, if $y =$

$(bx(0)0^{n_0+1}bx(1)0^{n_1+1}bx(k))^\frown 0^\omega \notin D(G)$, then $(bx(0)0^{n_0+1}bx(1)0^{n_1+1}bx(k))^\frown 0^{n_k+1}$ may only be extended in U by 0's.

Now by induction, there is a r. b. Π^0_1 class $P = [S]$ which is completely ranked up to β and a homeomorphism Ψ from G onto $D^\alpha(P)$ such that $y \leq_T \Psi(y) \leq_T y \oplus \mathbf{0}^{2\alpha-1}$. We may assume that y is always a subsequence of $\Psi(y)$ and that the values of y follow immediately after the occurrences of a in $\Psi(y)$.

It is immediate that Ψ is also a homeomorphism from $D(G)$ onto $D^\beta(P)$. Now let $\Phi(x) = \Gamma(\Psi(x))$. Then clearly Φ is a homeomorphism from Q onto $D^\beta(P)$.

It remains to show that P is completely ranked up to $\beta + 1$. Suppose first that z has infinitely many occurrences of b. Then, by the construction, $z = \Psi(y)$ for some $y \in G$ such that y has infinitely many occurrences of b. Thus $y \in D(G)$, so that $z \in D^\beta(P)$. Next suppose that $z \in D^\beta(P)$. Then $z = \Psi(y)$ for some $y \in D(G)$. Now y has infinitely many occurrences of b, so that z must also have infinitely many occurrences of b.

Finally, suppose that $z \in P$ has only finitely many occurrences of b, so that $z \notin D^\beta(P)$. Observe that by the construction, the occurrences of b may only follow immediately after occurrences of a. There are two cases.

First, suppose that $z \notin D^\alpha(P)$. Then z has only finitely many occurrences of a and by induction, there is some n such that no extension $\tau \in S$ of $z\lceil n$ has any occurrences of a past $\tau(n)$. It follows from the observation above that b may not occur in τ past $\tau(n)$ either.

Next, suppose that $z \in D^\alpha(P)$. Then $z = \Psi(y)$ for some $y \in G$ such that y has only finitely many occurrences of b. Thus by the construction, there is some n so that $y\lceil n$ may only be extended by 0's in U. Now choose m so that $z\lceil m$ includes the subsequence $y\lceil n$. It follows that no extension of $z\lceil m$ in S may contain any further occurrences of b. This concludes the proof that P is completely ranked up to $\beta + 1$.

We note that the construction of P and of Ψ are uniform.

Limit Case: Suppose that $\lambda = o(b)$ is a limit ordinal and that $Q = P_{e,\lambda+1} = [T]$. Let $a_0, a_1, \ldots$ enumerate the set of notations for ordinals less than λ and let $\alpha_n = o(a_n)$. By definition, we have $Q = \cap_n Q_n = \cap_n [T_n]$, where $T_n = U_{\phi_e(a_n),o(a_n)+1}$.

By induction, we have constructed r.b. classes $P_n = [U_n]$, completely ranked up to λ and homeomorphisms Φ_a from Q_n onto $D^{\alpha_n+1}(P_n)$. For each $x \in Q$ and each n, let

$$\Phi_n(x) = (a_n x(0)\sigma_{n,0} a_n x(1)\sigma_{n,1} \ldots,$$

where each witness $\sigma_{n,i}$ contains no occurrence of a_n. Now let

$$\Phi(x) = (bx(0)\sigma_{0,0} bx(1)\sigma_{1,0} bx(2)\sigma_{1,0} \ldots.$$

Here the sequence of witnesses $\sigma_{i,n}$ is enumerated in order, first by the sum $i+n$ and then by the value of n.

It is immediate that $x \leq_T \Phi(x)$ and it follows from the uniformity of the construction that $\Phi(x) \leq_T x \oplus \mathbf{0}^\beta$.

Define the Π^0_1 tree U to contain all initial segments of $\tau = (bx(0)\sigma_{0,0}bx(1)\sigma_{0,1}bx(2)\sigma_{1,0}\dots\sigma_{n,i}bx(k)\sigma)$ such that

1. For all m, j with $m + j < n + i$ or with $m + j = n + i$ and $m \leq n$, $a_m x(0)\sigma_{m,0}a_m x(1)\sigma_{m,1}\dots a_m x(j)\sigma_{m,j} \in U_n$ and $\sigma_{m,j}$ contains no occurrence of a_m.
2. If $n \neq 0$, then $a_{n-1}x(0)\sigma_{n-1,0}a_{n-1}x(1)\sigma_{n-1,i}\dots a_{n-1}x(i+1)\sigma \in U_{n-1}$
3. If $n = 0$, then $a_{i+1}x(0)\sigma \in U_{i+1}$ and σ contains no occurrence of a_{i+1}.

Then $P = [U]$ clearly contains $\Phi(x)$ for all $x \in Q$. It is easy to see that if $y \in P$ has infinitely many occurrences of b, then $y = \Phi(x)$ for some $x \in Q$. Now suppose that y has only finitely many occurrences of b and let

$$y = (bx(0)\sigma_{0,0}bx(1)\sigma_{0,1}bx(2)\sigma_{1,0}\dots\sigma_{n,i}bx(k))^\frown z,$$

where z has no occurrences of b. Let

$$\rho = (bx(0)\sigma_{0,0}bx(1)\sigma_{0,1}bx(2)\sigma_{1,0}\dots\sigma_{n,i}bx(k)).$$

This given, we can define a string ν as follows. There are two cases.

Case 1 If $n \neq 0$, then $u = (a_{n-1}x(0)\sigma_{n-1,0}a_{n-1}x(1)\sigma_{n-1,i}\dots a_{n-1}x(i+1))^\frown z \in P_{n-1}$. Let

$$\nu = a_{n-1}x(0)\sigma_{n-1,0}a_{n-1}x(1)\sigma_{n-1,i}\dots a_{n-1}x(i+1)).$$

Case 2 If $n = 0$, then $u = a_{i+1}x(0)^\frown z \in P_{i+1}$. Let

$$\nu = a_{i+1}x(0).$$

We will now establish several claims leading to the desired result that P is completely ranked and that Φ is a homeomorphism of Q onto $D^\beta(P)$. The proof depends on the definition of ν. We will give the proofs for Case 1 and leave the simpler Case 2 to the reader.

We claim first that there is some initial segment τ of y such that no extension of τ in U has any further occurrences of b. There are two subcases here.

(Subcase a): If z has an occurrence of a_{n-1}, then it follows from the definition of U that no further occurrence of b can occur after the first a_n in z has occurred in y. Let σ be an initial segment of z containing the first a_n.

(Subcase b): If z has no occurrences of a_{n-1}, then since U_{n-1} is completely ranked, there is some initial segment σ of z such that no extension of $\nu^\frown\sigma$ in U_{n-1} may contain any further occurrences of a_{n-1}. It follows from the definition of U that b can not occur in U past $\rho^\frown\sigma$.

Next, we claim that the rank $|y|_P$ of y in P equals the rank $|u|_{P_{n-1}}$ of u in P_{n-1}. We first observe that for any z', if $\nu^\frown z' \in U_{n-1}$, then $\rho^\frown z' \in U$. This implies that $|u|_{P_{n-1}} \leq |y|_P$. For the other inequality, observe that for any z' extending σ, if $\rho^\frown z' \in U$, then $\nu^\frown z' \in U_{n-1}$.

It now follows that $\Phi(x)$ has rank at least λ in P, since it is the limit of the sequence $\rho_n^\frown z_n$, where, for the appropriate value of k,
$\rho_n = (bx(0)\sigma_{0,0}bx(1)\sigma_{0,1}bx(2)\sigma_{1,0}\ldots\sigma_{n,i}bx(k))$,
$\nu_n = a_{n-1}x(0)\sigma_{n-1,0}a_{n-1}x(1)\sigma_{n-1,i}\ldots a_{n-1}x(i+1))$, and
$\Phi_n(x) = y = \nu_n^\frown z_n$.

Finally, we show that P is completely ranked up to $\beta+1$. We have already established that y has rank $\geq \beta$ if and only if y has infinitely many occurrences of b. Now suppose that $|y|_P < \beta$ and let ρ, ν and z be as above. Then for any a with $o(a) \leq b$, $|y|_P \geq o(a)$ if and only if $\nu^\frown z$ has rank $\geq o(a)$ in P_{n-1}, which is if and only if z has infinitely many occurrences of a, since U_{n-1} is completely ranked, and this is if and only if y has infinitely many occurrences of a. The previous discussion already established the other criterion for being completely ranked.

The uniformity of the proof shows that, using the Recursion theorem, we can actually compute indices for P and for Φ from an index for Q. We omit the details. □

Corollary V.4.11. *[39] For any computable ordinal λ which is either 0 or a limit and any finite n:*

(a) there is a $B \equiv_T 0^{(\lambda+2n)}$ with $rk(B) = \lambda + n$;

(b) for any degree $\mathbf{a}$ such that $\mathbf{0}^{(\lambda+2n+1)} \leq \mathbf{a} \leq \mathbf{0}^{(\lambda+2n+2)}$, there is a B of degree $\mathbf{a}$ with $rk(B) = \lambda + n + 1$.

Proof. □

Next we briefly consider sets which cannot be ranked.

Theorem V.4.12. *(Cenzer-Smith [46]) For any hyperimmune set A, there is a $C \equiv_T A$ which is not ranked.*

Proof. Let $A = \{f(0) < f(1) < \cdots\}$ be hyperimmune. Let $[T_0], [T_1], \ldots$ enumerate the Π^0_1 classes as in Lemma 1.2. We first define $B \leq_T A$ so that $[T_i]$ is uncountable whenever $B \in [T_i]$, which implies that B is not ranked. The characteristic function of B is the limit of a sequence of strings σ_n of length $f(n)$ which is computable in A. Let $\sigma_0 = 0$. Given σ_n, σ_{n+1} is defined in two cases.

(Case 1) There is some $i \leq f(n)$ and some σ of length $f(n)$ such that

1. $\sigma_n \in T_i$
2. $\sigma \notin T_i$
3. $\sigma\lceil n = \sigma_n\lceil n$
4. for any $j < i$, if $\sigma_n \notin T_i$, then $\sigma \notin T_j$.

Then we let i be the least for which there is a σ satisfying the conditions and we let σ_{n+1} be the (lexicographically) least corresponding to that i.

(Case 2) If there is no such i, then $\sigma_{n+1} = \sigma_n^\frown 0^{f(n+1)-f(n)}$.

Let B have characteristic function $\cup_n \sigma_n$. It is clear that $B \equiv_T A$. The proof that $B \in [T_i]$ implies $[T_i]$ uncountable is by induction. Suppose true for all $i < j$, suppose that $B \in [T_i]$, and choose m large enough that $B \notin [T_i]$ implies $B\lceil m \notin T_i$ for all $i < j$. Then for any $n > f(m)$, any extension $\sigma \notin T_i$ of $B\lceil n$ of length $f(n)$ will satisfy the conditions of Case 1. Thus the shortest extension σ of $B\lceil n$ not in T_i must have length $> f(n)$. If $[T_i]$ were countable, then every σ would have an extension not in T_i, so that we could define a function $h(n)$ to be the least k such that any string σ of length n has an extension of length k which is not in T_i. Then for $n > f(m)$, $h(n) > f(n)$, contradicting the assumption that A is hyperimmune.

It follows that B is unranked. Finally, let $C = A \oplus B$. Then $C \equiv_T A$ and $B \leq_{tt} C$, so that C is unranked by Lemma V.1.8. □

Of course this implies that every r.e. degree contains an unranked set. We say that **a** is *completely unranked* if every set A of degree **a** is unranked. Jockusch and Shore construct in [95] a Σ_2^0 degree which is completely unranked. Downey observed that since sets with the same truth-table degree have the same rank and since all sets in a hyperimmune-free degree have the same truth-table degree (see [163], p. 589), the construction of an unranked set of hyper-immune free degree will provide a completely unranked degree. This led to the following improvement in of the Jockusch-Shore result.

Theorem V.4.13. *(Downey [60]) There is a hyperimmune-free degree which is completely unranked.*

On the other hand, Downey also showed the following, again using a hyperimmune free degree **a**, so that every A of degree $\leq$ **a** is in fact $\leq_{tt}$ **a**.

Theorem V.4.14. *(Downey [60]) There exists a degree* $\mathbf{a} \leq \mathbf{0}''$ *such that every set* A *of degree* $\leq \mathbf{a}$ *is ranked.*

Cenzer and Smith consider in [46] the problem of sets below $\mathbf{0}'$ but with high rank. They showed that for every computable ordinal α, there is a Δ_2^0 set A of rank α. This was improved by Cholak and Downey in [51].

Theorem V.4.15. *(Cholak-Downey [51]) For each computable ordinal* α*, there is an r.e. set of rank* α*.*

V.5 Computable Trees with One or No Infinite Branches

Recall the notions of the height $ht(T)$ of a well-founded tree $T \subseteq \mathbb{N}^*$ and the height $ht_T(\sigma)$ of the nodes of T introduced in Section II.II.10. If T has a unique infinite branch, then following Clote [53], we let $\gamma_T = sup\{ht_T(\sigma) + 1 : T[s]$ is well-founded$\}$. Theorem II.II.10.19 of Clote showed that the hyperarithmetic complexity of the unique infinite branch is bounded in some sense by the height of T. In this section we consider the reverse result that every hyperarithmetic set is reducible to the unique infinite branch of some computable tree.

We also look at the complexity of the perfect kernel $K(Q)$ of a Π^0_1 class in $\{0,1\}^{\mathbb{N}}$.

In section III.II.5, we showed that any Π^0_1 class $P \subseteq \mathbb{N}^{\mathbb{N}}$ can be reduced to a Π^0_1 class $Q \subseteq \{0,1\}^{\mathbb{N}}$. In this section, we consider the connection between the height of a computable tree T such that $P = [T]$ and the rank of the tree S such that $Q = [S]$.

The following theorem is a variant of the result of Clote [53].

Theorem V.5.1. *For any computable ordinal α and any hyperarithmetic index $a \in H^{\alpha \cdot 2}$, there is a computable tree T with unique infinite branch x such that $\gamma_T \leq \omega\alpha$ and $H_a \leq_T x$.*

Proof. In fact, we will use the recursion theorem to obtain a function ψ such that for all hyperarithmetic indices $a \in H^{\alpha \cdot 2}$, $T_{\psi(a}$ is a computable tree with unique infinite branch x_a such that $\gamma_{T_{\psi(a)}} \leq \omega\alpha$ and $H_a \leq x_a$. Recall that $H_a = \cup_n \mathbb{N} - H_{\phi_a(n)}$, so that for $a \in H^{\alpha \cdot 2}$, we have

$$i \in H_a \iff (\exists n)(\forall k) i \in H_{\phi_{\phi_a(n)}(k)}(i).$$

Using Lemmas II.10.6, II.10.7 and II.10.8, we can find a function f such that

$$i \notin H_a \iff (\exists \infty n) i \in H_{f(a,n)},$$

where $f(a,n) \in H^{\beta \cdot 2}$ for some β with $\beta + 1 \leq \alpha$.

For the base case of $\alpha = 0$, we may assume that H_a is uniformly primitive recursive. Let p_i denote the ith prime number. Then for each i, we $x_a(p_i) = \langle \chi_{H_a}(i) \rangle$ and we let $x_a(j) = 0$ for all non-primes j. Let $T_{\psi(a)} = \{x_a \lceil n : n \in \mathbb{N}\}$. Certainly $\gamma(T_{\psi(a)}) = 0$ for all such a.

Now we may assume that for each a and n, there is a tree $T_{a,n} = T_{\psi(f(a,n))}$ with unique infinite branch $x_{a,n}$ and $\gamma_{T_{a,n}} \leq \omega \cdot \beta$ for some β with $\beta + 1 \leq \alpha$. Furthermore, for every i and n, $x_{a,n}(p_i) = \langle \tau \rangle$ for some sequence τ such that $\tau(0) = \chi_{H_{f(a,n)}}(i)$. Now define y_a and x_a as follows. For all i and n,

$$y_a(2^{i+1} \cdot 3^{n+1}) = x_{a,n}(i).$$

For all i, there are two cases in the definition of $y_a(p_i)$.

Case I: $y_a(p_i) = 0$. Then for all k,

$$y_a(p_i^{k+1}) = (\text{least } n > y_a(p_i^k)) i \in H_{f(a,n)}.$$

Here the sequence of values $y_a(p_i^{k+1})$ provides an infinite set of witnesses that $i \notin H_a$.

Case II: $y_a(p_i) = 1$. Then

$$y_a(p_i^2) = (\text{least } n)(\forall m \geq n) i \notin H_{f(a,m)}$$

and $y_a(p_i^k) = 0$ for all $k > 2$. Here $y_a(p_i^2)$ provides a witness that $i \in H_a$. Finally

$$x_a(j) = \langle y_a(j), y_a(j-1), \ldots, y_a(0)\rangle.$$

Observe that H_a may be computed from x_a by

$$i \in H_a \iff (x_a(i))_0 = 1 \iff \neg(x_a(i))_0 = 0.$$

The computable tree $T = T_{\psi(a)}$ with unique infinite branch x_a is defined as follows. Given a sequence $\sigma = (\sigma(0), \ldots, \sigma(m-1))$ which is a candidate for membership in T, we first require that there exists a sequence $e_0, e_1, \ldots, e_{m-1}$ such that for each $j < m$, $\sigma(j)$ has the form $\langle e_j, e_{j-1}, \ldots, e_0\rangle$. Then for $j = 2^{i+1} \cdot 3^{n+1} < m$, we require that

$$(e_{2^{i+1}.3}, e_{2^{i+1}.9}, \ldots, e_{2^{i+1}.3^{n+1}}) \in T_{a,n}.$$

These conditions mean that σ provides an answer to whether $i \in H_{f(a,n)}$ for a certain set of (i, m, n). For $j = p_i$, $e_j = 0$ indicates the guess that $i \notin H_a$ and then $\{e_{p_i^{k+1}}\}_k$ is intended to enumerate the infinite set $\{n : i \in H_{f(a,n)}\}$. Similarly $e_j = 1$ indicates the guess that $i \in H_a$. Then for $j = p_i^2$, $e_j = n$ is intended to be the least n such that $i \notin H_{f(a,m)}$ for all $m \geq n$. The string σ is in T if and only if the "witnesses" provided by σ as to whether $i \in H_a$ are confirmed by the coded subsequences from the trees $T_{a,n}$.

It follows that x_a is the unique infinite branch of T. That is, suppose that $x \in [T]$ and has the proper form. Then each of the coded subsequences must be in $T_{a,n}$ so that each infinite coded subsequence must be $x_{a,n}$. Now the coded values of $y_a(p_i^k)$ must be correct since the witnesses from $T_{a,n}$ are all correct.

It remains to check that $\gamma(T) \leq \omega \cdot \omega \cdot \alpha$. Let $\sigma \in T$ be a dead end of length m in the proper form and let $e_0, e_1, \ldots, e_{m-1}$ be given as above. There are two cases.

Case I: For some n, the coded subsequence $\tau = (e_{2^{i+1}.3}, e_{2^{i+1}.9}, \ldots, e_{2^{i+1}.3^{n+1}})$ is a dead end of $T_{a,n}$. Then it can be shown by induction that $ht_T(\sigma) \leq ht_{T_{a,n}}(\tau) \leq \omega \cdot \beta < \omega \cdot \alpha$ for some $\beta < \alpha$.

Case II: All coded subsequences from any $T_{a,n}$ are initial segments of $x_{a,n}$.

Case IIa: $e_j = y_a(j)$ for all primes j but some witness e_j where $j = p_i^k$ is different from $y_a(p_i^k)$. Suppose first that $y_a(i) = 0$ is correct and the witnesses should be $n_2 = y_a(p_i^2), n_3 = y_a(p_i^3), \ldots$ but for $j = p_i^k$, $e_j \neq n_j$. Now let $\sigma' \in T$ be any extension of σ long enough to predict the value of $x_{a,n}(n_j)$. Then τ must be incorrect as in Case I and thus $ht_T(\tau) \leq \omega\beta$. Thus $ht_T(\sigma) \leq \omega \times \beta + n < \omega \cdot \alpha$. A similar argument apples when $y_a(i) = 1$ is correct but the witness e_j where $j = p_i^2$ is incorrect.

Case IIb: For some i and for $j = p_i$, $e_j \neq y_a(j)$. Suppose first that $e_j = 0$ but $y_a(j) = 1$. Then there can be only a finite number K of witnesses in the sequence $y_a(p_i^k)$ and any extension of σ long enough to predict $y_a(p_i^K + 1)$ must code a dead end of $T_{a,K+1}$ as in Case IIa. A similar argument applies when $e_j = 1$ but $y_a(j) = 0$. □

Corollary V.5.2. *For any computable ordinal α, hyperarithmetic index $a \in H^{\alpha \cdot 2}$, there is a computable tree T with unique infinite branch x such that $\gamma_T \leq \omega\alpha$ and x is $\Sigma^0_{\alpha\cdot 2}$ complete.*

The next result explores further the relation between trees T in $\mathbb{N}^*$ with a unique infinite branch and the natural images $\Phi(T) \in \{0,1\}^*$, relating the height of T with the rank of $\Phi(T)$. Recall from the proof of Theorem III.III.7.3 the definition of $\Phi(x) = 0^{x(0)}10^{x(1)}\dots$ mapping $\mathbb{N}^{\mathbb{N}}$ into $\{0,1\}^{\mathbb{N}}$ and the corresponding mapping of trees so that

$$0^{\sigma(0)}10^{x(1)}1\dots0^{x(k-1)}10^{x(k)} \in S = \Phi(T) \iff (\sigma(0),\dots,\sigma(k-1)) \in T.$$

Then a dead end $\sigma \in T$ of length k corresponds to an isolated point

$$y_\sigma = 0^{\sigma(0)}10^{x(1)}1\dots0^{x(k-1)}10^{\infty} \in [S].$$

Theorem V.5.3. *Let $T \subset \mathbb{N}^*$ be a computable tree with no infinite paths and let $S = \Phi[T]$ be the image of T in $\{0,1\}^*$ and $Q = [S]$. Then for any $\sigma \in T$, $rk_Q(y_\sigma) \leq ht_T(\sigma)$.*

Proof. It is easy to see that $Q = \{y_\sigma : \sigma \in T$. The inequality is proved by induction on $\alpha = ht_T(\sigma)$. If $ht_T(\sigma) = 0$, then σ has no immediate extensions in T and hence y_σ is isolated in Q so that $rk_Q(y_\sigma) = 0$. Now let $\sigma = (\sigma(0),\dots,\sigma(k-1))$ and suppose that $ht_T(\sigma) = \alpha$, so that for any proper extension $\tau \in T$ of σ, $ht(T) < \alpha$. Let $y \in Q$ be any extension of $\sigma^* = 0^{\sigma(0)}10^{x(1)}1\dots0^{x(k-1)}1$ different from y_σ. Then $y = y_\tau$ for some extension $\tau \in T$ of σ, so that $ht_T(\tau) < \alpha$ and hence by induction $rk(y) < \alpha$. It follows that $rk_Q(y_\sigma) \leq \alpha$. □

Exercises

V.5.1. Show that the reverse inequality in Theorem V.5.3 does not hold. In fact, for any computable ordinal α, there is a computable tree T with no infinite path and $\sigma \in T$ such that the rank of y_σ in $\Phi[T]$ is 0 but $ht_T(\sigma) = \alpha + 1$.

V.5.2. Combine Theorem V.5.3 and Corollary V.5.2 to show that for any computable ordinal α, there is a real $x \in \{0,1\}^{\mathbb{N}}$ of degree $\mathbf{0}^{\alpha\cdot 2}$ and a Π^0_1 class Q such that $rk_Q(x) \leq \omega \cdot \alpha$.

V.6 Logical Theories revisited

In this section, we apply the basis results for countable Π^0_1 classes to axiomatizable theories.

Theorem V.6.1. *Let Γ be an axiomatizable first order theory with only countably many complete consistent extensions. Then*

(a) Γ has a decidable complete consistent extension.

(b) *If Γ has only finitely many complete consistent extensions, then every complete consistent extension is decidable.*

(c) *Every complete consistent extension of Γ is hyperarithmetic.*

Proof. Let Γ be an axiomatizable theory with countably many complete consistent extensions. By Theorem III.III.9.1, the set of complete consistent extensions of Γ may be represented as a Π^0_1 class P. It follows from Theorem V.2.3 that P has a computable member, which represents a decidable complete consistent extension of Γ. Similarly, it follows from Theorem V.4.4 that every complete consistent extension of Γ is hyperarithmetic. If Γ has only finitely many complete consistent extensions, then P has only finitely many elements and therefore all of the elements of P are computable by Theorem V.2.2 and thus all of the complete consistent extensions of Γ are decidable. □

Given an axiomatizable theory Γ, let us say that an extension Δ of Γ is a *finite extension* of Γ if there is a finite set F of sentences such that Δ is logically equivalent to $\Gamma \cup F$. Then it is easy to see that a complete consistent extension Δ of Γ is a finite extension if and only if Δ is isolated in the Π^0_1 class of complete consistent extensions of Γ. In particular, any complete consistent finite extension of Γ must be decidable. Thus if Δ is an undecidable complete consistent extension of Γ, then Δ is not a finite extension. We now want to focus on complete consistent extensions of rank one.

Theorem V.6.2. *Let Γ be an axiomatizable first-order theory which has a unique complete consistent, non-finite extension Δ. Then $\Delta \leq_T \mathbf{0}''$ and if Γ is decidable, then $\Delta \leq_T \mathbf{0}'$.*

Proof. This follows from Theorem V.4.3. □

Here is an existence result.

Theorem V.6.3. (a) *For any degree $\mathbf{b} \leq 0'$, there is a decidable theory Γ with unique complete consistent, non-finite extension Δ and Δ has degree $\mathbf{b}$.*

(b) *For any degree $\mathbf{b}$ such that $\mathbf{0}' \leq \mathbf{b} \leq \mathbf{0}'$, there is an axiomatizable theory Γ with unique complete consistent, non-finite extension Δ of degree $\mathbf{b}$.*

Proof. (a) Let $\mathbf{b} \leq 0'$. Then by Theorem V.4.5, there is a decidable Π^0_1 class P with unique nonisolated element B of degree $\mathbf{b}$. By Theorem III.III.9.3, there is a decidable theory Γ such that P represents the set of complete consistent extensions of Γ and thus B represents the unique non-finite complete consistent extension Δ of Γ.

(b) This follows from Corollary V.5.2 and Theorem III.III.9.3 as in (a). □

Chapter VI

Index Sets

The notions of *enumeration* and of an *index set* are fundamental in the study of the computable functions and computably enumerable sets. The complexity (in the arithemetic hierarchy) of many properties can be measured using index sets. For example, the index set $Inf = \{a : W_a \text{ is infinite}\}$ is Π^0_2 complete, so that from this point of view, the property of being infinite is Π^0_2. The chapter begins with a brief list of such results for c.e. index sets, together with their complexity.

We present an enumeration of the Π^0_1 classes and then classify several index sets for Π^0_1 classes. In particular, we study index sets for properties related to cardinality, computable cardinality, measure and category. We then indicate how these index sets will play an important role in the application of Π^0_1 classes to various mathematical problems in Part 2.

A set A is said to be an *index set* (for c. e. sets) if for any a, b, $a \in A$ and $\phi_a = \phi_b$ imply that $b \in A$. We can also define a *co-index set* to be a set A such that for any a, b, $(a \in A \ \& \ b \in A \ \& \ \phi_a = \phi_b)$ implies that $a = b$. Thus in particular, $\emptyset$ and ω are index sets. Rice's Theorem ([198], p. 21) states that these are the only two computable index sets. We have defined the index sets K and K_0 in Section II.3. In fact, it is the case that if A is an index set other than $\emptyset, \omega$, then $K \leq_1 A$ or $K \leq_1 \bar{A}$ where $K = \{a : a \in W_a\}$. Here are some other examples of index sets which we will employ:

- $K_1 = \{a : W_a \neq \emptyset\}$;
- $Fin = \{a : W_a \text{ is finite}\}$;
- $Inf = \{a : W_a \text{ is infinite}\}$;
- $Cof = \{a : \omega \setminus W_a \text{ is finite}\}$;
- $Coinf = \{a : \omega \setminus W_a \text{ is infinite}\}$;
- $Rec = \{a : W_a \text{ is a computable set}\}$;
- $Tot = \{a : \phi_a \text{ is total}\}$;

- $Ext = \{a : \phi_a$ is extendible to a total computable function$\}$;
- $Ext_2 = \{a : \phi_a$ is extendible to a total $\{0,1\}$-valued computable function$\}$;
- $Comp = \{e : W_e \equiv_T K\}$;
- $U_1^1 = \{a : (\exists x)(\forall n) < x\lceil n >\notin W_a\}$.

Following Soare [198], p. 66, we define $(\Sigma_n^m, \Pi_n^m) \leq_m (B, C)$ for a disjoint pair of sets B and C if for some Σ_n^m complete set A, there is a computable function f such that, for any a, $a \in A \iff f(a) \in B$ and $a \notin A \iff f(a) \in C$. If B is Σ_n^m, C is Π_n^m and $(\Sigma_n^m, \Pi_n^m) \leq_m (B, C)$, then we will say that the pair (B, C) is (Σ_n^m, Π_n^m) *complete*.

The index sets described above all turn out to be complete for some level of the arithmetical hierarchy. Here is a brief list of such complexity results, most taken from Soare [198], where the reader can find a further discussion of index sets. We give some details of the proofs as preparation for the work on index sets for Π_1^0 classes.

Theorem VI.0.4. *(i) K, K_0 and K_1 are Σ_1^0 complete sets;*

(ii) Tot is a Π_2^0 complete set;

(iii) (Fin, Inf) is (Σ_2^0, Π_2^0) complete.

(iv) $(Cof, Coinf)$ is (Σ_3^0, Π_3^0) complete;

(v) Ext, Ext_2, and Rec are Σ_3^0 complete sets;

(vi) $Comp$ is Σ_4^0 complete;

(vii) U_1^1 is a Σ_1^1 complete set.

Sketch. (i) It is clear that these are all Σ_1^0 sets and that K_0 is complete. Let $W = Dom(\phi)$ be any c. e. set and define the partial recursive function ϕ_e so that $\phi_e(m,i) = \phi(m)$. Let $f(m) = S_1^1(e,m)$ so that $\phi_{f(m)}(i) = \phi_e(m,i) = \phi(m)$. Then

$$m \in W \iff \phi(m)\downarrow \iff W_{f(m)} = \omega$$

and

$$m \notin W \iff \phi(m)\downarrow \iff W_{f(m)} = \emptyset.$$

Thus $m \in W \iff f(m) \in W_{f(m)} \iff W_{f(m)} \neq \emptyset$.

(ii,iii) $a \in Tot \iff (\forall m)(\exists s) m \in W_{a,s}$ and $a \in Fin \iff (\exists m)(\forall n > m)(\forall s) n \notin W_{a,s}$.

For the completeness, let B be a Π_2^0 set and let R be a computable relatio such that

$$i \in B \iff (\forall m)(\exists n) R(i,m,n).$$

It is an important observation that we may assume that $R(i,m,n) \rightarrow R(i,m,n+1)$ and that $R(i,m+1,n) \rightarrow R(i,m,n)$. That is, let $R'(i,m,n) \iff (\forall m' \leq m)(\exists n' \leq n) R(i,m',n')$. Then R' has the desired properties.

Now let $\phi_a(i,m) = (\mu n)R(i,m,n)$ and let $g(i) = S^1_1(a,i)$, so that $\phi_{g(i)}(m) = (\mu n)R(i,m,n)$. If $i \in B$, then $W_{g(i)} = \omega$, so that $g(i) \in Tot$ and also $g(i) \in Inf$. If $i \notin B$, then $W_{g(i)}$ is finite, so that $g(i) \in Fin$ and $g(i) \notin Tot$.

(iv,v) Let A be a Σ^0_3 set and let R be a computable relation so that, for all a,

$$a \in A \iff (\exists m)(\forall n)(\exists k)R(a,m,n,k)$$

We will define a primitive recursive function f such that $a \in A$ if and only if $W_{f(a)}$ is cofinite, which will be if and only if $W_{f(a)}$ is computable. Let the standard noncomputable c. e. set K have a computable enumeration $K = \cup_s K_s$. The c. e. set $W_{f(a)}$ in stages $W_{f(a),s}$ so that $\omega \setminus W_{f(a),s} = \{\{b^s_{a,0} < b^s_{a,1} < \dots\}$.

Stage 0: $W_{f(a),s} = \emptyset$.

Stage $s+1$: For each $m \leq s$ such that either $m \in K_{s+1} \setminus K_s$ or such that there is some $n \leq s$ such that $(\forall n' \leq n)(\exists k \leq s)R(a,m,n',k)$ but $\neg(\forall n' \leq n)(\exists k < s)R(a,m,n',k)$, enumerate $b^s_{a,m}$ into $W_{f(a),s+1}$.

If $a \in A$, then for some m, $b^s_{a,m}$ is put into $W_{f(a),s+1}$ infinitely often, so that $\lim_s b^s_{a,m} = \infty$ and hence $W_{f(a)}$ is cofinite. If $a \notin A$, then for every m, $\lim_s b^s_{a,m} = b_{a,m} < \infty$. Thus $W_{f(a)}$ is coinfinite. Furthermore, $K \leq W_{f(a)}$ (so that $W_{f(a)}$ is not computable), since $m \in K \iff m \in K_{b_{a,m}}$ and $b_{a,m}$ can be uniformly computed from $W_{f(a)}$.

This argument can be modified to show that Ext and Ext_2 are both Σ^0_3 complete, as follows. Here we want to say that $a \in A$ if and only if ϕ_a is extendible. As before, put $m \in W_{f(a)}$ at stage $s+1$ (by defining $\phi_{f(a)}(m) = 0$) if the list of n such that $(\exists k)R(a,m,n,k)$ becomes longer at stage s. Also, replace the action associated with the set K with the following. If $\phi_{m,s}(b^s_{a,m}) = j$ and

$$(\forall i < b^s_{a,m})\phi_{f(a),s}(i) = \phi_{m,s}(i),$$

then define $\phi_{f(a)}(b^s_{a,m}) = 1 - j$, thus putting $b^s_{a,m} \in W_{f(a)}$. (By the usual convention, $a - b = 0$ if $a < b$.) If $a \in A$, then $W_{f(a)}$ is cofinite as before, so that $\phi_{f(a)}$ is extendible. If $a \notin A$, then for each m, $\phi_{f(a)}(b_{a,m})$ is either undefined or not equal to $\phi_m(b_{a,m})$, so that $\phi_{f(a)}$ is not extendible.

(vi) A proof is given in Soare [198, Ch. XII].

(vii) A proof can be found in Hinman [87, p. 84]. □

Each of the results above can be relativized. That is, let $W^x_e = \{n : \Phi^x_e(n)\downarrow\}$. Then for example, $Fin^x = \{a : W^x_a \text{ is finite}\}$ is $\Sigma^{0,x}_2$ complete. In particular, if we let $x = \emptyset^{(n)}$ denote the n-th jump of the emptyset, then Post's theorem (Theorem II.6.7) implies that a set is $\Sigma^{0,x}_k$ if and only if it is Σ^0_{n+k}, see Soare [198]. It follows that, for example, Fin^K is Σ^0_3 complete.

VI.1 Index sets for Π^0_1 classes

There are several different ways to define index sets to Π^0_1 classes. We use here an approach from [40] based on primitive recursive trees.

Let σ_n denote the string $\sigma \in \mathbb{N}^*$ such that $\langle\sigma\rangle = n$. Then $\sigma_0, \sigma_1, \dots$ enumerate $\mathbb{N}^*$, and furthermore, whenever $\sigma_i \prec \sigma_j$, it must be the case that $i < j$.

Then a tree T is primitive recursive, computable, etc. if the corresponding set $\{i : \sigma_i \in T\}$ is itself primitive recursive, computable, etc..

Definition VI.1.1. *Let π_e denote the eth primitive recursive function and let $\sigma \in T_e \iff (\forall \tau \preceq \sigma)\pi_e(\langle\tau\rangle) = 1$; let $P_e = [T_e]$.*

Lemma VI.1.2. *(a) For each e, Π_e is a Π^0_1 class;*

(b) For each Π^0_1 class P, there are infinitely many e such that $P = P_e$.

Proof. Part (a) is clear. Part (b) follows from Proposition III.3.1 and the observation that every primitive recursive function has infinitely many indices. □

An enumeration of the strong Π^0_n classes can be given based on the enumeration of the Σ^0_n sets. Let W^n_e be the eth Σ^0_n set. To be more precise, W^n_e is the domain of the function ϕ^n_e where $\phi^n_e(m) = \Phi_e(m, \emptyset^{(n)})$. Then the eth strong Π^0_{n+1} class is defined as follows, where we identify $\sigma \in \mathbb{N}^*$ with $\langle\sigma\rangle$ as usual for simplicity of expression.

Definition VI.1.3. $T^{n+1}_e = \{\sigma : (\forall \tau \preceq \sigma)\tau \in W^n_e\}$*;* $P^{n+1}_e = [T^{n+1}_e = \{x : (\forall m) x\lceil m \in W^n_e\}$*.*

Next we look at the complexity of the various notions of boundedness.

Theorem VI.1.4. *Let $g \geq 2$ be a computable function.*

(a) $\{e : T_e$ is g-bounded$\}$ is Π^0_1 complete;

(b) $\{e : T_e$ is almost g-bounded$\}$ is Σ^0_2 complete.

Proof. (a) Let $g : \mathbb{N}^* \to \mathbb{N}$ be an arbitrary function such that $g(\sigma) \geq 2$ for all σ.

The set is Π^0_1 since T_e is g-bounded if and only if

$$(forall\sigma)(\forall i)[\sigma^\frown i \in T_e \to i < g(\sigma)].$$

For the completeness, we will define a primitive recursive function h such that $T_{h(e)}$ is g-bounded if and only if $e \notin K$. Let

$$\sigma \in T_{h(e)} \iff (\forall t < |\sigma|)[\phi_{e,t}(e) \uparrow \to \sigma(t) = 0].$$

It follows from the Master Enumeration Theorem II.2.5 and the s-m-n Theorem II.2.7 that h is primitive recursive. If $e \notin K$, then $T_{h(e)} = \{0^t : t \in \mathbb{N}\}$ and is clearly g-bounded. If $e \in K$ and $\phi_{e,t}(e)\downarrow$, then $0^{t\frown} i \in T_{h(e)}$ for all i, so that $T_{h(e)}$ is not g-bounded.

(b) This set is Σ^0_2, since if $g = \phi_a$, then T_e is almost g-bounded if and only if

$$(\exists k)(\forall i)(\forall \sigma)[(|\sigma| \geq k \ \& \ \sigma^\frown i \in T_e) \to i < \phi_a(\sigma)].$$

For the completeness, we define a reduction of Fin as follows. For each e and s, recall that $W_{e,s} = \{i : \phi_{e,s}(i)\downarrow\}$ and that $\phi_{e,s}(i)\downarrow$ implies that $i \leq s$. Thus $e \in Fin$ if and only if $\{s : W_{e,s+1} \setminus W_{e,s} \neq \emptyset\}$ is finite. For $|\sigma| = s$, let

$$\sigma \in T_{h(e)} \iff (\forall n < s)[W_{e,n+1} \setminus W_{e,n} = \emptyset \to \sigma(n+1) < g(\sigma)].$$

If $e \in Fin$ and k satisfies $W_{e,k} = W_e$, then $T_{h(e)}$ is g-bounded above k. If $e \notin Fin$, then for each n such that $W_{e,n+1} \setminus W_{e,n} \neq \emptyset$, we have $0^n{}^\frown i \in T_{h(e)}$ for every i, so that $T_{h(e)}$ is not almost bounded by g. □

Theorem VI.1.5. *(a) $\{e : T_e$ is c. b.$\}$ is Σ^0_3 complete.*

(b) $\{e : T_e$ is almost c. b.$\}$ is Σ^0_3 complete.

Proof. (a) The first set is Σ^0_3, since T_e is c. b. if and only if T_e is ϕ_a-bounded for some total computable function ϕ_a.

For the completeness, we define a reduction f of *Rec* to our set. This will be done so that $[T_{f(e)}]$ is empty if W_e is finite and $[T_{f(e)}]$ has a single element if W_e is infinite. The primitive recursive tree $T_{f(e)}$ is defined as follows: Put $\sigma = (s_0, s_1, \dots, s_{k-1}) \in T_{f(e)}$ if and only if $s_0 < s_1 < \dots < s_{k-1}$ and there exists a sequence $m_0 < m_1 < \dots < m_{k-1}$ such that, for each $i < k$, $m_i \in W_{e,s_i} \setminus W_{e,s_i-1}$ and m_i is the least element of $W_{e,s_{k-1}} \setminus \{m_0, \dots, m_{i-1}\}$. We observe that if W_e is finite, then $T_{f(e)}$ is also finite and therefore recursively bounded. Now fix e and suppose that W_e is infinite. Then we may define canonical sequences $n_0 < n_1 < \dots$ of elements of W_e and corresponding stages $t_0 < t_1 < \dots$ such that, for each i, $n_i \in W_{e,t_i} \setminus W_{e,t_i-1}$ and $(t_0, t_1, \dots, t_i) \in T_{f(e)}$ as follows. Let n_0 be the least element of W_e and, for each k, let n_{k+1} be the least element of $W_e \setminus W_{e,t_k}$. Then for each k, $(t_0, \dots, t_k) \in T_{f(e)}$ and $n_k \in W_{e,t_k}$. Furthermore, we see by induction on k that

$$k \in W_e \to k \in W_{e,t_k}.$$

For $s = 0$, this is because $n_0 = 0$ if $0 \in W_e$. Assuming the statement to be true for all $i < k$, we see that if $k \in W_e$, then either $k \in W_{e,t_{k-1}}$, or else $n_k = k$. In either case, we have $k \in W_{e,t_k}$. The key fact here is that for any $(s_0, \dots, s_k) \in T_{f(e)}$, $s_k \leq t_k$. To see this, let $(s_0, \dots, s_k) \in T_{f(e)}$, let $(m_0, \dots, m_k)$ be the associated sequence of elements of W_e, and suppose by way of contradiction that $s_k \geq t_k$. It follows from the definitions of $T_{f(e)}$ and of $t_0, \dots, t_k$ that in fact $s_i = t_i$ and $m_i = n_i$ for all $i \leq k$. Thus $T_{f(e)}$ has the sequence $(t_0 + 1, t_1 + 1, \dots)$ as a bounding function.

Suppose now that W_e is computable. Then the sequence $t_0 < t_1 < \dots$ is also computable and thus $T_{f(e)}$ is computably bounded by. Now suppose that $T_{f(e)}$ is bounded by some computable function h. Then we must have $t_k < h(k)$ for each k. It follows that $k \in W_e \iff k \in W_{e,h(k)}$, so that W_e is computable.

(b) This set is Σ^0_3, since T_e is a. c. b. if and only if T_e is ϕ_a-almost bounded for some total computable function ϕ_a.

For the completeness, use the argument given in (3) above. We may assume that W_e is infinite, since otherwise the argument goes through trivially. Clearly, if W_e is computable, then $T_{f(e)}$ is computably bounded and therefore a. c. b. as well. If $T_{f(e)}$ is almost bounded by the computable function g, let k be large enough so that for $|\sigma| > k$, $\sigma^i \in T_{f(e)} \to i < g(\sigma)$ and let $\tau = (t_0, t_1, \dots, t_k)$. Then we can recursively define a bounding function $h(i) \geq t(i)$ by letting $h(i) = t(i)$ for $i \leq k$ and, for each $j \geq k$,

$$h(j+1) = max\{g(\langle \tau^\frown (s_{k+1}, \dots, s_j) \rangle) : s_i \leq h(i) \text{ for each } i \text{ with } k < i \leq j\}.$$

It follows as above that W_e is computable. $\square$

Theorem VI.1.6. *(a) $\{e : T_e$ is bounded$\}$ is Π^0_3 complete;*

(b) $\{e : T_e$ is almost bounded$\}$ is Σ^0_4 complete.

Proof. (a) This set is Π^0_3, since

$$T_e \text{is bounded} \iff (\forall\sigma)(\exists n)(\forall m > n)(\sigma^\frown m \notin T_e).$$

For the completeness, we define a reduction of $\omega \backslash Cof$ as follows. Let $\phi(e, m, s) =$ (least $n > m)(n \notin W_{e,s})$. This is a primitive recursive definition since $n \in W_{e,s} \rightarrow n \leq s$. Then, using the s-m-n Theorem, define the tree $T_{f(e)}$ by

$$T_{f(e)} = \{0^m : m \in \mathbb{N}\} \cup \{0^{m\frown}(s+1) : \phi(e, m, s+1) > \phi(e, m, s)\}.$$

Then $T_{f(e)}$ will be a finite-branching tree if and only if, for each m, there are only finitely many s such that $0^{m\frown}(s+1) \in T_{f(e)}$. Now if W_e is not cofinite, then for each m there is a minimal $n > m$ such that $n \notin W_e$. It follows that $\lim_s \phi(e, m, s) = n$, so that $\phi(e, m, s+1) > \phi(e, m, s)$ for only finitely many s, which will make $T_{f(e)}$ finite-branching. On the other hand, if W_e is cofinite and we choose m so that $n \in W_e$ for all $m > n$, then it is clear that there will be infinitely many s such that $\phi(e, m, s+1) > \phi(e, m, s)$, so that 0^m will have infinitely many successors and $T_{f(e)}$ will not be finite-branching. Thus we have

$$e \notin Cof \iff T_{f(e)} \text{ is bounded.}$$

(b) This set is Σ^0_4, since

$$T_e \text{ is almost bounded} \iff (\exists k)(\forall\sigma)(\exists n)(\forall m > n)(|\sigma| > k \rightarrow \sigma^\frown m \notin T_e).$$

For the completeness, first modify the proof of part (a) by letting $T_{g(e)}$ contain 0^m for each m together with $0^{m\frown}(s+1)$ if m is the least such that $\phi(e, m, s+1) > \phi(e, m, s)\}$ This modification ensures that $T_{g(e)}$ is always almost bounded, since only for the largest $m \notin W_e$ will there be infinitely many s with $0^{m\frown}s+1 \in T_{g(e)}$. By the previous argument, $T_{g(e)}$ will be bounded if and only if $e \notin Cof$. Now S be an arbitrary Σ^0_4 set and suppose that $a \in S \iff (\exists k)R(a, k)$, where R is Π^0_3. By the usual quantifier methods, we may assume that $R(a, k)$ implies that $R(a, j)$ for all $j > k$. By the argument above, there is a computable function h such that $R(a, k)$ if and only if $T_{h(a,k)}$ is bounded and such that $T_{h(a,k)}$ is almost bounded for every a and k. Now simply define

$$T_{\phi(a)} = \{0^n : n < \omega\} \cup \{(0^k 1)^\frown \sigma : \sigma \in T_{h(a,k)}\}.$$

If $a \in S$, then $T_{h(a,k)}$ is bounded for all but finitely many k and is almost bounded for the remainder. Thus $T_{\phi(a)}$ is almost bounded. If $a \notin S$, then, for every k, $T_{h(a,k)}$ is not bounded, so that $T_{\phi(a)}$ is not almost bounded. $\square$

Index sets for decidable Π^0_1 classes will use the alternate definition that $P = [T]$ for some computable tree T with no dead ends.

Theorem VI.1.7. *(i) $\{e : P_e$ is decidable is Π_2^0 complete.*

(ii) For any recursive $g \geq 2$, $\{e : P_e$ is decidable and g-bounded is Π_1^0 complete.

(iii) For any computable $g \geq 2$, $\{e : P_e$ is decidable and g-a.b. is D_2^0 complete.

(iv) $\{e : P_e$ is decidable and c. b. is Σ_3^0 complete.

(v) $\{e : P_e$ is decidable and almost c. b. is Σ_3^0 complete.

(vi) $\{e : P_e$ is decidable and bounded is Π_3^0 complete.

(vii) $\{e : P_e$ is decidable and almost bounded is Σ_4^0 complete.

Proof. (i) This set is Π_2^0 since T_e has no dead ends if and only if

$$(\forall \sigma \in T_e)(\exists i)(\sigma^\frown i \in T_e).$$

For the completeness, let C be a Π_2^0 set and R be a computable relation so that

$$e \in C \iff (\forall m)(\exists n)R(e, m, n).$$

Put $\emptyset$ and (m) in $T_{f(e)}$ for all m and, for any k, put $(m,n)^\frown 0^k \in T_{f(e)}$ if and only if $R(e, m, n)$. Thus $T_{f(e)}$ has no dead ends if and only if $e \in C$. Note that $T_{f(e)}$ is g-a.b. for any g.

(ii) This index set is Π_1^0, since a g-bounded tree T_e has no dead ends if

$$(\forall \sigma)(\sigma \in T_e \rightarrow (\exists i \leq g(\sigma)(\sigma \frown i \in T_e).$$

For the completeness, observe that the proof given in Theorem VI.1.4 in fact defines a tree with no dead ends.

(iii) This index set is D_2^0 since the property of being g-bounded is Σ_2^0 and for any tree U_e, T_e has no dead ends if and only if

$$(\forall \sigma \in T_e)(\exists i)\sigma^\frown i \in T_e.$$

For the completeness, let $A = B \cap C$, where B is a Σ_2^0 set and C is a Π_2^0 set. The tree $T_{j(e)}$ is constructed in two parts. First, modify the construction of part (a) by putting $1^\frown\sigma \in T_{j(e)} \iff \sigma \in T_{f(e)}$. Then $T_{f(e)} \cap I((1))$ is always g-a.b. and has no dead ends if and only if $e \in C$. Then we use the function h defined in Theorem VI.1.6 which has the property that $e \in B$ if and only if $T_{h(e)}$ is g-a.b.. Note that $T_{h(e)}$ always has no dead ends. Then put $0^\frown\sigma \in T_{j(e)}$ if and only if $0^\frown\sigma \in T_{h(e)}$.

(iv) For this and the remaining cases, the upper bound on the complexity follows from part (0) above and complexity of the corresponding parts of Theorems VI.1.5 and VI.1.6. The completeness of the remaining cases follows from a simple modification of the reductions used to prove the corresponding theorems above. That is, one needs only ensure that corresponding trees used in the reductions have no dead ends. This is easily accomplished by modifying any given recursive tree T to construct a new computable tree T' such that (i) $0^k \in T'$ for all $k \geq 0$ and (ii) for any $n \geq 1$, $(\sigma_1, \ldots, \sigma_n) \in T$ iff $(\sigma_1{+}1, \ldots, \sigma_n{+}1) \frown 0^k \in T'$ for all $k \geq 0$. □

Notions of boundedness for strong Π^0_2 classes are considered in the exercises.

Exercises

VI.1.1. Show that $\{e : T^2_e$ is g-bounded$\}$ is Π^0_1 complete and $\{e : T^2_e$ is computably bounded$\}$ is Σ^0_3 complete.

VI.1.2. Show that for any Δ^0_2 function $g \geq 2$, $\{e : T^2_e$ is g-bounded$\}$ is Π^0_2 complete and $\{e : T^2_e$ is highly bounded$\}$ is Σ^0_4 complete. (Hint: relativize from Theorems VI.1.4.

VI.1.3. Give the details for the proofs of Theorem VI.1.7(4-7).

VI.2 Cardinality

In this section we classify index sets corresponding to cardinality properties of Π^0_1 classes.

Theorem VI.2.1. *Let $g \geq 2$ be a computable function from $\mathbb{N}^*$ to $\mathbb{N}$.*

(a) $\{e : P_e$ is g-bounded and nonempty$\}$ is Π^0_1 complete;

(b) $\{e : P_e$ is g-bounded and empty$\}$ is D^0_1 complete;

(c) $(\{e : P_e$ is g-bounded and nonempty$\}, \{e : P_e$ is g-bounded and empty$\})$ is (Π^0_1, Σ^0_1) complete.

Proof. We observe first that the relation $\sigma \in Ext(T_e)$ has a Π^0_1 characterization. That is,

$$\sigma \in Ext(T_e) \iff (\forall n)(\exists \tau)[|\tau| = n \ \&\ \sigma \prec \tau \ \&\ \tau \in T_e],$$

where the quantifier "$(\exists\tau)$" is bounded by g in the following sense. Let $h(0) = g(\emptyset)$ and for each n, let $h(n+1) = max\{g(\sigma) : \sigma \in \{0, 1, \dots, h(n)\}^n\}$. Then $\sigma \in T_e$ implies $\sigma(n) \leq h(n)$ for all n, so that the quantifier "$\exists\tau$" above may be replaced by "$(\exists\tau \in \{0, 1, \dots, h(n-1)\}^n)$".

(a,c) Now P_e is nonempty if and only if $\emptyset \in Ext(T_e)$.

For the double completeness, define a reduction f for a given Π^0_1 set A so that $P_{h(e)}$ is always g-bounded and is nonempty if and only if $e \in A$. Let R be a computable relation so that $e \in A \iff (\forall n)R(e, n)$. Then the map may be defined by putting $0^n \in T_{f(e)} \iff R(e, n)$ and putting no other strings in $T_{h(e)}$.

(b) We see that this set is D^0_1 by part (a) and Theorem VI.1.4.

For the completeness, let $C = B \setminus A$, where A and B are Π^0_1 sets and let R and S be computable relations so that

$$e \in A \iff (\forall n)R(e, n) \ \text{ and } e \in B \iff (\forall n)S(e, n).$$

Then a reduction f of C to our set is given by putting $\sigma \in T_{f(e)}$ if and only if either

(i) $(\forall i < |\sigma|)[R(e,i)\ \&\ \sigma(i+1) < g(\sigma\lceil i))]$ or

(ii) $\sigma = (1 + g(\emptyset) + n)$ where not $S(e,n)$ and $(\forall i < n)(S(e,i))$.

Clearly $T_{f(e)}$ is g-bounded if and only if if $e \in B$. Similarly $T_{f(e)}$ is nonempty if and only if $e \in A$. Thus $T_{f(e)}$ is g-bounded and empty if and only if $e \in B - A$. □

Theorem VI.2.2. *For any computable $g \geq 2$,*

(a) $\{e : P_e$ is almost g-bounded and nonempty$\}$ is Σ^0_2 complete;

(b) $\{e : P_e$ is almost g-bounded and empty$\}$ is D^0_2 complete.

Proof. It follows from the proof of Theorem VI.1.4 that the relation "T_e is g-bounded above k" is Π^0_1. Now modify the proof of Theorem VI.2.1 to define h for $\sigma \in T_e$ with $|\sigma| = k$ by $h(\sigma,k) = max\{\sigma(i) : i < k\}$ and for all n, $h(\sigma, k+n+1) = max\{g(\sigma) : \sigma \in \{0,1,\dots,h(k+n)\}^{k+n}\}$. Then for $\sigma| = k$,

$$\sigma \in Ext(T_e) \iff (\forall n)(\exists \tau \in \{0,1,\dots,h(\sigma,k+n)\}^{k+n})(\sigma \prec \tau\ \&\ \tau \in T_e).$$

Thus the relation "$\sigma \in T_e$" is Π^0_1 when restricted to σ of length k when T_e is g-bounded above k. Then for g-bounded T_e, P_e is nonempty if and only if there exists k such that T_e is g-bounded above k and there exists $\sigma \in T_e$ such that $\sigma \in Ext(T_e)$. This gives the upper bound on the complexity for both parts.

(a) For the completeness, use the same reduction as in the proof of Theorem VI.1.4.

(b) For the completeness, let $A = B \cap C$, where B is a Σ^0_2 set and C is a Π^0_2 set. Suppose that $b \in B \iff (\exists m)(\forall n)R(b,m,n)$ and $c \in C \iff (\forall m)(\exists n)S(c,m,n)$, where R and S are computable. Define the function $\phi(b,n)$ to be the least $m < n$ such that $R(b,m,n)$ for all $n' < n$ (or $\phi(b,n) = n$ if there is no such m), so that $b \in B$ if and only if $\phi(b,n)$ is eventually constant. Define the tree $T_{f(b)}$ recursively as follows. Every string (m) of length 1 is in $T_{f(b)}$. If $\sigma \in T_{f(b)}$ is of odd length $2s+1$, then $\sigma^\frown i \in T_{f(b)}$ if either $i < g(\sigma)$ or $\phi(b,s+1) > \phi(b,s)$. If $\sigma \in T_{f(b)}$ is of even length $2s+2$ and $\sigma(0) = m$, then $\sigma^\frown i \in T_{f(e)}$ if $i = 0$ and either $s < m$ or, for all $n \leq s$, $\neg S(b,m,n)$. Observe that allowing an extension when $s < m$ in the second part of the definition of $T_{f(b)}$ means that we have always have arbitrarily long strings in $T_{f(b)}$.

Suppose first that $b \in B$. Then there is some k such that $\phi(b,s+1) = \phi(b,s)$ for all $s \geq k$. It follows that $T_{f(b)}$ is g-bounded above k. Next suppose that $b \in C$. Then, for any m, choose n_m such that $S(b,m,n)$. It follows that there is no σ of length $2n_m + 3$ beginning with $\sigma(0) = m$ in $T_{f(b)}$. It follows that $P_{f(b)}$ is empty in this case. Thus if $b \in A$, then $P_{f(b)}$ is g almost bounded and is empty. If $b \notin B$, then, since $T_{f(b)}$ has arbitrarily long strings, it will not be almost bounded by g. If $b \notin C$, then $P_{f(b)}$ will be nonempty, since for any m such that $\neg S(b,m,n)$ for all n, we will have $m^\frown 0^\omega \in P_{f(b)}$. □

Theorem VI.2.3. *(a) $\{e : P_e$ is c. b. and empty$\}$ is Σ^0_2 complete;*

(b) $\{e : P_e$ *is c. b. and nonempty*$\}$ *is* Σ^0_3 *complete.*

Proof. (a) The case of c. b. empty classes is equivalent to bounded empty classes and is treated in Theorem VI.2.10 below.

(b) This set is Σ^0_3, since P_e is c. b. and nonempty if and only if

$$(\exists a)[a \in Tot \ \& \ e \in P_e \text{ is } \phi_a\text{-bounded and nonempty})].$$

For the completeness, modify the reduction f from the proof of Theorem VI.1.5 as follows. For any $\sigma = (s_0, s_1, \ldots, s_{k-1}) \in T_{f(e)}$, add $\sigma ^\frown 0^k$ to $T_{f'(e)}$ whenever there is no $s < k$ such that $\sigma ^\frown s \in T_{f(e)}$. It is clear that $P_{f'(e)}$ will contain exactly one element for each e. □

Theorem VI.2.4. *(a)* $\{e : P_e$ *is bounded and empty*$\}$ *is* Σ^0_2 *complete;*

(b) $\{e : P_e$ *is bounded and nonempty*$\}$ *is* Π^0_3 *complete.*

Proof. (a) The case of bounded empty classes is a special one, since $[T]$ is bounded and empty if and only if T is finite, that is, if and only if

$$(\exists n)(\forall \sigma)[\sigma \in T_e \rightarrow < \sigma > < n].$$

For the completeness, define a reduction f of Fin to by letting

$$T_{f(e)} = \{\emptyset\} \cup \{(\langle n, s\rangle) : n \in W_{e,s+1} \setminus W_{e,s}\}.$$

(b) Recall that $\{e : T_e$ is finite-branching$\}$ is Π^0_3 complete. Now if T_e is finite-branching, then for any σ,

$$\sigma \in Ext(T_e) \iff (\forall i)(\exists \tau)[\sigma \prec \tau \ \& \ \tau \in T_e \ \& \ |\tau| \geq i].$$

Thus our set is Π^0_3. For the completeness, use the same reduction f as given in the proof of Theorem VI.1.6, since $P_{f(e)} = \{0^\omega\}$ for every e. □

Theorem VI.2.5. $\{e : P_e$ *is a. b. and empty*$\}$ *and* $\{e : P_e$ *is a. b. and nonempty*$\}$ *are both* Σ^0_4 *complete.*

Proof. For the "nonempty" case, the set is Σ^0_4, since P_e is a. b. and nonempty if and only if

$$(\exists k)(\exists \sigma)[B(k, e) \ \& \ |\sigma| \geq k \ \& \ \sigma \in Ext(T_e)].$$

For the completeness, use the same reduction as given in Theorem VI.1.6(b).

For the "empty" case, the set is Σ^0_4, since P_e is a. b. and empty if and only if

$$(\exists k)[B(k, e) \ \& \ (\forall \sigma)(|\sigma| = k \rightarrow \sigma \notin Ext(T_e))].$$

For the completeness, modify the proof of Theorem VI.1.6(b). First define

$$T_{g'(e,k)} = \{m^i : i \leq m + k\} \cup \{(m^{m+k})^\frown s + 1) : (0^m)^\frown s + 1 \in T_{g(e)}\}.$$

where g is the function defined in Theorem VI.1.6(b). Note that $[T_{g'(e,k)}]$ is always empty and that $T_{g'(e,k)}$ is always almost bounded. $T_{g'(e,k)}$ is never actually bounded because the empty string has infinitely many successors (m) for each m. However, $T_{g'(e,k)}$ clearly has the following properties.

(i) If $e \notin Cof$, then every node except $\emptyset$ has finitely many successors.

(ii) If $e \in Cof$, then for some m, m^{m+k} has infinitely many successors.

Now let S be any Σ^0_4 set and suppose that $a \in S \iff (\exists k)R(a,k)$, where R is Π^0_3. By the usual quantifier methods, we may assume that $R(a,k)$ implies that $R(a,j)$ for all $j > k$. Since Cof is Σ^0_3 complete set, it follows from the above discussion above that there is a recursive function h' such that $T_{h'(a,k)}$ is almost bounded for all a,k, $[T_{h'(a,k)}]$ is empty for all a,k and

(iii) if $R(a,k)$, then every node in $T_{h'(a,k)}$ except $\emptyset$ has finitely many successors and

(iv) if $\neg R(a,k)$, then for some m, m^{m+k} has infinitely many successors.

Now define $T_{\psi(a)} = \{(k)^\frown\sigma : \sigma \in T_{h'(a,k)}\}$. $[T_{\psi(a)}]$ is empty since each $[T_{h'(a,k)}]$ is empty. If $a \in S$, then for all but finitely many k, every node in $T_{h'(a,k)}$ except $\emptyset$ has finitely many successors, and for the remainder, $T_{h'(a,k)}$ is almost bounded. Thus $T_{\psi(a)}$ is almost bounded. If $a \notin S$, then, for every k, $T_{h'(a,k)}$ has a string of length $\geq k$ with infinitely many successors, so that $T_{\psi(a)}$ is not almost bounded. □

Theorem VI.2.6. *$(\{e : P_e = \emptyset\}, \{e : P_e \neq \emptyset\})$ is (Σ^1_1, Π^1_1) complete.*

Proof. The upper bounds on the complexity follow from the fact that

$$P_e \neq \emptyset \iff (\exists x)(\forall n) x\lceil n \in T_e.$$

For the completeness, let A be a Σ^1_1 set, so that, by the normal form theorem (see Hinman [87, p. 84]), there is a primitive recursive relation R such that, for all a,

$$a \in A \iff (\exists x)(\forall n) R(a, x|n).$$

Then we may define $T_{f(a)} = \{\sigma : R(a,\sigma)\}$ by the s-m-n Theorem. Then $a \in A \iff P_{f(a)} \neq \emptyset$, as desired. □

Next we consider index sets for the cardinality of strong Π^0_2 classes.

Theorem VI.2.7.

(i) $(\{e : P^2_e$ *is g-bounded & empty*$\}), \{e : P^2_e$ *is g-bounded & nonempty*$\})$ *is* (Σ^0_2, Π^0_2) *complete for any computable* $g \geq 2$.

(ii) $\{e : P^2_e$ *is c. b. and nonempty*$\}$ *is* Σ^0_3 *complete and* $\{e : P^2_e$ *is c. b. and empty*$\}$ *is* Σ^0_2 *complete.*

(iii) $\{e : P^2_e$ *is bounded and nonempty*$\}$ *is* Π^0_3 *complete and* $\{e : P^2_e$ *is bounded and empty*$\}$ *is* Σ^0_2 *complete.*

(iv) $\{e : P^2_e \neq \emptyset\}, \{e : P^2_e = \emptyset\})$ *is* (Σ^1_1, Π^1_1) *complete.*

(v) ($\{e : P_e^2$ *is g-bounded and empty*$\}, \{e : P_e^2$ *is g-bounded and non empty*$\}$) *is* (Σ_2^0, Π_2^0) *complete for any* $g \geq 2$ *which is computable in* $\mathbf{0}'$.

(vi) $\{e : P_e^2$ *is highly bounded and nonempty*$\}$ *is* Σ_4^0 *complete and* $\{e : P_e^2$ *is highly bounded and empty*$\}$ *is is* Σ_2^0 *complete.*

Proof. The upper bounds on the complexity are routine to check.

(i) For the completeness, we define a reduction f such that $P_{2,f(e)}$ is always a class of sets and such that $e \in Inf$ if and only if $P_{2,f(e)}$ is nonempty. Simply let $0^n \in T_{2,f(e)}$ if and only if there exist $a_0 < \cdots < a_{n-1}$ each in W_e.

(ii,iii,iv) In each case, the completeness follows exactly as in Theorems VI.2.3, VI.2.4 .

(v.vi) These are simply relativizations of Theorems VI.2.1 and VI.2.4. □

Next we consider finite cardinality. Results related to almost boundedness are relegated to the exercises.

Theorem VI.2.8. *For any positive integer c and any computable function* $g \geq 2$,

(a) ($\{e : P_e$ *is g-bounded* & $|P_e| > c\}, \{e : P_e$ *is g-bounded* & $|P_e| \leq c\}$) *is* (Σ_2^0, Π_2^0) *complete;*

(b) $\{e : P_e$ *is g-bounded and* $Card(P_e) = c + 1\}$ *is* D_2^0 *complete;*

c) $\{e : P_e$ *is g-bounded and* $Card(P_e) = 1)$ *is* Π_2^0 *complete.*

Proof. $\{e : P_e$ is g-bounded and $Card(P_e) > c\}$ is Σ_2^0, since if P_e is g-bounded, then $Card(P_e) > c)$ if and only if there exist k and incomparable $\sigma_1, \sigma_2, \ldots, \sigma_{c+1} \in \omega^k$ such that each $\sigma_i \in Ext(T_e)$. For $c = 0$, this set is in fact Π_1^0 by Theorem VI.2.1. These facts imply the upper bounds on the complexity.

To prove the Σ_2^0 completeness for cardinality $> c$, we define a reduction f of $\omega \setminus Tot$, as follows. For each e, let $\sigma = 0^{m_0}1^r0^{m_1}1^r \ldots 0^{m_{k-1}}1^r0^{m_k}1^t \in T_{f_c(e)}$ if and only if the following conditions are satisfied.

(i) $1 \leq r \leq c$ and $t \leq r$.

(ii) for each $i < k$, if $\phi_{e,|\sigma|}(i)\downarrow$, then $\phi_{e,|\sigma|}(i) = m_i$.

(iii) if $\phi_{e,|\sigma|}(k)\downarrow$, then $\phi_{e,|\sigma|}(k) \geq m_k$.

Thus if ϕ_e is total, then $P_{f(e)}$ has exactly c elements, $0^{\phi_e(0)}1^r0^{\phi_e(1)}1^r \ldots$ for $1 \leq r \leq c$. On the other hand, if ϕ_e is not total, then $P_{f_c(e)}$ will be infinite. Note that the tree $T_{f_c(e)}$ is always g-bounded, since it is a binary tree. This reduction shows both the double completeness result as well as the completeness for cardinality $= 1$. Note that since Tot is $|Pi_2^0$ complete, it follows that for any Π_2^0 set C, there is a reduction h_c of C so that $card(P_{h_c(e)}) = c$ if $e \in C$ and $P_{h_c(e)}$ is infinite otherwise.

To prove the D_2^0 completeness for cardinal $= c + 1$, let $A = B \cap C$ where B is Σ_2^0 and C is Π_2^0, let h_1 be a reduction of $\omega - B$ (as above) so that $P_{h_1(e)}$ is infinite if $e \in B$ and $card(P_{f(e)}) = 1$ otherwise. Let h_{c+1} be the reduction of C described above. Then a reduction ϕ of A to $\{e : P_e$ is g-bounded $= c + 1\}$ may be given by defining $T_{\phi(e)} = T_{h_1(e)} \oplus T_{h_{c+1}(e)}$. □

Remark. It follows from Theorem VI.2.8 that $\{e : card(P_e \cap \{0,1\}^\omega) > c\}$ is Σ^0_2 complete, that $\{e : card(P_e \cap \{0,1\}^\omega) = 1\}$ and $\{e : card(P_e \cap \{0,1\}^\omega) \leq c\}$ are both Π^0_2 complete, and that $\{e : card(P_e \cap \{0,1\}^\omega) = c+1\}$ is D^0_2 complete.

Theorem VI.2.9. *For any positive integer c, $\{e : P_e$ is c. b. and $Card(P_e) > c\}$, $\{e : P_e$ is c. b. and $Card(P_e) \leq c\}$, and $\{e : P_e$ is c. b. and $Card(P_e) = c\}$, are all Σ^0_3 complete.*

Proof. The g-bounded case above is uniformly Σ^0_2. Then

$$P_e \text{ is c. b. } \& \ |P_e| > c \iff (\exists a)[a \in Tot \ \& \ P_e \text{is } \phi_a\text{-bounded } \& \ |P_e| > c.$$

A similar argument gives the upper bound for cardinality $\leq c$.

For the Σ^0_3 completeness of cardinality $> c$, recall the modified function g from the proof of Theorem VI.2.3 such that $card(P_{g(e)}) = 1$ for all e and such that $P_{g(e)}$ is c. b. if and only if $e \in Rec$. Fix c and let T be a binary tree such that $[T]$ has exactly $c+1$ elements. Then let $T_{k(e)} = T_{g(e)} \otimes T$. Then k is a reduction of Rec to $\{e : P_e$ is c. b. and $Card(P_e) > c\}$. This same reduction k also works for cardinality $= c+1$ and cardinality $\leq c+1$. Note here that since T is binary, $T_{g(e)} \otimes T$ will be r. b. if $T_{g(e)}$ is r. b. and since $[T]$ is nonempty, $T_{g(e)} \otimes T$ will be not r. b. if $T_{g(e)}$ is not computably bounded. □

Theorem VI.2.10. *For any positive integer c,*

(a) $\{e : P_e$ is bounded & $|P_e| \leq c\}$ and $\{e : P_e$ is bounded & $|P_e| = 1\}$ are both Π^0_3 complete;

(b) $\{e : P_e$ is bounded & $|P_e| > c\}$ and $\{e : P_e$ is bounded & $|P_e| = c+1\}$ are both D^0_3 complete.

Proof. Let us define here a Σ^0_3 relation $C(c,k,e)$ such that $C(c,k,e)$ holds iff there is a $j \geq k$ such that there exist distinct $\sigma_1, \ldots, \sigma_{c+1} \in \omega^j \cap T_e$ such that for all $n > j$ there exists $\tau_1, \ldots, \tau_{c+1} \in \omega^n \cap T_e$ which extend $\sigma_1, \ldots, \sigma_{c+1}$ respectively. Note that if T_e is bounded above k, then $C(c,k,e)$ implies that $card(P_e) \geq c$.
$\{e : P_e$ is bounded and $Card(P_e) \leq c\}$ is Π^0_3, since

$$(P_e \text{ is bounded } \& \ |P_e| \leq c) \iff P_e \text{ is bounded } \& \ \neg C(c,0,e)].$$

The upper bounds on the complexity for the other cases follows easily.

For the completeness results, let $A = B \cap C$, where B is a Π^0_3 set and C is a Σ^0_3 set. It follows from Theorem VI.1.6 that there is a reduction f such that $card(P_{f(e)}) = 1$ for all e and such that $T_{f(e)}$ is bounded if and only if $e \in B$. This gives the Π^0_3 completeness in the cases of cardinality $\leq c$ and cardinality $= 1$.

Suppose now that $e \in C \iff (\exists m)(\forall n)(\exists k)R(e,m,n,k)$, where R is computable. We will define, uniformly in e, a computable tree $T_{g(e)}$ such that $T_{g(e)}$ is bounded for all e and such that $P_{g(e)}$ has exactly 2 elements if $e \in C$ and exactly one element (0^ω) otherwise. Let $T_{g(e)}$ consist of all strings 0^m together with all strings $(0^m)^\frown(r+1, k_1, k_2, \dots k_n)$ such that

(i) either $m = r = 0$ or $m > 0$ and $\neg R(e, m-1, r, k)$ for all $k < n$;

(ii) for all $i \leq n$, $R(e, m, i, k_i)$ and $\neg R(e, m, i, j)$ for all $j < k_i$.

Clearly $T_{g(e)}$ is always bounded and has at least one element 0^ω. There will be another element $(0^m)^\frown(r+1, k_1, k_2), \dots)$ when m is the least number such that $(\forall n)(\exists k)R(e, m, n, k)$. Thus $e \in C$ if and only if $P_{g(e)}$ contains exactly two elements and $e \notin C$ if and only if $P_{g(e)}$ contains exactly one element. By taking a disjoint union with a fixed set containing exactly c elements, we may obtain a recursive function g_c such that $e \in C$ if and only if $P_{g_c(e)}$ contains exactly $c+1$ elements and $e \notin C$ if and only if $P_{g_c(e)}$ contains exactly c elements.

The reduction of A for cardinality $> c$ is then given by $T_{h(e)} = T_{f(e)} \otimes T_{g_c(e)}$; this also works for the case of cardinality $= c+1$. □

Theorem VI.2.11. *For any positive integer c,*

(a) $(\{e : Card(P_e) > c\}), (\{e : Card(P_e) \leq c\})$ (Σ^1_1, Π^1_1) *complete;*

(b) $\{e : Card(P_e) = c\}$ *is* Π^1_1 *complete.*

Proof. $I_P(> c)$ is Σ^1_1 uniformly in c since $e \in I_P(\geq c)$ if and only if there exist distinct $x_1, \dots, x_c \in P_e$. It then immediately follows that $I_P(\leq c)$ is Π^1_1 uniformly in c.

For $I_P(= c)$, we recall from Theorem 1.3 that any countable Π^0_1 class contains a hyperarithmetic member. Thus we have

$$e \in I_P(= c) \iff e \in I_P(\leq c) \ \& \ (\exists x_1, \dots, x_c \in HYP)(x_c \in P_e).$$

It then follows from the Spector-Gandy Theorem II.10.5 that $I_P(= c)$ is Π^1_1.

For the completeness, let A be a Σ^1_1 set and let f be the function from Theorem VI.2.6 which reduces A to $\{e : P_e \neq \emptyset\}$ and its complement to $\{e : P_e = \emptyset\}$. Let T be a primitive recursive tree such that $card([T]) = c$. Then a reduction of A to $\{e : Card(P_e) > c\}$ may be defined by $T_{g(e)} = T_{f(e)} \oplus T$ and this simultaneously reduces $\mathbb{N} \setminus A$ to $\{e : Card(P_e) \leq c\}$ and in fact reduces $\mathbb{N} \setminus A$ to $\{e : Card(P_e) = c\}$. □

Theorem VI.2.12. *Let c be a positive integer.*

(a) For any computable function $g \geq 2$,

(i) $(\{e : P_e$ *is decidable, g-bounded and* $Card(P_e) > c\}$ *is* D^0_1 *complete;*

(ii) $\{e : P_e$ *is decidable, g-bounded and* $Card(P_e) \leq c\}$ *is* Π^0_1 *complete;*

(iii) $\{e : P_e$ *is decidable, g-bounded and* $Card(P_e) = c+1\}$ *is* D^0_1 *complete.*

(b) $\{e : P_e$ *is decidable* & $|P_e| \leq c\}$, $\{e : P_e$ *is decidable* & $|P_e| > c\}$ *and* $\{e : P_e$ *is decidable* & $|P_e| = c\}$ *are all* Π^0_2 *complete.*

Proof. For a decidable tree T_e, P_e has $> c$ members if and only if there T_e contains incomparable $\sigma_1, \ldots, \sigma_{c+1}$. The upper bounds on the complexity now follow from Theorem VI.1.7.

(a) (i) For the D_1^0 completeness of this set as well as the set in (iii), let $A = B \cap C$, where B is a Π_1^0 set and C is a Σ_1^0 set. Let R be a computable relation so that $e \in C \iff (\exists n)R(e,n)$. It follows from the proof of Theorem VI.1.4 that there is a computable function g such that if $e \in B$, then $P_{g(e)} = \{0^\omega\}$ and is 2-bounded and if $e \notin B$, then $P_{g(e)}$ is not g-bounded. Let $P_{h(e)} = P_{g(e)} \cup \{0^n 1^{i+1} 0^\omega : i < c \;\&\; R(e,n) \;\&\; (\forall m < n)\neg R(e,m)\}$.

(ii) Let Q be a fixed decidable Π_1^0 class having exactly c elements. For the Π_1^0 completeness of this set, just let $P_{g(e)} = P_{h(e)} \otimes Q$.

(iii) For the Π_1^0 completeness, use the reduction h given in the proof of Theorem VI.1.4.

(b) For the Π_2^0 completeness when cardinality equals 1, modify the reduction given for the Π_2^0 set C in the proof of part (0) of Theorem VI.1.7 by putting $\emptyset$ in $T_{f(e)}$ and by putting $(n_0, n_1, \ldots, n_{k-1}) \in T_{f(e)}$ if and only if, for all $i < k$, n_i is the least such that $R(e,i,n)$. Then if $e \in C$, $T_{f(e)}$ will be a decidable tree with exactly one element and if $e \notin C$, then $T_{f(e)}$ will be a finite tree and hence will not be decidable.

For the Π_2^0 completeness in the other two sets, let $P_{g(e)} = P_{f(e)} \otimes Q$ as in (ii) above. □

There are five cases to consider when we examine the possible infinite cardinality of a Π_1^0 class P. P may be finite, countably infinite, or uncountable. Negations of two of these adds the notions of infinite and of countable. Our first result deals with the problem of finite versus infinite sets.

Theorem VI.2.13.

(a) For any computable function $g \geq 2$, $(\{e : P_e$ is g-bounded and infinite$\}, \{e : P_e$ is g-bounded and finite$\})$ is (Π_3^0, Σ_3^0) complete.

(b) $\{e : P_e$ is c. b. and infinite$\}$ is D_3^0 complete and $\{e : P_e$ is c. b. and finite$\}$ is Σ_3^0 complete.

(c) $(\{e : P_e$ is bounded and infinite$\}, \{e : P_e$ is bounded and finite$\})$ is (Π_4^0, Σ_4^0) complete.

(d) $(\{e : P_e$ is infinite$\}, \{e : P_e$ is finite$\})$ is (Σ_1^1, Π_1^1) complete.

Proof. In each case, the upper bound on these complexities follows from the uniformity of previous results (Theorems VI.2.8, VI.2.9, VI.2.10 and VI.2.11, since P_e is infinite if and only if $Card(P_e) > c$ for all c.

(a) For the completeness, we define a reduction of Cof to $\{e : P_e$ is g-bounded and finite$\}$ which simultaneously reduces $\omega \backslash Cof$ to $\{e : P_e$ is g-bounded and infinite$\}$. Let

$$T_{f(e)} = \{0^n : n \in \omega\} \cup \{0^n 1 0^k : n \notin W_{e,k}\}.$$

Then $T_{f(e)}$ is always a binary tree and it is easy to see that $P_{f(e)} = \{0^\omega\} \cup \{0^n10^\omega : n \notin W_e\}$, so that $f(e) \in I_P(g\text{-bounded} < \aleph_0) \iff e \in Cof\}$.
(b) For the Σ^0_3 completeness in the finite case, use the same reduction f given in part (a) above.

For the D^0_3 completeness in the infinite case, let $A = B \cap C$ where B is a Σ^0_3 set and C is a Π^0_3 set and let f be the reduction given above applied to $\omega \setminus C$, so that $T_{f(a)}$ is always a binary tree and such that $P_{f(a)}$ is finite if and only if $a \notin C$. It follows from the proof of Theorem VI.2.3 that there is a recursive function g such that $P_{g(a)}$ is always a singleton and is r. b. if and only if $a \in B$. Then the reduction of A to $\{e : P_e$ is c. b. and infinite$\}$ is given by $T_{h(a)} = T_{f(a)} \otimes T_{g(a)}$.

(c) For the double completeness, let A be a Π^0_4 set and let C be a Σ^0_3 set so that $e \in A \iff (\forall m)\langle e, m\rangle \in C$. We may assume that if $e \notin A$, then $\langle e, m\rangle \in C$ for only finitely many m. Let the reduction f be given by the proof of Theorem VI.2.13, so that $T_{f(e,m)}$ is always bounded and $P_{f(e,m)}$ has one element if $\langle e, m\rangle \notin C$ and has two elements if $\langle e, m\rangle \in C$. Define the reduction h by $T_{h(e)} = \otimes_m T_{f(e,m)}$. Then $T_{h(e)}$ is always bounded and $card(P_{h(e)} = \prod_m card(P_{f(e,m)}$, so that if $e \notin C$, then $T_{h(e)}$ is finite, and, if $e \in C$, then $T_{h(e)}$ is uncountable.

(d) For the completeness, just let f be the reduction of Theorem VI.2.6 and let $T_{h(a)} = T_{f(a)} \otimes T$, where T is a primitive recursive tree such that $[T]$ is infinite. □

Remark. It follows from part (a) that $\{e : P_e \cap \{0,1\}^\omega$ is infinite$\}$ is Π^0_3 complete and $\{e : P_e \cap \{0,1\}^\omega$ is finite$\}$ is Σ^0_3 complete.

Again we give only two cases for decidable classes.

Theorem VI.2.14.

(a) For any computable function $g \geq 2$, ($\{e : P_e$ is g-bounded, decidable$\}$, $\{e : P_e$ is g-bounded, decidable and finite$\}$) is (Π^0_2, Σ^0_2) complete.

(b) $\{e : P_e$ is decidable and infinite$\}$ is Π^0_2 complete and $\{e : P_e$ is decidable and finite$\}$ is D^0_2 complete.

Proof. The upper bounds on the complexities of the classes in both parts easily follow from the uniformity of the proof of Theorem VI.2.12 as in the previous result.

For the Σ^0_2 completeness in the g-bounded infinite case, we define a reduction f so that $P_{f(e)}$ is always 2-bounded and decidable and is finite if and only if W_e is finite. Let $P_{f(e)}$ contain 0^ω and, for each m, contain 0^m10^ω if $m = [k, s]$ where $k \in W_{e,s+1} \setminus W_{e,s}$. This also gives the Π^0_2 completeness in the (unbounded) infinite case.

For the D^0_2 completeness in the finite case, let $A = B \cap C$ where B is Σ^0_2 and C is Π^0_2. Using the reduction f, it follows that there is a reduction k of B such that $P_{k(e)}$ is always 2-bounded and decidable and is finite if and only if $e \in B$. Let h be the reduction from Theorem VI.2.12 so that $P_{h(e)}$ is decidable and has cardinality 1 if $e \in C$ and otherwise $P_{h(e)}$ is not decidable. Then $P_{j(e)} = P_{k(e)} \otimes P_{h(e)}$ defines a reduction of A to $\{e : P_e$ is finite and decidable$\}$. □

The other three notions of infinity produce the same level of complexity independent of the level of boundedness and of the decidability.

Theorem VI.2.15. *($\{e : P_e$ is uncountable$\}, \{e : P_e$ is countable$\}$) is (Σ^1_1, Π^1_1) complete and $\{e : P_e$ is countably infinite$\}$ is Π^1_1 complete and the same result holds for bounded classes, for c. b. classes, and for g-bounded classes, and also for decidable classes and for strong Pi^0_2 class, under each possible notion of boundedness.*

Proof. Recall from Theorem VI.1.6 that in each case the underlying set of e such that P_e is suitably bounded is a Σ^0_4 set. (For strong Π^0_2 classes, this also holds.) Then the property of being uncountable is Σ^1_1, since for any tree T_e, P_e is uncountable if and only if P_e has a perfect subset, i.e. if and only if there exists an embedding f from $\{0,1\}^{<\omega}$ into T_e which preserves the partial order $\prec$. It follows that the property of being countable is Π^1_1 and, by Theorem VI.2.13, that the property of being countably infinite is also Π^1_1.

For the completeness of $\{e : P_e$ is uncountable$\}$, we define a reduction of $\{e : P_e \neq \emptyset\}$ to $\{e : P_e$ is 2-bounded and uncountable$\}$ as follows. Define the binary tree $T_{f(e)}$ to consist of all strings $0^{n_0}\frown\tau_0\frown 0^{n_1}\frown\tau_1\frown\cdots\frown 0^{n_{k-1}}\frown\tau_{k-1}0^n$, where $(n_0,\dots,n_{k-1}) \in U_e$ and for $i < k$, $\tau_i = (1)$ or $\tau_i = (1,1)$. Then for any path $x \in [T_e]$, $T_{f(e)}$ will contain uncountably many paths, so that if P_e is nonempty, then $P_{f(e)}$ will be uncountable. If P_e is empty, then every path in $P_{f(e)}$ will end in 0^ω, so that $P_{f(e)}$ will be countable. Note that f also reduces $\{e : P_e = \emptyset\}$ to $\{e : P_e$ is 2-bounded and countable$\}$. A reduction g of $\{e : P_e = \emptyset\}$ to $\{e : P_e$ is 2-bounded and countably infinite$\}$ is then given by $T_{g(e)} = T_{f(e)} \oplus T$, where T is some primitive recursive binary tree with $[T]$ countably infinite.

It is clear that these reductions work for each of the notions of boundedness and also for decidable classes, since the trees constructed have no dead ends. □

Exercises

VI.2.1. A Π^0_1class P is said to be *intrinsically bounded* by g if the tree $T_P = \{\sigma : P \cap \sigma \neq \emptyset\}$ is g-bounded. Show that $\{e : P_e$ is c. b.$\}$ is Π^1_1 complete, and similarly for all other notions of boundedness.

VI.2.2. Show that $\{e : P_e$ is a. c. b. and empty$\}$ and $\{e : P_e$ is a. c. b. and nonempty$\}$ are both Σ^0_3 complete. Hint: For the completeness in the "empty" case, modify the reduction f from the proof of Theorem VI.1.6 by letting $T_{f'(e)}$ contain strings $(n, s_0, \dots, s_{k-1})$ such that $(s_0,\dots,s_{k-1}) \in T_{f(e)}$ and $k < n$, so that $T_{f'(e)}$ is a. r. b. if and only if $T_{f(e)}$ is a. c. b. and $P_{f'(e)}$ is always empty.

VI.2.3. Let c be a positive integer and $g \geq 2$ a computable function. Show that $\{e : P_e$ is g-almost bounded and $Card(P_e) > c\}$ is Σ^0_2 complete and both $\{e : P_e$ is g-almost bounded and $Card(P_e) \leq c\}$ and $\{e : P_e$ is g-almost bounded and $Card(P_e) = c\}$ are D^0_2 complete.

VI.2.4. For any positive integer c, show that $\{e : P_e$ is a. c. b. and $Card(P_e) > c\}$, $\{e : P_e$ is a. c. b. and $Card(P_e) \leq c\}$, and $\{e : P_e$ is a. c. b. and $Card(P_e) = c\}$, are all Σ^0_3 complete.

VI.2.5. For any positive integer c, show that $\{e : P_e$ is a. b. and $Card(P_e) > c\}$, $\{e : P_e$ is a. b. and $Card(P_e) \leq c\}$ and $\{e : P_e$ is a. b. and $Card(P_e) = c\}$ are all Σ^0_4 complete.

VI.2.6. Show that $(\Sigma^0_1, \Pi^0_1) \leq_m (\{e : P_e$ is decidable, g-bounded and $Card(P_e) > c\}, \{e : P_e$ is decidable, g-bounded and $Card(P_e) \leq c\})$.

VI.2.7. For any positive integer c and any computable function $g \geq 2$,

(a) $(\{e : P^2_e$ is g-bounded and $Card(P_e) \leq c\})$ is Π^0_3 complete.

(b) $\{e : P^2_e$ is g-bounded and $Card(P^2_e) = c + 1\}$ is D^0_3 complete.

c) $\{e : P^2_e$ is g-bounded and $Card(P^2_e) = 1)$ is Π^0_3 complete.

(a) $(\{e : P^2_e$ is g-bounded and finite$\})$ is Σ^0_4 complete

VI.3 Computable Cardinality

The *computable cardinality* of a class P is the cardinality of the set of computable members of P. Also, we say that P is *computably nonempty* if it has a computable member and *computably empty* otherwise. In this section, we classify the various index sets of classes with given computable cardinality conditions. The first theorem extends the result of Gasarch and Martin [77] that the property of being computably nonempty is Σ^0_3 complete for c. b. Π^0_1 classes.

Theorem VI.3.1. *For any computable $g \geq 2$,*

(a) $(\{e : P_e$ *is g-bounded and computably nonempty*$\}, \{e : P_e$ *is g-bounded and computably empty*$\})$ *is* (Σ^0_3, Π^0_3) *complete*

(b) $\{e : P_e$ *is g-bounded, nonempty, and computably empty*$\}$ *is* Π^0_3 *complete.*

Proof. The upper bounds on the complexity follow from Theorem VI.2.1 and the fact that P_e has a computable member if and only if

$$(\exists a)[a \in Tot \ \& \ (\forall n)(\phi_a \lceil n \in U_e)].$$

For the completeness of the first two sets, we define a reduction of Ext_2 by letting $P_{f(a)}$ equal

$$\{x \in \{0,1\}^\omega : \phi_a \prec x\} = \{x : (\forall m)(\forall s)(\forall i)[\phi_{a,s}(m) = i \rightarrow x(m) = i]\}.$$

For the other completeness, let Q be a nonempty, binary Π^0_1 class with no computable members and let $P_{h(a)} = P_{f(a)} \oplus Q$. $\square$

Theorem VI.3.2. *(a)* $\{e : P_e$ *is c. b. and computably nonempty*$\}$ *is* Σ^0_3 *complete;*

(b) $\{e : P_e$ *is c. b. and computably empty*$\}$ *is* D^0_3 *complete;*

(c) $\{e : P_e$ *is c. b., nonempty and computably empty*$\}$ *is* D^0_3 *complete.*

Proof. The upper bounds on the complexity are easily checked.

For the completeness in (a) use the reduction from Theorem VI.3.1. For the completeness of the other two sets, let $A = B \cap C$, where B is Π^0_3 and C is Σ^0_3. It follow from the proof of Theorem VI.3.1 that there is a computable function f such that $P_{f(a)}$ is always c. b. nonempty and has a computable member if and only if $a \notin B$. It follows from the proof of Theorem VI.1.5 that there is a computable function g such that $P_{g(a)}$ is c. b. if and only if $a \in C$. Then a reduction h of A to the sets in (b) and (c) may be given by $P_{h(a)} = P_{f(a)} \otimes (P_{g(a)} \cup \{0^\omega\})$. □

Proofs of the next two theorems are left for the exercises.

Theorem VI.3.3. *(a)* $\{e : P_e$ *is bounded and computably nonempty*$\}$ *is* D^0_3 *complete;*

(b) $\{e : P_e$ *is bounded and computably empty*$\}$ *is* Π^0_3 *complete;*

(c) $\{e : P_e$ *is bounded, nonempty and computably empty*$\}$ *is* Π^0_3 *complete.* □

Theorem VI.3.4. *(a)* $\{e : P_e$ *is and computably nonempty*$\}$ *is* Σ^0_3 *complete;*

(b) $\{e : P_e$ *is computably empty*$\}$ *is* Π^0_3 *complete;*

(c) $\{e : P_e$ *is nonempty and computably empty*$\}$ *is* Σ^1_1 *complete.* □

Theorem VI.3.5. *Let c be a positive integer and let $g \geq 2$ be a computable function.*

(a) $(\{e : P_e$ *is g-bounded and has computable cardinality* $> c\}$,
$\{e : P_e$ *is g-bounded and computable cardinality* $\leq c\})$ *is* (Σ^0_3, Π^0_3) *complete;*

(b) $\{e : P_e$ *is g-bounded, nonempty, and has computable cardinality* $= c\}$ *is* D^0_3 *complete.*

Proof. The upper bounds on the complexity are easily checked.

(a) For the completeness, let f be the reduction from Theorem VI.3.1 such that $T_{f(a)}$ is always a binary tree and such that $P_{f(a)}$ has a computable member if and only if $a \in Ext_2$. Let T_b be a fixed binary tree such that P_b consists of exactly $c+1$ computable elements. Then let $P_{h(a)} = P_{f(a)} \otimes P_b$.

(b) We begin with a construction for unbounded classes using the Σ^0_3 completeness of *Rec*.

Define the *modulus function* μ_a for the c. e. set W_a by

$$\mu_a(i) = (\text{least } s)[W_a \cap \{0, 1, \dots, i\} = W_{a,s} \cap \{0, 1, \dots, i\}].$$

It is easy to see that W_a is computable if and only if μ_a is computable. We shall define a tree $T_{f(a)}$ so that $P_{f(a)} = \{\mu_a\}$ and hence P_a has exactly one computable element if and only if $a \in Rec$.

The tree $T_{f(a)}$ is defined so that a string σ of length n is in $T_{f(a)}$ if and only if each of the following three conditions is satisfied.

1. $(\forall i < n)(i \in W_{a,n} \iff i \in W_{a,\sigma(i)})$,
2. $\sigma_0 > 0 \to 0 \in W_{a,\sigma(0)} \setminus W_{a,\sigma(0)-1}$, and
3. $(\forall 0 < m < n)[\sigma(m) > \sigma(m-1) \to m \in W_{a,\sigma(m)} \setminus W_{a,\sigma(m)-1})]$.

Now let $A = B \cap C$, where B is Σ^0_3 and C is Π^0_3. It follows from the completeness of Rec and the above construction that $P_{f(a)}$ is a singleton for each a and has a (unique) computable member if and only if $a \in B$. Let h be the reduction from (a) such that $P_{h(a)}$ has no computable members if $a \in C$ and has at least 2 computable members if $a \notin C$. Let S be a fixed class with exactly c computable members. Define the computable function ψ so that

$$P_{\psi(a)} = S \otimes (P_{f(a)} \oplus P_{h(a)}).$$

This gives a reduction of A to $\{e : P_e \text{ has exactly } c \text{ computable members}\}$.

Finally, let k be the primitive recursive function given in Theorem III.7.7 so that for any e, $P_{k(e)}$ is a Π^0_1 class of sets such that there is a one-to-one correspondence between the members of P_e and the computable members of $P_{k(e)}$. Then the composite function $k(\psi(a))$ gives a desired reduction of A to

$$\{e : P_e \text{ is } g\text{-bounded, nonempty, and has computable cardinality} = c\},$$

so that A is D^0_3 complete. □

The next three theorems essentially follow from the proof of Theorem VI.3.5. Details are left for the exercises.

Theorem VI.3.6. *Let c be a positive integer.*

(a) $\{e : P_e$ *is c. b. and has computable cardinality* $> c\}$ *is* Σ^0_3 *complete;*

(b) $\{e : P_e$ *is c. b. and has computable cardinality* $\leq c\}$ *is* D^0_3 *complete;*

(c) $\{e : P_e$ *is c. b. and has computable cardinality* $= c\}$ *is* D^0_3 *complete.* □

Theorem VI.3.7. *Let c be a positive integer.*

(a) $\{e : P_e$ *is bounded and has computable cardinality* $\leq c\}$ *is* Π^0_3 *complete;*

(b) $\{e : P_e$ *is bounded and has computable cardinality* $> c\}$ *is* Dp_3 *complete;*

(c) $\{e : P_e$ *is bounded and has computable cardinality* $= c\}$ *is* D^0_3 *complete.* □

Theorem VI.3.8. *Let c be a positive integer.*

(a) ($\{e : P_e$ *has computable cardinality* $> c\}, \{e : P_e$ *has computable cardinality* $\leq c\}$) *is* (Σ^0_3, Π^0_3) *complete;*

(b) $\{e : P_e$ *has computable cardinality* $= c\}$ *is* D^0_3 *complete.* □

For finite versus infinite computable cardinality, all versions of boundedness produce index sets of the same complexity (excluding almost boundedness).

Theorem VI.3.9. ($\{e : P_e$ *has finite computable cardinality*$\}, \{e : P_e$ *has finite computable cardinality*$\}$) *is* (Σ^0_4, Π^0_4) *complete and the similar result is true for g-bounded, c. b. and bounded classes and also for strong* Π^0_2 *classes.*

Proof. The upper bounds follow from the uniformity of Theorems VI.3.5, VI.3.6, VI.3.7 and VI.3.8.

For the completeness results, let A be a Π^0_4 set, so that for some Σ^0_3 relation R,

$$a \in A \iff (\forall i) R(i, a).$$

As usual, R may be defined so that if $a \notin A$, then $R(i, a)$ for only finitely many values of i. By the proof of part (i) of Theorem VI.3.1, there is a computable function f so that for each a and i, $R(i, a)$ if and only if $P_{f(i,a)}$ has a computable member and $P_{f(i,a)}$ is a binary class. Now let

$$T_{\phi(a)} = \{0^n : n \geq 0\} \cup \{0^i 1 \frown \sigma : \sigma \in U_{f(i,a)}.$$

Then it is clear that $a \in A$ if and only if $P_{\phi(a)}$ has infinitely many computable members and $P_{\phi(a)}$ is always a binary class. □

Next we consider the problem of whether a Π^0_1 class has a member computable in $\mathbf{0}'$, or equivalently whether it has an element in Δ^0_2. We omit the almost computably bounded classes, since by Exercise 4.1.3, an a. c. b. Π^0_1 class has a member computable in $\mathbf{0}'$ if and only if it is nonempty.

Theorem VI.3.10. *(a)* ($\{e : P_e$ *has a* Δ^0_2 *member*$\}, \{e : P_e$ *has no* Δ^0_2 *member*$\}$) *is* (Σ^0_4, Π^0_4) *complete and* $\{e : P_e$ *is nonempty but has no* Δ^0_2 *member*$\}$ *is* Σ^1_1 *complete.*

(b) $\{e : P_e$ *is bounded and has a* Δ^0_2 *member*$\}$ *is* Σ^0_4 *complete and*

$$\{e : P_e \textit{ is bounded and has no } \Delta^0_2 \textit{ member}\}$$

and

$$\{e : P_e \textit{ is bounded and nonempty but has no } \Delta^0_2 \textit{ member}\}$$

are both Π^0_4 *complete.*

Proof. (a) By relativization of Theorem VI.0.4, the set $Tot(\mathbf{0}') = \{e : \phi^{\mathbf{0}'}_e$ is total is a Π^0_3 complete set and

$$P_e \text{ has a } \Delta^0_2 \text{ member} \iff (\exists a)[a \in Tot(0') \;\&\; (\forall n)(\phi'_a \lceil n \in T_e)].$$

The upper bounds on the complexity follow easily.

For the completeness of the first two sets, let S be an arbitrary Σ^0_4 set and suppose that

$$a \in S \iff (\exists m)(\forall n)(\exists i)(\forall j) R(a, i, j, m, n),$$

for some recursive relation R. Define the reduction f so that

$$(m, i_0, i_1, \dots) \in P_{f(a)} \iff (\forall n)(\forall j) R(a, i_n, j, m, n).$$

It is clear that if $a \notin A$, then $P_{f(a)}$ is empty and therefore has no member computable in $\mathbf{0}'$. On the other hand, if $a \in A$, then we may choose m so that $(\forall n)(\exists i)(\forall j) R(a, i, j, m, n)$ and define an infinite path $(m, i_0, i_1, \dots) \in P_{f(a)}$ which computable in $\mathbf{0}'$ by letting i_n be the least i such that $(\forall j) R(a, i, j, m, n)$.

For the Σ^1_1 completeness result, let A be a complete Σ^1_1 set and let f be the reduction given in Theorem thm:iue so that $P_{f(a)}$ is nonempty iff $a \in A$. Then let g be the computable function such that $P_{g(a)} = P_{f(a)} \oplus Q$, where Q is a Π^0_1 class with no members computable in $0'$.

(b) The upper bounds on the complexity of the three sets follows as in part (a). For the completeness of the first two sets, let h be the function given in Theorem III.7.7. Then P_e has a (respectively, no) Δ^0_2 member if and only if $P_{h(e)}$ is bounded and has a (resp. no) Δ^0_2 member.

For the Π^0_4 completeness of the third set, let Q be a nonempty bounded Π^0_1 class Q with no Δ^0_2 members and let $P_{k(e)} = Q \oplus P_{h(e)}$. $\square$

Finally, we consider the computable and Δ^0_2 cardinality of strong Π^0_2 classes.

Theorem VI.3.11. *(i) For any $g \geq 2$ which is computable in $\mathbf{0}'$,*

$$(\{e : P^2_e \textit{ is g-bounded and comp. empty}\}, \{e : P^2_e \textit{ is g-bounded and comp. nonempty}\})$$

is (Σ^0_3, Π^0_3) complete and

$$\{e : P^2_e \textit{ is g-bounded, nonempty, and computably empty}\}$$

is Π^0_3 complete.

(ii) $\{e : P^2_e$ is c. b. and computably nonempty is Σ^0_3 complete and

$$\{e : P^2_e \textit{ is c. b. and computably empty}\}$$

and

$$\{e : P^2_e \textit{ is c. b., nonempty, and computably empty}\}$$

are D^0_3 complete.

(iii) $\{e : P^2_e$ is bounded and computably nonempty is D^0_3 complete and

$$\{e : P^2_e \textit{ is bounded and computably empty}\}$$

and

$$\{e : P^2_e \textit{ is bounded, nonempty, and computably empty}\}$$

are Π^0_3 complete.

(iv) ($\{e : P^2_e$ *is computably nonempty*$\}, \{e : P^2_e$ *is computably empty*$\}$) *is* (Σ^0_3, Π^0_3) *complete and* $\{e : P^2_e$ *is nonempty but computably empty*$\}$ *is* Π^0_3 *complete.*

(v) $\{e : P^2_e$ *is highly bounded and computably nonempty*$\}$ *is* Σ^0_4 *complete,*

$$\{e : P^2_e \text{ is highly bounded and computably empty}\},$$

and

$$\{e : P^2_e \text{ is highly bounded, nonempty and computably empty}\}$$

are all Σ^0_4 *complete.*

Proof. The upper bounds on the complexity are routine. The completeness of parts (i)–(iv) follow from previous results on Π^0_1 classes. The completeness in part (v) follows easily from the Σ^0_4 completeness of the property of being highly bounded. □

Exercises

VI.3.1. Prove Theorem VI.3.3. Hint: Combine the reductions from Theorems VI.1.6 and VI.3.1.

VI.3.2. Prove Theorem VI.3.4. Hint: Combine the reductions from Theorems VI.2.6 and VI.3.1.

VI.3.3. Prove Theorems VI.3.6, VI.3.7 and VI.3.8.

VI.3.4. Show that the computable cardinality of a decidable class P is the same as the cardinality of P, except when P is uncountable but has a countable infinite number of computable elements. Then formulate and prove decidable versions of Theorems VI.3.5, VI.3.6, VI.3.7 and VI.3.8.

VI.3.5. Prove Theorem VI.3.11(v).

VI.3.6. Show that $(\{e : P^2_e$ has a Δ^0_2 member$, \{e : P^2_e$ has no Δ^0_2 member$\})$ is (Σ^0_4, Π^0_4) complete and likewise for g-bounded, c. b. and bounded classes.

VI.3.7. Show that $\{e : P^2_e$ is bounded and nonempty but has no Δ^0_2 member is Π^0_4 complete and likewise for g-bounded and c. b. classes. However, for unbounded classes the corresponding set is Σ^1_1 complete.

VI.4 Index Sets and Lattice Properties

In this section, we consider in particular the complexity of the inclusion relation and the lattice operations on the family of Π^0_1 classes.

Lemma VI.4.1. *There are primitive recursive functions* ψ_i*,* ψ_u*,* ψ_s *and* ψ_p *such that, for all a and b, (a)* $P_{\psi_i(a,b)} = P_a \cap P_b$*; (b)* $P_{\psi_u(a,b)} = P_a \cup P_b$*; (c)* $P_{\psi_s(a,b)} = P_a \oplus P_b$*; (d)* $P_{\psi_p(a,b)} = P_a \otimes P_b$

Proof. (a) Here we define ψ_i so that $T_{\psi_i(a,b)} = T_a \cap T_b$, that is, $\pi_{\psi_i(a,b)}(\sigma) = \pi_a(\sigma) \cdot \pi_b(\sigma)$.

(b) Similarly, $T_{\psi_i(a,b)} = T_a \cup T_b$.

(c) Here $T_{\psi_i(a,b)} = \{\emptyset\} \cup \{0^\frown\tau : \tau \in T_a\} \cup \{1^\frown\tau : \tau \in T_b\}$, so that $\pi = \pi_{\psi_u(a,b)}$ is defined by $\pi(\emptyset) = 1$ and

$$\pi(\sigma) = \begin{cases} \pi_a(\sigma(1), \ldots, \sigma(|\sigma| - 1)), & \text{if } \sigma(0) = 0 \\ \pi_b(\sigma(1), \ldots, \sigma(|\sigma| - 1)), & \text{if } \sigma(0) = 1 \end{cases}$$

A formal definition of ψ_u can now be obtained by the s-m-n Theorem.

(d) Here $T = T_{\psi_p(a,b)}$ is defined to contain σ if and only if $(\sigma(0), \sigma(2), \ldots) \in T_a$ and $\sigma(1), \sigma(3), \dots \in T_b$. Details are left to the reader. □

Next we consider some aspects of the *Verification Problem* for Π^0_1 classes, that is, $\{\langle i, j\rangle : P_i \subseteq P_j\}$. This problem has been studied for various families of ω-languages by Klarlund [107], Staiger [205] and others. More generally, Cenzer and Remmel [43] investigated index set problems concerning the size of the difference of two classes.

Theorem VI.4.2. *(Staiger) Let $g \geq 2$ be a computable function.*

(i) $\{\langle a, b\rangle : P_a, P_b$ *are g-bounded and* $P_a \subseteq P_b\}$ *and* $\{\langle a, b\rangle : P_a, P_b$ *are g-bounded and* $P_a = P_b\}$ *are* Π^0_2 *complete.*

(ii) $\{\langle a, b\rangle : P_a, P_b$ *are g-bounded and* $P^2_a \subseteq P^2_b\}$ *and* $\{\langle a, b\rangle : P^2_a, P^2_b$ *are g-bounded and* $P^2_a = P^2_b\}$ *are* Π^0_3 *complete.*

Proof. (i) The first set is Π^0_2 (and hence also the second set), since

$$P_a \subseteq P_b \iff (\forall\sigma)(\sigma \in Ext(T_a) \to \sigma \in T_b).$$

For the completeness, let b be given so that $P_b = \{0^\omega\}$. Let A be a Π^0_2 set and let R be a computable relation such that, for all i

$$i \in A \iff (\forall m)(\exists n)R(i, m, n)$$

Define the tree $T_{f(i)}$ to contain 0^m for all m and also

$$0^m1^n \in T_{f(i)} \iff (\forall j < n)\neg R(i, m, j)$$

Then it is clear that

$$P_{f(i)} \subseteq P_b \iff P_{f(i)} = P_b \iff i \in A.$$

(ii) Recall that $\sigma \in T^2_e \iff (\forall\tau \preceq \sigma)\tau \in W_e)$, which is a Σ^0_1 relation. Now let $G_n = \{\sigma : |\sigma| = n \;\&\; (\forall m < n)\sigma(m) < g(m)\}$. Then

$$\sigma \in Ext(T^2_e) \iff (\forall n)(\exists\tau \in G_n)(\sigma \preceq \tau \& \; \tau \in T^2_e),$$

and this relation is Π^0_2, since the quantifier $(\exists\tau \in G_n)$ is bounded. It follows as in (i) that the two sets are Π^0_3. For the completeness, simply relativize the argument from (i). That is, let R now be a Π^0_1 relation and note that the corresponding class $P^2_{f(i)}$ will indeed by a strong Π^0_2 class. □

The complexity of being "almost equal" or "almost a subset" is covered by considering differences of classes. The following results are taken from [43].

Theorem VI.4.3. *For any computable function g and any finite $k \geq 1$:*

(i) $\{\langle a,b\rangle : P_a$ *and* P_b *are g-bounded and* $card(P_a - P_b) \leq k\}$ *is* Π^0_2 *complete.* *[(ii)]* $\{\langle a,b\rangle : P^2_a$ *and* P^2_b *are g-bounded and* $card(P^2_a - P^2_b) \leq k\}$ *is* Π^0_2 *complete.*

Proof. The completeness follows from Theorem VI.2.8 in (i) and from Exercise 7 in (ii). Thus we need only see that index sets have the appropriate complexity.

(i) To see that (i) is Π^0_2, we claim that

$$card(P_a - P_b) \leq k \iff (\forall e)[P_b \cap P_e = \emptyset \ \rightarrow \ card(P_a \cap P_e) \leq k].$$

Certainly if the condition is false, then $card(P_a - P_b) > k$. On the other hand, suppose that $card(P_a - P_b) > k$. Then there are $k+1$ elements $x_0, x_1, \ldots, x_k$ in $P_a - P_b$. For each i, there is a basic open set U_i such that $x_i \in U_i$ and $U_i \cap P_b = \emptyset$. Then $P_e = U_0 \cup \cdots \cup U_k$ contradicts the condition.

(ii) The argument is similar to (i). □

Theorem VI.4.4. *For any computable function $g \geq 2$:*

(i) $\{\langle a,b\rangle : P_a$ *and* P_b *are g-bounded and* $P_a - P_b$ *is finite*$\}$ *is* Σ^0_3 *complete.* *[(ii)]* $\{\langle a,b\rangle : P^2_a$ *and* P^2_b *are g-bounded and* $P^2_a - P^2_b$ *is finite*$\}$ *is* Π^0_4 *complete.*

Proof. In each case, the upper bound on the complexity follows from the uniformity of Theorem VI.4.3. The completeness follows from Theorem VI.2.13 and Exercise 3.7. □

Theorem VI.4.5. *For any computable function $g \geq 2$ and any finite $k \geq 0$:*

(i) $\{\langle a,b\rangle : P_a$ *and* P_b *are g-bounded and* $P_a - P_b$ *has* $\leq k$ *computable members*$\}$ *and* $\{\langle a,b\rangle : P^2_a$ *and* P^2_b *are g-bounded and* $P^2_a - P^2_b$ *has* $\leq k$ *computable members*$\}$ *are* Π^0_3 *complete.*

(ii) $\{\langle a,b\rangle : P_a,\ P_b$ *are g-bounded and* $P_a - P_b$ *has* $< \aleph_0$ *computable members*$\}$ *and* $\{\langle a,b\rangle : P^2_a,\ P^2_b$ *are g-bounded and* $P^2_a - P^2_b$ *has* $< \aleph_0$ *computable members*$\}$ *are* Σ^0_4 *complete.*

Proof. (i) The completeness follows from Theorem VI.3.5. For the upper bounds on the complexity, we claim that

$$P_a - P_b \text{ has } \leq k \text{ computable members} \iff (forall e)[P_b \cap P_e = \emptyset \rightarrow card(P_a \cap P_e) \leq k].$$

The key here is that if there are $k+1$ computable elements $x_0, x_1, \ldots, x_k$ in the difference, then $\{x_0, \ldots, x_k\}$ is a Π^0_1 class. Details are left to the reader. A similar argument covers strong Π^0_2 classes.

(ii) The upper bounds follow from the uniformity of (i) and completeness follows from Theorem VI.3.9. □

Theorem VI.4.6. *$\{e : P_e \subseteq \{0,1\}^{\mathbb{N}}$ & P_e is thin$\}$ and $\{e : P_e \subseteq \{0,1\}^{\mathbb{N}}$ & P_e is thin$\}$ are Π^0_4 complete sets.*

Proof. First observe that P_e is minimal if and only if, for all a, either $P_e \cap P_a$ is finite or $P_e - P_a$ is finite. Thus the property of being minimal is Π^0_4 by Theorem VI.4.4. P_e is thin if and only if, for every a, there exist $\sigma_1, \dots, \sigma_k$ such that $P_e \cap P_a = P_e \cap (I(\sigma_1) \cup \dots \cup I(\sigma_k))$. Thus the property of being thin is Π^0_4 by Theorem VI.4.2.

For the completeness, the proof of Theorem III.8.3 may be modified for a given Π^0_4 set C to define a reduction f so that $P_{f(c)}$ is thin and minimal if $c \in C$ and otherwise is neither. The modification uses a uniform version of Theorem VI.2.13 that $\{e : P_e$ is finite$\}$ is Σ^0_3 complete to add a new limit point to P if $c \notin C$ and otherwise to add only isolated points. That is, let $c \in C$ if and only if $P_{g(c,e)}$ is finite for all e. Now define the computable function f so that, at each stage s of the construction of $T_{f(c)}$, there is a copy of $T^s_{f(c,e)}$ below τ^s_e but not below τ^s_{e+1}. If τ^s_e is abandoned, we just extend the finitely many branches with 0's. Now if $c \in C$, then only finitely many new points have been added below any τ_e, so that no new limit point has been added. Then $P_{f(c)}$ will be a minimal thin class as before. If $c \notin C$, then for some e, we have attached an infinite Π^0_1 class, a copy of $P_{g(c,e)}$ below τ_e. Thus there is a second limit point below τ_e. It follows that $P_{f(c)}$ is not minimal or thin. □

Exercises

VI.4.1. Show that $\{e : \{0,1\}^{\mathbb{N}} \subseteq P_e\}$ is Π^0_1 complete.

VI.4.2. Give the details in the proof of Lemma VI.4.1.

VI.4.3. Show that for any cardinal c, $\{\langle a, b\rangle : card(P_a \cap P_b) \leq c\}$ has the same complexity as $\{a : card(P_a) \leq c\}$ and similarly for cardinality $= c$.

VI.4.4. Show that $\{\langle a, b\rangle : P_a, P_b$ are g-bounded and $P_a \cap P_b \neq \emptyset\}$ is Π^0_1 complete and $\{\langle a, b\rangle : P^2_a, P^2_b$ are g-bounded and $P^2_a \cap P^2_b \neq \emptyset\}$ is Π^0_2 complete.

VI.4.5. Show that $\{\langle e, \langle\sigma\rangle\rangle : I(\sigma) \subseteq P_e\}$ and $\{\langle e, \langle\sigma\rangle\rangle : P_e$ is g-bounded and $I(\sigma) \subseteq P_e\}$ are Π^0_1 complete (for any computable g).

VI.4.6. Show that $\{\langle e, \langle\sigma\rangle\rangle : I(\sigma) \subseteq P^2_e\}$ and $\{\langle e, \langle\sigma\rangle\rangle : P^2_e$ is g-bounded and $I(\sigma) \subseteq P_e\}$ are Π^0_2 complete (for any computable g).

VI.4.7. Show that $\{\langle a, b\rangle : P_a \subseteq P_b\}$ is Π^1_1 complete.

VI.4.8. For any computable function g and any finite k:

(i) $\{\langle a, b\rangle : P_a$ and P_b are g-bounded and $card(P_a - P_b) = 1\}$ is Π^0_2 complete. [(ii)] $\{\langle a, b\rangle : P_a$ and P_b are g-bounded and $card(P_a - P_b) = k + 1\}$ is D^0_2 complete.

VI.4.9. For any computable function g and any finite k:

(i) $\{\langle a,b\rangle : P^2_a$ and P^2_b are g-bounded and $card(P^2_a - P^2_b) = 1\}$ is Π^0_3 complete. [(ii)] $\{\langle a,b\rangle : P^2_a$ and P^2_b are g-bounded and $card(P^2_a - P^2_b) = k+1\}$ is D^0_3 complete.

VI.4.10. For any computable function $g \geq 2$, $\{\langle a,b\rangle : P_a$ and P_b are g-bounded and $P_a - P_b$ is countable $\}$ and $\{\langle a,b\rangle : P^2_a$ and P^2_b are g-bounded and $P^2_a - P^2_b$ is countable$\}$ are Π^1_1 complete.

VI.5 Separating Classes

Recall that, for any two sets A and B, the class $S(A,B)$ contains those *separating* sets C such that $A \subset C$ and $B \cap C = \emptyset$. When A and B are c. e. sets, $S(A,B)$ is a Π^0_1 class of sets. There are twp special cases here. The class of supersets of W_e is $S(W_e,\emptyset)$ and the class of sets disjoint from W_e, is $S(\emptyset, W_e)$. Note that the class $S(A,B)$ of separating sets has the following property, which we shall refer to as being *closed under between-ness*, that, for any sets X,Y,Z, if $X \subset Y \subset Z$ and $X,Z \in P$, then $Y \in P$.

Lemma VI.5.1.

1. *For any nonempty Π^0_1 class $P \subseteq \{0,1\}^{\mathbb{N}}$, the following are equivalent.*
   (a) *P is the class of subsets of a Π^0_1 set A.*
   (b) *P is the class of subsets of some set A.*
   (c) *P is closed under subsets and under union.*
2. *For any nonempty Π^0_1 class $P \subseteq \{0,1\}^{\mathbb{N}}$, the following are equivalent.*
   (a) *P is the class of supersets of a Σ^0_1 set A*
   (b) *P is the class of supersets of some set A*
   (c) *P is closed under supersets and under intersection.*
3. *For any Π^0_1 class $P \subseteq \{0,1\}^{\mathbb{N}}$, the following are equivalent.*
   (a) *P is the class of separating sets of some pair A,B*
   (b) *P is the class of separating sets of some pair A,B of r. e. sets.*
   (c) *P is closed under union, intersection and between-ness.*

Proof. (i) Certainly (a) implies (b) and (b) implies (c). To show that (c) implies (a), suppose that P is closed under subsets and under union and let

$$A = \{n : (\exists x)[x \in A \ \&\ x(n) = 1]\}.$$

We claim that $P = \mathcal{P}(A)$. First we show by induction that $A \cap \{0,1\ldots,n-1\} \in P$ for all n. For $n=0$, this follows from the subset property and the fact that A is nonempty. Now suppose that $A \cap \{0,\ldots,n-1\} \in P$. If $n \notin A$, then

$A \cap \{0, \ldots, n\} = A \cap \{0, \ldots, n-1\} \in P$ by assumption. If $n \in A$, then by definition there is some $B \in P$ with $n \in B$ and then, by closure under union, $A \cup B \in P$ and by closure under subset, $A \cap \{0, \ldots, n\} \in P$. Finally P is a closed set and $lim_{n \to \infty} A \cap \{0, \ldots, n\} = A$, so that $A \in P$ as desired.

Now let $P = \mathcal{P}(A) = [T]$ for some computable tree and recall that $Ext(T)$ is a Π^0_1 set. Then observe that A may be defined by:

$$n \in A \iff \iff (\exists \sigma \in \{0,1\}^n + 1)[\sigma \in Ext(T) \ \& \ \sigma(n) = 1].$$

(ii) This is left as an exercise.

(iii) Observe that $S = S[A, B]$ if and only if S is the intersection of the class of supersets of A with the class of subsets of $\{0,1\}^{\mathbb{N}} - B$. Details are left as an exercise. □

Lemma VI.5.2. *Suppose that $P = [T]$ where T is a tree with no dead ends. Then*

1. *P is closed under subsets if and only if for every $\sigma \subset \tau$, if $\tau \in T$, then $\sigma \in T$.*
2. *P is closed under supersets if and only if for every $\sigma \subset \tau$, if $\sigma \in T$, then $\tau \in T$.*
3. *P is closed under union if and only if, for every σ and τ in T, $\sigma \cup \tau \in T$.*
4. *P is closed under intersection if and only if, for every σ and τ in T, $\sigma \cap \tau \in T$.*

Proof. The proof is left as an exercise. □

Theorem VI.5.3.

1. *$Sub = \{e : P_e = S(\emptyset, W_b)$ for some $b\}$ is Π^0_2 complete.*
2. *$Sup = \{e : P_e = S(W_a, \emptyset)$ for some $a\}$ is Π^0_2 complete.*
3. *$Sep = \{e : P_e = S(W_a, W_b)$ for some $a, b\}$ is a Π^0_2 complete set.*

Proof. (i) By Lemma VI.5.1, $e \in Sub$ if and only if P_e is closed under subsets and under intersection. Thus, by Lemma VI.5.2, $e \in Sub$ if and only if P_e is 2-bounded and

$$(\forall \sigma, \tau \in \{0,1\}^*)[[(\sigma \subset \tau \ \& \ \tau \in Ext(T_e)) \to \sigma \in Ext(T_e)] \ \& \\ [(\sigma \in Ext(T_e) \ \& \ \tau \in Ext(T_e)) \to \sigma \cap \tau \in Ext(T_e)]].$$

For the completeness, let A be a Π^0_2 set and R a recursive relation so that

$$a \in A \iff (\forall m)(\exists n) R(a, m, n).$$

Define the Π^0_1 class $P_{f(a)}$ as follows:

$$x \in P_{f(a)} \iff (\forall m)[x(2m) = x(2m{+}1) = 0 \vee (x(2m) = x(2m{+}1) = 1 \ \& \ (\forall n) \neg R(a, m, n).$$

Now if $a \in A$, then $P_{f(a)} = \{0^\omega\}$ and $f(a) \in Sub$. If $a \notin A$, then choose m such that $(\forall n)\neg R(a,m,n)$. Then $\{2m, 2m+1\} \in P_{f(e)}$, but $\{2m\} \notin P_{f(e)}$, so $f(e) \notin Sub$.

(ii) For any sets B, C,

$$C \in S(\emptyset, B) \iff C \cap B = \emptyset \iff B \subset \omega \setminus C.$$

Thus $e \in Sub \iff f(e) \in Sup$, where $\phi_{f(e)}(n) = 1 - \phi_e(n)$ and similarly $e \in Sup \iff f(e) \in Sub$. The result now follows from (i).

(iii) It follows easily from Lemma VI.5.1 that Sep is a Π^0_2 set and the completeness follows from part (i). □

Theorem VI.5.4. *(i)* $\{e \in Sep : P_e \neq \emptyset\}$ *is* Π^0_2 *complete.*

(ii) $\{e \in Sep : P_e$ *is nonempty but has no computable members*$\}$ *is* Π^0_3 *complete.*

(iii) $\{e \in Sep : P_e$ *has a computable member*$\}$ *is* Σ^0_3 *complete.*

Proof. (i) This set is Π^0_2 by Theorem VI.5.3 and Theorem VI.2.1. The completeness follows by the proof of part (i) of Theorem VI.5.3.

(ii) This set is Π^0_3 by Theorem VI.5.3 and Theorem VI.3.1. For the completeness, we define a reduction of Rec to Sep. This is done by uniformizing the proof from Odifreddi [163] of Shoenfield's theorem that every noncomputable c. e. Turing degree contains a recursively inseparable pair of c. e. sets. That is, define computable functions $f(e)$ and $g(e)$ so that

$n \in W_{f(e)} \iff (\exists s)(n)_1 \in W_{e,s+1} \setminus W_{e,s}$ & $\phi_{(n)_2,s}(n) \simeq 0$, and

$n \in W_{g(e)} \iff (\exists s)(n)_1 \in W_{e,s+1} \setminus W_{e,s}$ & $\phi_{(n)_2,s}(n) \simeq 1$.

Then $W_{f(e)}$ and $W_{g(e)}$ are a disjoint pair of c. e. sets with the following two properties:

(a) $W_{f(e)}$ and $W_{g(e)}$ have the same Turing degree as W_e;

(b) For any separating set D such that $W_{f(e)} \subset D$ and $W_{g(e)} \cap D = \emptyset$, we have W_e computable in D.

It follows from (a) that if W_e is computable, then the pair $W_{f(e)}$ and $W_{g(e)}$ have the computable separating set $W_{f(e)}$. It follows from (b) that if W_e is not computable, then there is no computable separating set for $W_{f(e)}, W_{g(e)}$. Finally, define the computable function h by letting $\phi_{h(e)}(\sigma) = 1$ if and only if

$$(\forall i < |\sigma|)[(i \in W_{f(e),|\sigma|} \to \sigma(i) = 1) \;\&\; (i \in W_{g(e),|\sigma|} \to \sigma(i) = 0)]/.$$

Then we have $P_{h(e)} = S(W_{f(e)}, W_{g(e)})$. It then follows from the discussion above that

$$e \in Rec \iff e \in Sep \;\&\; P_{h(e)} \text{ has a computable member.}$$

(iii) This follows from the proof of (ii), since $P_{h(e)}$ is always a nonempty class of separating sets. □

For many applications of Π^0_1 classes, one demonstrates the difficulty of finding a solution to a certain type of computable problem by constructing c. e. sets W_a and W_b and a corresponding separating class $P_e = S(W_a, W_b)$ such that the set of solutions to the given problem corresponds to the class P_e. Thus we want to consider for a given property $\mathcal{R}$ of classes, such as the property of being finite, $\{e \in Sep : P_e \text{ has property } \mathcal{R}\}$. We note that there is a primitive recursive function ψ such that $S(W_a, W_b) = P_{\psi(a,b)}$ for each a and b and conversely, there is a partial computable function ϕ such that for all $e \in Sep$, $P_e = S(W_{(\phi(e))_0}, W_{(\phi(e))_1})$.

Theorem VI.5.5.

(i) $(\{\langle a,b\rangle : S(W_a, W_b) = \emptyset\}, \{\langle a,b\rangle : S(W_a, W_b) \neq \emptyset\})$ *is* (Σ^0_1, Π^0_1) *complete.*

(ii) For any positive integer c, $(\{\langle a,b\rangle : card(S(W_a, W_b)) > c\}, \{\langle a,b\rangle : card(S(W_a, W_b) \leq c\})$ *is* (Σ^0_2, Π^0_2) *complete,* $\{\langle a,b\rangle : card(S(W_a, W_b)) = c+1\}$ *is* D^0_2 *complete and* $\{\langle a,b\rangle : card(S(W_a, W_b)) = 1\}$ *is* Π^0_2 *complete.*

(iii) $(\{\langle a,b\rangle : S(W_a, W_b)$ *is finite*$\}, \{\langle a,b\rangle : S(W_a, W_b)$ *is infinite*$\})$ *is* (Σ^0_3, Π^0_3) *complete.*

Proof. In each case, an upper bound on the complexity is given by the reduction ψ noted above together with previous results, Theorems VI.2.1, VI.2.8, VI.2.13 and VI.3.1. For the rest of the proof, we set $W_b = \emptyset$.

(i) Observe that $S(W_e, W_b)$ is empty if and only if W_e is empty.

(ii) Observe that $card(S(W_e, W_b)) = 2^c$ if and only if $card(\mathbb{N} \setminus W_e) = c$. Thus only powers of 2 need to be considered. Now $e \in Tot \iff \mathbb{N} \setminus W_e = \emptyset$, which is $\iff Card(S(W_e, W_b)) = 1$ and also $\iff Card(S(W_e, W_b)) \leq 1$, which gives the completeness for cardinality > 1 as well. If we let $W_{\phi(e,c)} = \{n + c : n \in W_e\}$, then $card(\mathbb{N} \setminus W_e) = c + card(\mathbb{N} \setminus W_e)$. Thus $card(S(W_e, W_b) \leq 1 \iff card(W_\phi(e,c), b) \leq 2^c$ and similarly for $SS(> 2^c)$.

Next we show the D^0_2 completeness for cardinality 2^{c+1}. It follows from the reduction above that, for a given Π^0_2 set A, there is a computable function f such that if $a \in A$, then $card(\mathbb{N} \setminus W_a) = c$ and if $a \notin A$, then $card(\mathbb{N} \setminus W_a) > c$. Let B be a Σ^0_2 set. We will obtain a reduction g such that if $e \in B$, then $card(\mathbb{N} \setminus W_e) = 0$ and if $e \notin B$, then $card(\mathbb{N} \setminus W_e) = 1$. Of course it suffices to define such a reduction for the Σ^0_2 complete set Fin, which we do as follows. Given an index e, construct the c. e. set $W_{g(e)}$ in stages $W_{g(e),s}$ along with a number x_s which is intended to be the unique member of $\mathbb{N} \setminus W_e$, if any. We assume as usual that at most one element comes into W_e at any stage s. The construction begins with $W_{g(e),0} = \emptyset$ and $x_0 = 0$. At stage $s+1$, there are two cases.

(Case 1) If no element comes into W_e, or if an element $x < x_s$ comes into W_e, then we let $x_{s+1} = x_s$ and we put $s+1 \in W_{g(e),s+1}$. In this case, $W_{g(e),s+1} = \{0, 1, \ldots, s+1\} \setminus \{x_s\}$.

(Case 2) If an element $x \geq x_s$ comes into W_e, then we put $x_s \in W_{g(e),s+1}$ and let $x_{s+1} = s+1$; in this case $W_{g(e),s+1} = \{0, 1, \ldots, s\}$.

If W_e is finite, then at some stage, we obtain x_s greater than every element of W_e, so that Case 1 applies at every later stage t. Thus $x_t = x_s$ for all $t > s$ and $\omega \setminus W_{g(e)} = \{x_s\}$. If W_e is infinite, then Case 2 applies infinitely often and $W_{g(e)} = \omega$. Finally, we define a reduction of the D^0_2 set $A \cap B$ by letting $W_{h(e)} = W_{f(e)} \oplus W_{g(e)}$.

(iii) Observe that $S(W_a, W_b)$ is finite if and only if W_a is cofinite and apply Theorem VI.0.4. □

Exercises

VI.5.1. Prove Lemma VI.5.2.

VI.5.2. Give the details in the proof of Lemma VI.5.1.

VI.5.3. Show that that there is a primitive recursive function ψ such that $S(W_a, W_b) = P_{\psi(a,b)}$ for each a and b and there is a partial computable function ϕ such that for all $e \in Sep$, $P_e = S(W_{(\phi(e))_0}, W_{(\phi(e))_1})$.

VI.5.4. Show that $\{e \in Sep : card(P_e) = c+1\}$ is D^0_2 complete and similarly for the Sep versions of cardinality 1, finite or infinite from Theorem VI.5.5.

VI.6 Measure and Category

In this section, we consider properties such as being perfect, being meager, and having measure $> r$ or $\geq r$ for some fixed real r.

Recall that a closed set C is *perfect* if every element of C is a limit point of C, that is, if $D(C) = C$. In particular, ω^ω, $\{0, 1, \ldots, k\}^\omega$ (for any k) and $\emptyset$ are all perfect; some authors exclude the empty set. We can use the method of Theorem VI.2.15 to classify index sets of perfect classes.

Theorem VI.6.1. *(i) For any computable function $g \geq 2$, $\{e : P_e$ is g-bounded and perfect$\}$ and $\{e : P_e$ is g-bounded, nonempty and perfect$\}$ are Π^0_3 complete.*

(ii) $\{e : P_e$ is c. b. and perfect$\}$ and $\{e : P_e$ is c. b. , nonempty and perfect$\}$ are D^0_3 complete.

(iii) $\{e : P_e$ is bounded and perfect$\}$ and $\{e : P_e$ is bounded, nonempty and perfect$\}$ are Π^0_4 complete.

(iv) $\{e : P_e$ is perfect$\}$ and $\{e : P_e$ is nonempty and perfect$\}$ are Σ^1_1 complete.

Proof. We first observe that if $P = [T]$ where U is a tree with no dead ends, then P is perfect if and only if $d(T) = T$. Thus P_e is perfect if and only if

$$(*)(\forall\sigma)[\sigma \in Ext(T_e) \rightarrow (\exists\tau)(\exists i, j)(\sigma \prec \tau \;\&\; \tau^\frown i \in Ext(T_e) \;\&\; \tau^\frown j \in Ext(T_e))].$$

We also observe that there is a Π^0_2 relation $B \subset \mathbb{N} \times \mathbb{N}^*$ such that if T_e is finite branching, then $\sigma \in Ext(T_e) \iff B(e,\sigma)$. (This is left as an exercise.)

(i) The upper bounds now follow easily from (*) and Theorem VI.2.1.

For the completeness of both index sets, modify the proof of part (i) of Theorem VI.2.13 by letting $T_{h(e)}$ contain $\{0^n : n \in \omega\}$ together with all strings $0^n 1^\frown \sigma_1 \dots \sigma_k$ where $n \notin W_{e,k}$ and each σ_i is either (010) or (011). It is then easy to see that $T_{h(e)}$ is perfect if and only if $e \notin Cof$.

(ii) It follows from Theorem VI.1.5 and the uniform proof of part (i) that $\{e : P_e$ is c. b. and not perfect$\}$ is Σ^0_3. The upper bounds on the complexity now follow from (*).

For the completeness of each set, let $A = B \cap C$ be a D^0_3 set where B is Σ^0_3 and C is Π^0_3. It follows from the proof of part (iii) of Theorem VI.2.3 that there is a computable function g' such that $P_{g'(e)}$ is always a singleton and is c. b. if and only if $e \in B$.

It follows from our proof of part (i) above that there is a comptable function h' such that $P_{h'(e)}$ is always c. b. and $e \in C$ if and only if $P_{h'(e)}$ is nonempty perfect. For each e, let

$$P_{\phi(e)} = P_{g'(e)} \otimes P_{h'(e)}.$$

Then the desired reduction is given by

$$e \in A \iff P_{\phi(e)} \text{ is c. b. and perfect.}$$

and this also works for nonempty perfect.

(iii) The upper bound on the complexity follows from (*) and Theorem VI.2.4 as above.

For the completeness results, let A be an arbitrary Π^0_4 set and let R be a computable relation such that for all a,

$$a \in A \iff (\forall m)(\exists n)(\forall j)(\exists k) R(a,m,n,j,k).$$

We assume as usual that $R(a,m,n,j,k) \to R(a,m,n+1,j,k)$. The desired reduction of A is defined as follows. First, for each m, n, and a, let $T_{f(m,n,a)}$ consist of all strings $(k_0+1, k_1+1, \dots, k_t+1)$, where for each $j \le t$, k_j is the least k such that $R(a,m,n,j,k)$. Then let $T_{f(a)}$ contain all strings of the form 0^n together with all strings of the form $0^{n_0} * \sigma_0^\frown 0 * \sigma_1^\frown 0 * \dots * \sigma_r$, where for each $m \le r$, $\sigma_m \in U_{f(m,n_m,a)}$ and $n_{m+1} = |\sigma_m|$. Each $T_{f(m,n,a)}$ is finite-branching, so that $T_{f(a)}$ is always finite-branching. $0^\omega \in U_{f(a)}$, so that $P_{f(a)}$ is always nonempty. Elements of $P_{f(a)}$, other than 0^ω, have one of two forms:

(a) $0^{n_0} * \sigma_0^\frown 0 * \sigma_1^\frown 0 * \dots \sigma_t^\frown 0 * x$, where for each $m \le t$, $\sigma_m \in T_{f(m,n_m,a)}$ and $n_{m+1} = |\sigma_m|$ and $x \in P_{f(m+1,n_{m+1},a)}$.

(b) $0^{n_0} * \sigma_0^\frown 0 * \sigma_1^\frown 0 * \dots$, where for each m, $\sigma_m \in T_{f(m,n_m,a)}$ and $n_{m+1} = |\sigma_m|$.

Suppose that $a \in A$. Then for infinitely many n, there exists $x_n \in P_{f(0,n,a)}$ and we have $0^n * x_n \in P_{f(a)}$. Thus 0^ω is not isolated. Similarly any string $\sigma = 0^{n_0} * \sigma_0^\frown 0 * \sigma_1^\frown 0 * \dots * \sigma_r \in Ext(U_{f(a)})$, will have infinitely many extensions $0^{n_0} * \sigma_0^\frown 0 * \sigma_1^\frown 0 * \dots * \sigma_r^\frown 0 * x_n$ in $P_{f(a)}$.

On the other hand, suppose that $a \notin A$ and let M be the least m such that $\neg(\exists n)(\forall j)(\exists k)R(a,m,n,j,k)$. Then there will be an isolated path $0^{n_0} * \sigma_0^\frown 0 * \sigma_1^\frown 0 * \cdots * \sigma_{M-2}^\frown 0 * x$ in $P_{f(a)}$, where $x \in P_{f(M-1,|\sigma_{M-2}|,a)}$.

Thus we have

$$a \in A \iff P_{f(a)} \text{ is bounded, nonempty perfect.}$$

The same reduction applies for bounded, perfect.

(iv) First define the Π^1_1 relation $Isol(x,e)$ which says that x is isolated in P_e by

$$Isol(x,e) \iff x \in P_e \ \& \ (\exists n)(\forall y)[(x\lceil n = y\lceil n \ \& \ x \neq y) \to y \notin P_e].$$

Next recall from Theorem V.2.2 that every isolated point in P_e must be hyperarithmetic. Thus $\{e : P_e$ is perfect is seen to be Σ^1_1 by the Spector-Gandy Theorem II.10.5, since

$$P_e \text{ is perfect} \iff (\forall^{HYP} x)\neg Isol(x,e).$$

It follows from Theorem VI.2.6 that $\{e : P_e \text{ is nonempty perfect}\}$ is also Σ^1_1.

For the completeness in the nonempty perfect case, let f be the reduction given in Theorem VI.2.6 so that, for an arbitrary Σ^1_1 set A, $P_{f(a)}$ is nonempty if and only if $a \in A$, and let $T_{g(a)} = T_{f(a)} \otimes \{0,1\}^{<\omega}$. Then $P_{g(a)}$ is nonempty perfect if $a \in A$ and is empty otherwise. For the other case, let g be as above and let

$$T_{h(a)} = \{0^{n\frown}(\sigma(0)+1, \ldots, \sigma(k-1)+1) : n \in \omega \ \& \ \sigma \in T_{g(a)}\}.$$

Thus

$$P_{h(a)} = \{0^\omega\} \cup \{0^{n\frown}(x(0)+1, x(1)+1, \ldots) : n \in \mathbb{N} \ \& \ x \in P_{g(a)}\}.$$

For $a \in A$, $P_{h(a)}$ is clearly a perfect set, and for $a \notin A$, $P_{h(a)} = \{0^\omega\}$. □

Next we consider the notions of category. We begin with a few definitions. A set $K \subset \mathbb{N}^{\mathbb{N}}$ is said to be *dense in* another set M if $M \subset Cl(K)$. For a closed set K, K is dense in M if and only if $M \subset K$. K is said to be *nowhere dense* in ω^ω if there is no string σ such that K is dense in the interval $I(\sigma)$. Similarly, $K \subset \{0,1\}^\omega$ is nowhere dense if there is no $\sigma \in \{0,1\}^{<\omega}$ such that K is dense in $I(\sigma) \cap \{0,1\}^\omega$. Thus a closed set K is nowhere dense if and only if it includes no interval. Note that a nonempty open set can never be nowhere dense. A set is said to be *meager* or *first category* if it is the countable union of nowhere dense sets. A meager set includes no interval, by the Baire Category Theorem, and thus a closed meager set is itself nowhere dense. Thus a closed set K is meager if and only if it includes no interval. A set is said to be *non-meager* or *second category* if it is not meager. Thus a closed set K is second category if and only if it includes an interval. Note that a nonempty open set always contains an interval and thus is always non-meager. Finally, a set is said to be *comeager* if it is the complement of a meager set. It follows that a closed set K is comeager if and only if $K = \omega^\omega$ (or $\{0,1\}^\omega$).

Theorem VI.6.2.

(i) For all $\sigma \in \omega^{<\omega}$, $\{e : I(\sigma) \subset P_e\}$ is Π^0_1 complete and for all $\sigma \in \{0,1\}^{<\omega}$, $\{e : I(\sigma) \cap \{0,1\}^\omega \subset P_e \cap \{0,1\}^\omega\}$ is Π^0_1 complete.

(ii) $\{e : P_e$ is meager$\}$ and $\{e : P_e$ is meager in $\{0,1\}^{\mathbb{N}}\}$ are both Π^0_2 complete.

Proof. (i) is left as an exercise.

(ii) The upper bound on the complexity follows from the fact that P_e is non-meager if and only if $I(\sigma) \subset P_e$ for some σ.

For the completeness, let A be a Σ^0_2 set and let R be a computable relation so that

$$a \in A \iff (\exists m)(\forall n)R(m, n, a).$$

Then a reduction of A to $\{e : P_e$ is meager$\}$ given by

$T_{f(a)} = \{0^m : m \in \omega\} \cup \{(0^m)^\frown 1^\frown \tau : \tau \in \{0,1\}^{<\omega} \ \&\ (\forall n < |\tau|)R(m, n, a)\}$. □

It follows that, for example, $\{e : P_e \neq \mathbb{N}^{\mathbb{N}}\}$ is Σ^0_1 complete and $\{e : P_e$ is non-meager$\}$ is Σ^0_2 complete. Also note that $\{e : P_e$ is co-meager$\} = \{e : \mathbb{N}^{\mathbb{N}} = P_e\}$ and is Π^0_1 complete.

Next we consider the complexity of index sets associated with measure. Recall that the measure on $\{0,1\}^{\mathbb{N}}$ is defined by setting $\mu(I(\sigma) = 2^{-|\sigma|}$ and the measure on $\mathbb{N}^{\mathbb{N}}$ is defined (with $\lambda(\mathbb{N}^{\mathbb{N}}) = 1$) by setting the measure of $\{x : x(m) = n\}$ to be 2^{-n-1}, so that $I(\sigma)$ has measure $2^{-(m_0+m_1+\cdots+m_{k-1}+k)}$. Recall from section II.II.4 that a real number r is said to be Π^0_1 (respectively, Σ^0_1, etc.) if $\{q \in \mathbb{Q} : q < r\}$ is a Π^0_1 (resp. Σ^0_1, etc.) set. We note that the ordered ring $\mathbb{Q}$ of rationals is a computable structure and can be coded into $\mathbb{N}$ for computability purposes.

Lemma VI.6.3.

(a) For any Π^0_1 class P, $\mu(P)$ is a Π^0_1 real number.

Proof. Let T be a computable tree such that $P = [T]$, let $\tau_0, \tau_1, \ldots$ be a computable enumeration of $\mathbb{N}^* - T$. For each $n \in \mathbb{N}$, let $K_n = \mathbb{N}^{\mathbb{N}} - \bigcup_{i \leq n} I(\tau_i)$. The result now follows from the fact that $\mu(P)$ is the decreasing limit of the computable sequence $\langle \mu(K_m) \rangle_{m\in\omega}$ of dyadic rationals. □

It is an exercise to show that $\mu(P)$ need not be computable.

Theorem VI.6.4. *(i) For any Σ^0_1 real $r \in (0,1]$, $(\{e : \mu(P_e) < r\}, \{e : \mu(P_e) \geq r\})$ is (Σ^0_1, Π^0_1) complete if r is not computable, then $\{e : \mu(P_e) \leq r)$ is Σ^0_1 complete.*

(ii) For any Π^0_1 real $r < 1$, $(\{e : \mu(P_e) > r\}, \{e : \mu(P_e) \leq r\})$ is (Σ^0_2, Π^0_2) complete and $\{e : \mu(P_e) = r)$ is Π^0_2 complete. If r is Π^0_1 complete, then $(\{e : \mu(P_e) < r\}, \{e : \mu(P_e) \geq r\})$ (Σ^0_2, Π^0_2) complete.

(iii) For any Σ_1^0 *real* $r \in (0,1]$, $(\{e : \mu(P_e \cap \{0,1\}^{\mathbb{N}}) < r\}, \{e : \mu(P_e \cap \{0,1\}^{\mathbb{N}}) \geq r\})$ *is* (Σ_1^0, Π_1^0) *complete if* r *is not computable, then* $\{e : \mu(P_e \cap \{0,1\}^{\mathbb{N}}) \leq r)$ *is* Σ_1^0 *complete.*

(iv) For any Π_1^0 *real* $r < 1$, $(\{e : \mu(P_e \cap \{0,1\}^{\mathbb{N}}) > r\}, \{e : \mu(P_e \cap \{0,1\}^{\mathbb{N}}) \leq r\})$ *is* (Σ_2^0, Π_2^0) *complete and* $\{e : \mu(P_e \cap \{0,1\}^{\mathbb{N}}) = r)$ *is* Π_2^0 *complete. If* r *is* Π_1^0 *complete, then* $(\{e : \mu(P_e \cap \{0,1\}^{\mathbb{N}}) < r\}, \{e : \mu(P_e \cap \{0,1\}^{\mathbb{N}}) \geq r\})$ *is* (Σ_2^0, Π_2^0) *complete.*

Proof. (i) Let $\sigma_0, \sigma_1, \ldots$ enumerate $\mathbb{N}^*$ and let

$$P_{e,n} = \mathbb{N}^{\mathbb{N}} - \bigcup \{I(\sigma_i) : i < n \ \&\ \sigma_i \notin T_e\}.$$

Then the function $\mu(P_{e,n})$ is computable and we have for any rational q:

$$\mu(P_e) \geq q \iff (\forall n)\mu(P_{e,n}) \geq q.$$

If r is Σ_1^0 and *not* rational, then

$$\mu(P_e) \geq r \iff \mu(P_e) > r \iff (\forall q \in \mathbb{Q})[q < r \to \mu(P_{e,n}) \geq q.$$

For the completeness, let A be a Π_1^0 set and R a computable relation such that

$$a \in A \iff (\forall n)R(n,a).$$

The necessary reduction f of A is defined so that $P_{f(a)}) = \{0,1\}^{\mathbb{N}}$ when $a \in A$ and $P_{f(a)} = \emptyset$ if $a \notin A$. Just let The reduction f is defined by $T_{f(a)} = \{\sigma : (\forall n < |\sigma|)R(n,a)\}$.

(ii) Let r be a Π_1^0 real. Then we have

$$\mu(P_e) \leq r \iff (\forall q \in \mathbb{Q})(q \leq \mu(P_e) \to q \leq r)$$

and similarly

$$\mu(P_e) \geq r \iff (\forall q \in \mathbb{Q})(q \leq r \to q \leq \mu(P_e)).$$

It follows that $\{e : \mu(P_e) \leq r\}$, $\{e : \mu(P_e) \geq r\}$, and $\{e : \mu(P_e) = r\}$ are all Π_2^0 sets. Next we show the completeness of the latter two sets. Let B be a Π_1^0 set so that $r = \sum_{i \in B} 2^{-i-1}$ and let $P_B = \{0^\omega\} \cup \bigcup_{i \in B} I(0^i 1)$, so that $\mu(P_B) = r$. Since $r \neq 1$, we may assume that B is co-infinite. Let A be a Π_2^0 set and R a computable relation so that

$$a \in A \iff (\forall m)(\exists n)R(m,n,a).$$

Here we assume as usual that if $a \notin A$, then $(\exists n)R(m,n,a)$ for only finitely many m. Now define the reduction g by

$$P_{g(a)} = \{0^\omega\} \cup \bigcup \{I(0^m 1) : i \in B \ \text{ or } \ (\forall n)\neg R(m,n,a)\}.$$

If $a \in A$, then clearly $P_{g(a)} = P_B$ so that $\mu(P_{g(a)}) = r$. If $a \notin A$, then $P_{g(a)}$ includes P_B together with cofinitely many intervals $I((m))$, so that $\mu(P_{g(a)}) > r$.

For the completeness of measure $\geq r$ when r is Π^0_1 complete, let B be a Π^0_1 set such that $\mu(P_B) = r$. Let A be a Π^0_2 set and, by the completeness, let f be a computable functions such that, for any a,

$$a \in A \iff (\forall m) f(a, m) \notin B.$$

Define the uniformly Π^0_1 set $C_a = B \setminus \{f(a, m) : m \in \mathbb{N}\}$, so that for any a, we have $a \in A \iff C(a) = B$ and otherwise, C_a is a proper subset of B. Then define

$$P_{g(a)} = \{0^\omega\} \cup \bigcup \{I((0^n 1)) : n \in C(a)\}.$$

If $a \in A$, then $P_{g(a)} = P_B$, so that $\mu(P_{g(a)}) = r$ and if $a \notin A$, then $P_{g(a)}$ is a subset of $P_B - I(0^n 1)$ for some $n \in B$ and thus $\mu(P_{g(a)}) < r$.

Parts (iii) and (iv) follow immediately. □

Exercises

VI.6.1. Define a Π^0_2 relation $B \subset \mathbb{N} \times \mathbb{N}^*$ such that if T_e is finite branching, then $\sigma \in Ext(T_e) \iff B(e, \sigma)$.

VI.6.2. For any computable function $g \geq 2$, show that $\{e : P_e \text{ is } g\text{-a.b. and perfect}\}$ and $\{e : P_e \text{ is } g\text{-a.b., nonempty and perfect}\}$ are Π^0_3 complete.

VI.6.3. $\{e : P_e \text{ is a.c.b. and perfect}\}$ and $\{e : P_e \text{ is a.c.b., nonempty and perfect}\}$ are D^0_3 complete.

VI.6.4. $\{e : P_e \text{ is a.b. and perfect}\}$ and $\{e : P_e \text{ is a.b., nonempty and perfect}\}$ are D^0_4 complete.

VI.6.5. Prove part (i) of Theorem VI.6.2.

VI.6.6. Define a Π^0_1 class P such that $\mu(P)$ is a Π^0_1 complete real.

VI.7 Derivatives

In this section we consider the uniform (arithmetic) complexity of $D^\alpha(P_e)$ and the complexity of various cardinality properties of $D^\alpha(P_e)$. These problems were first studied in the context of Polish spaces by Kuratowski, see [120], where the Cantor-Bendixson derivative is viewed as a mapping from the space of compact subsets of $\{0,1\}^\omega$ to itself. Kuratowski showed that the derivative is a Borel map of class exactly two. In particular, he showed that the family $D^{-1}(\{\emptyset\})$ of finite closed sets is a universal $\mathbf{\Sigma}^0_2$ class and posed the problem of determining the exact Borel class of the iterated operator D^α. Cenzer and Mauldin showed in [30, 31] and that the iterated operator D^n is of Borel class exactly $2n$ for finite n and that for any limit ordinal λ and any finite n, $D^{\lambda+n}$ is of Borel class exactly $\lambda + 2n + 1$. In particular it is shown that for any α, the family T_α of closed sets K such that $D^\alpha(K) = \emptyset$ is a universal $\mathbf{\Sigma}^0_{2\alpha}$ set. Lempp [127] gives an effective version of this result.

We first observe that the basic results on the cardinality of Π^0_1 classes can be relativized. For any fixed set X, let P^X_e enumerate the binary classes which are Π^0_1 in X. That is, let $P^X_e = [T^X_e]$, where

$$T^X_e = \{\sigma : (\forall \tau \preceq \sigma)(\langle e, \tau\rangle \notin W^X_e\}.$$

Theorem VI.7.1. *For any set X,*

1. $(\{e : P^X_e$ *is empty*$\}, \{e : P^X_e$ *is nonempty*$\})$ *is* $({\Sigma^0_1}^X, {\Pi^0_1}^X)$ *complete,*
2. $\{e : card(P^X_e) = 1\}$ *is* ${\Pi^0_2}^X$ *complete.*
3. *For any positive integer c,* $(\{e : card(P^X_e) > c\}, \{e : card(P^X_e) \leq c\})$ *is* $({\Sigma^0_2}^X, {\Pi^0_2}^X)$ *complete and* $\{e : card(P^X_e) = c+1\}$ *is* ${D^0_2}^X$ *complete.*
4. $(\{e : P^X_e$ *is finite*$\}, \{e : P^X_e$ *is infinite*$\})$ *is* $({\Sigma^0_3}^X, {\Pi^0_3}^X)$ *complete.*

Proof. This follows from the proofs of Theorems VI.2.1, VI.2.8 and VI.2.13. □

The strong Π^0_n classes were defined in Section III.III.3. Here we need a uniform definition of the strong Π^0_β classes for any computable ordinal β. Let

$$T_{e,\alpha} = \{\sigma : (\forall \tau \preceq \sigma)(\langle e, \tau\rangle \notin \mathbf{O}^\alpha\}.$$

and let

$$P_{e,\alpha} = [T_{e,\alpha}].$$

Then a closed set P is said to be Π^0_α if it equals $P_{e,\alpha}$ for some index e. It follows that P is $\Pi^0_{\alpha+1}$ if and only if P is Π^0_1 in $\mathbf{O}^\alpha$. Furthermore, for any ordinal β, P is a strong Π^0_β class if and only if T_P is a Π^0_β set. (See the exercises.)

The following result from [39] is now immediate for successor ordinals.

Theorem VI.7.2. *For any computable ordinal α,*

1. $(\{e : P_{e,\alpha+1}$ *is empty*$\}, \{e : P_{e,\alpha+1}$ *is nonempty*$\})$ *is* $(\Sigma^0_{\alpha+1}, \Pi^0_{\alpha+1})$ *complete,*
2. $\{e : card(P_{e,\alpha+1}) = 1\}$ *is* $\Pi^0_{\alpha+2}$ *complete.*
3. *For any positive integer c,* $(\{e : card(P_{e,\alpha+1}) > c\}, \{e : card(P_{e,\alpha+1}) \leq c\})$ *is* $(\Sigma^0_{\alpha+2}, \Pi^0_{\alpha+2}$ *complete and* $\{e : card(P_{e,\alpha+1}) = c+1\}$ *is* $D^0_{\alpha+2}$ *complete.*
4. $(\{e : P_{e,\alpha+1}$ *is finite*$\}, \{e : P_{e,\alpha+1}$ *is infinite*$\})$ *is* $(\Sigma^0_{\alpha+3}, \Pi^0_{\alpha+3})$ *complete.*

We need a uniform version of Lemma V.V.4.1.

Lemma VI.7.3. *[[39]] For any computable limit ordinal λ and any finite $n > 0$,*

(a) $\{\langle e, \sigma\rangle : \sigma \in d^n(T_e \cap \{0,1\}^*)\}$ *is* Σ^0_{2n}*;*

(b) $\{\langle e,\sigma\rangle : \sigma \in d^{\lambda}(T_e \cap \{0,1\}^*)\}$ *is* $\Pi^0_{\lambda+1}$*;*

(c) $\{\langle e,\sigma\rangle : \sigma \in d^{\lambda+n}(T_e \cap \{0,1\}^*)\}$ *is* $\Sigma^0_{\lambda+2n}$*.*

By the uniform proof of Theorem V.V.4.8, we have the following.

Theorem VI.7.4. *There is a primitive recursive function ϕ such that, for any computable ordinal α, if Q is the $\Pi^0_{2\alpha+1}$ class with index e, then $P_{\phi(e)}$ is the index of a Π^0_1 class P of sets such that there is a homeomorphism H from Q onto $D^\alpha(P)$ with $x \leq_T H(x) \leq x \oplus 0^{2\alpha-1}$ for all $x \in Q$.*

Theorem VI.7.5. *For any recursive ordinal α,*

1. $(\{e : D^\alpha(P_e)$ *is empty*$\}, \{e : D^\alpha(P_e)$ *is nonempty*$\})$ *is* $(\Sigma^0_{2\alpha+1}, \Pi^0_{2\alpha+1})$ *complete.*

2. $\{e : card(D^\alpha(P_e)) = 1)\}$ *is* $\Pi^0_{2\alpha+2}$ *complete.*

3. *For any positive integer c,* $(\{e : card(D^\alpha(P_e)) \leq c), \{e : card(D^\alpha(P_e)) > c\})$ *is* $(\Sigma^0_{2\alpha+2}, \Pi^0_{2\alpha+2})$ *complete and* $\{e : card(D^\alpha(P_e)) = c+1)$ *is* $D^0_{2\alpha+2}$ *complete.*

4. $(\{e : D^\alpha(P_e)$ *is finite*$\}, \{e : D^\alpha(P_e)$ *is infinite*$\})$ *is* $(\Sigma^0_{2\alpha+3}, \Pi^0_{2\alpha+3})$ *complete.*

Proof. The upper bound on the complexity follows from Lemma VI.7.3 and Theorem VI.7.2. That is, for example, fix $\alpha = \lambda + n$, where λ is a limit and $n > 0$. Then $D^\alpha(P_e) = [d^{\lambda+n}(T_e)]$ and it follows from Lemma VI.7.3 that this equals $P_{f(e),\lambda+2n+1}]$ for some computable function f. Since $\lambda+2n+1 = 2\alpha+1$, the complexity follows from Theorem VI.7.2.

The completeness follows from Theorems VI.7.2 and VI.7.5. That is, for example, $P_{e,\lambda+1}$ is finite if and only if $D^\lambda(P_{f(e)})$ is finite, and $\{e : P_{e,\lambda+1}$ is finite$\}$ is $\Sigma^0_{\lambda+3}$ complete, therefore $\{e : D^\lambda(P_{f(e)})$ is finite$\}$ is also $\Sigma^0_{\lambda+3}$ complete. □

Lempp used different methods in [127] to prove parts (i) and (iv). He gave weaker versions of parts (ii) and (iii), showing that
$(\Sigma^0_{2\alpha+1}, \Pi^0_{2\alpha+1}) \leq (I^{(\alpha)}_P(\text{empty}), I^{(\alpha)}_P(=1))$.

We now consider the complexity of the perfect kernel $K(P_e)$. It follows from Theorem VI.2.15 that $\{e : K(P_e) = \emptyset\}$ is Π^1_1 complete. It follows from Theorem VI.7.5 that, for every computable ordinal α, there exists e such that $D^\alpha(P_e)$ is nonempty but $D^{\alpha+1}(P_e) = \emptyset$. This gives us the following.

Theorem VI.7.6. *There is a Π^0_1 class $P \subseteq \{0,1\}^{\mathbb{N}}$ such that*

(i) $rk(P) = \omega_1^{C-K}$.

(ii) $T_{K(P)}$ *is* Σ^1_1 *complete, that is,* $\{\sigma : K(P) \cap I(\sigma) \neq \emptyset\}$ *is* Σ^1_1 *complete.*

Proof. Let

$$P = \bigcup_e \{0^e 1x : x \in P_e\}.$$

Then

$$K(P_e) = \{0^\omega \cup \bigcup_e \{0^e 1x : x \in K(P_e)\}$$

and, for each α,

$$D^\alpha(P) = \{0^\omega \cup \bigcup_e \{0^e 1x : x \in D^\alpha(P_e)\}.$$

We know that $rk(P) \leq \omega_1^{C-K}$ by Theorem V.V.1.4 and it now follows from Theorem VI.7.5 that $rk(P) = \omega_1^{C-K}$. It also follows from TheoremV.V.1.4 that $K(P)$ is a Σ_1^1 class, so that $T_{K(P)}$ is a Σ_1^1 set. For the completeness, observe that $K(P_e) \neq \emptyset$ if and only if P_e is uncountable and that $\{e : P_e \text{ is uncountable}\}$ is Σ_1^1 complete by Theorem VI.2.15. Then we have

$$K(P_e) \neq \emptyset \iff K(P) \cap I(0^e 1) \neq \emptyset,$$

which shows that $K(P)$ is Σ_1^1 complete. □

Exercises

VI.7.1. Give a careful proof of Lemma VI.7.3.

VI.7.2. Show that, for any ordinal β, P is a strong Π_β^0 class if and only if T_P is a Π_β^0 set.

VI.8 Index Sets for Logical Theories

In this section we define index sets for (propositional) logical theories and consider the complexity of properties associated with the consistency and completeness of such theories.

Let *Sent* denote the set of sentences $\{\gamma_0, \gamma_1, \dots\}$ of the propositional language with variables $\{A_0, A_1, \dots\}$, enumerated first by length and then lexicographically. The e'th axiomatizable theory $\Gamma_e \subseteq Sent$ may be defined as the set of consequences of $\{\gamma_i : i \in W_e\}$. The following lemma is left as an exercise.

Lemma VI.8.1. *$\{i : \gamma_i \in \Gamma_e\}$ is a c. e. set and in fact there is a computable function f such that $\{i : \gamma_i \in \Gamma_e\} = W_{f(e)}$.*

As in section III.III.9, a Π_1^0 class $P \subseteq \{0,1\}^{\mathbb{N}}$ represents a class G of subsets of *Sent* if, for any $x \in TN$, $x \in P$ if and only if $\{A_i : x(i) = 1\} \in G$.

The next result now follows easily from the uniformity of the proof of Theorem III.9.1.

Lemma VI.8.2. *There is a primitive recursive function f such that, for all e, $P_{f(e)}$ represents the set of complete consistent extensions of Γ_e. Furthermore, if Γ_e is a decidable theory, then $P_{f(e)}$ is a decidable Π^0_1 class, that is, $\{\sigma : P_{f(e)} \cap I(\sigma) \neq \emptyset\}$ is a computable set.*

Note here that when Γ_e is decidable, we do not necessarily have $T_{f(e)}$ to be a tree without dead ends; there simply *exists* a tree T without dead ends such that $P_{f(e)} = [T]$.

On the other hand, Theorem III.9.3 may be uniformized as follows.

Lemma VI.8.3. *There is a primitive recursive function g such that, for all e, P_e represents the set of complete consistent extensions of $\Gamma_{g(e)}$. Furthermore, if P_e is a decidable Π^0_1 class, then $\Gamma_{g(e)}$ is a decidable theory.*

Note again that when P_e is decidable (which is true whenever T_e has no dead ends), then $\Gamma_{g(e)}$ is a decidable theory but it not necessarily true that $W_{g(e)}$ is a computable set.

We can now apply the index set result sets of this chapter to obtain some complexity results for axiomatizable theories.

Theorem VI.8.4. *1. $\{e : \Gamma_e$ is consistent$\}$ is Π^0_1 complete.*

2. $\{e : \Gamma_e$ is consistent and complete$\}$ is Π^0_2 complete.

3. $\{e : \Gamma_e$ is essentially undecidable$\}$ is Π^0_3 complete.

Proof. (1) Using the function f from Lemma VI.8.2, Γ_e is consistent if and only if $P_{f(e)}$ is nonempty, and this is a Π^0_1 condition by Theorem VI.2.1. For the completeness, P_e is nonempty if and only if $\Gamma_{g(e)}$ is consistent, where g is the function from Lemma VI.8.3. The completeness now follows from Theorem VI.2.1.

(2) Γ_e is consistent and complete if and only if it has a unique complete consistent extension, that is, if and only if $card(P_{f(e)}) = 1$, which is a Π^0_2 condition by Theorem VI.2.8. The completeness follows from Theorem VI.2.8 since $card(P_e) = 1$ if and only if $\Gamma_{g(e)}$ is consistent and complete.

(3) Γ_e is essentially undecidable if and only if it has no computable complete consistent extension, that is, if and only if $P_{f(e)}$ has no computable element, which is a Π^0_3 complete condition by Theorem VI.3.1. The completeness follows from Theorem VI.3.1 since P_e has no computable element if and only if $\Gamma_{g(e)}$ is essentially undecidable. □

We can also classify the index sets of theories with a given number of complete consistent extensions (and similarly for computable complete consistent extensions). The next theorem follows from Theorems VI.2.8, VI.2.13 and VI.2.15 as above. Let us abbreviate "computable consistent extensions" by CCEs.

Theorem VI.8.5. *Let $c > 0$ be finite.*

1. $(\{e : \Gamma_e$ has $> c$ CCEs$\}, \{e : \Gamma_e$ has $\leq c$ CCEs$\}$ is (Σ^0_2, Π^0_2) complete.

2. *$\{e : \Gamma_e$ has exactly c CCEs$\}$ is D^0_2 complete.*

3. *$(\{e : \Gamma_e$ has finitely many CCEs$\}, \{e : \Gamma_e$ has infinitely many CCEs$\}$ is (Σ^0_3, Π^0_3) complete.*

4. *$\{e : \Gamma_e$ has exactly $\aleph_0$ CCEs$\}$ is Π^1_1 complete.*

5. *$(\{e : \Gamma_e$ has uncountably many CCEs$\}, \{e : \Gamma_e$ has countably many CCEs$\}$ is (Σ^1_1, Π^1_1) complete.*

For a given number of computable complete consistent extensions, we apply Theorems VI.3.5 and VI.3.9.

Theorem VI.8.6. *Let $c > 0$ be finite.*

1. *$(\{e : \Gamma_e$ has $> c$ computable CCEs$\}, \{e : \Gamma_e$ has $\leq c$ computable CCEs$\}$ is (Σ^0_3, Π^0_3) complete.*

2. *$\{e : \Gamma_e$ has exactly c computable CCEs$\}$ is D^0_3 complete.*

3. *$(\{e : \Gamma_e$ has $< \aleph_0$ computable CCEs$\}, \{e : \Gamma_e$ has $\geq \aleph_0$ computable CCEs$\})$ is (Σ^0_4, Π^0_4) complete.*

Finally, we consider thin Martin–Pour-El (MPE) theories.

Theorem VI.8.7. *$\{e : \Gamma_e$ is $MPE\}$ is a Π^0_4 complete set.*

Proof. Let f be the function from Lemma VI.8.2 so that $P_{f(e)}$ represents the set of complete consistent extensions of Γ_e. Then Γ_e is MPE if and only if $P_{f(e)}$ is thin, and this is a Π^0_4 condition by Theorem VI.4.6. For the completeness, let g be the function from Lemma VI.8.3 such that P_e represents the class of complete consistent extensions of $\Gamma_{g(e)}$. Then P_e is thin if and only if Γ_e is Martin–Pour-El and the Π^0_4 completeness now follows from Theorem VI.4.6. □

.

Exercises

VI.8.1. Prove Lemma VI.8.1

VI.8.2. Prove Lemma VI.8.3.

VI.8.3. Prove Lemma VI.8.2.

VI.8.4. Let f be the function from Lemma VI.8.1. Show that $\{e : W_{f(e)} = W_e\}$ is Π^0_2 complete. That is, the property of being a logical theory is Π^0_2 complete for sets of sentences.

Chapter VII

Reverse Mathematics

There is a close connection between Π_1^0 classes and certain subsystems of second order arithmetic which are used in the so-called Reverse Mathematics developed by Friedman and Simpson (see [192]). In particular, the system WKL_0 (Weak Konig's Lemma) corresponds roughly to the statement that every infinite tree in $\{0,1\}^*$ has an infinite. The system ACA_0 arithmetic comprehension) corresponds to the statement that every infinite, finitely branching tree has an infinite path. Thus the representation theorems from Part B may be used to show that certain standard infinite combinatorial theorems are logically equivalent, over the base theory RCA_0, to either WKL_0 or to ACA_0.

For example, consider the completeness theorem for (countable) propositional logic. Given a consistent theory Γ, the set of complete consistent extensions of Γ can be viewed as the infinite paths through a certain infinite binary tree and thus Weak Konig's Lemma can be used to prove that a complete consistent extension exists. On the other hand, given an arbitrary infinite tree $T \subset \{0,1\}^*$, we showed that there exists a consistent theory Γ such that T represents the class of complete consistent extensions of Γ. The completeness theorem tells us that a complete consistent extension exists and therefore T possesses an infinite path. This gives an (informal) proof of Weak Konig's Lemma from the completeness theorem and demonstrates that the two are logically equivalent.

We also present in this chapter the reverse mathematics of propositional logic. Later, in Part B, we will we will consider the proof-theoretic strenth of theorems from various areas of algebra, analysis and combinatorics. This will includee the Cantor-Schroder-Bernstein Theorem and related theorems about symmetric marriages in a highly computable society which are equivalent, variously, to Weak Konig's Lemma or to Arithmetic Comprehension. We also examine several results on infinite partially ordered sets, including Dilworth's theorem that any poset of width n can be covered by n chains.

VII.1 Subsystems of Second Order Arithmetic

In this section, we discuss the language of second order arithmetic, models of second order arithmetic and the basic axiom system for second order arithmetic as well as certain subsystems closely related to Π^0_1 classes. These are RCA_0, WKL_0 and ACA_0. For details, see Simpson's [192].

A second order structure includes both objects and sets of objects. Thus a model of second order arithmetic includes a model of first order arithmetic, with a set of objects intended as natural numbers together with the usual operations of addition and subtraction, as well as a collection of *sets* of numbers and the membership relation ($n \in X$) between the objects and the sets.

The language $\mathcal{L}_2$ of second order arithmetic thus includes the usual language of first order arithmetic, with constant symbols 0 and 1, intended to denote the corresponding natural numbers, with binary function symbols $+$ and $\cdot$, intended to denote the addition and multiplication functions on the natural, and with a relation symbol $<$ intended to denote the ordering of the natural numbers, as well as the usual equality symbol $=$ from predicate logic.

There is also a relation symbol $\in$ which denotes the membership relation. There are two sorts of variables intended to range over numbers and over sets. Number variables $i, j, k, m, n, \dots$ are intended to range over the set $N = \{0, 1, \dots\}$ of natural numbers and set variables $X, Y, Z, \dots$ are intended to range over subsets of $\mathbb{N}$.

Terms are defined as in first order arithmetic to compose the smallest set of strings containing the two constant symbols and all number variables and closed under $t = t_1 + t_2$ and $t = t_1 \cdot t_2$. Atomic formulas are $t_1 = t_2$, $t_1 < t_2$ and $t_1 \in X$, where t_1 and t_2 are terms and X is a set variable. Formulas compose the smallest set containing all atomic formulas and closed under the propositional connectives, number quantifiers ($\forall n$) and $\exists n$) and also under *set quantifiers* ($\forall X$) and ($\exists X$).

A model for the language $\mathcal{L}_2$ has the form

$$\mathcal{M} = \langle M, S_M, +_M, \cdot_M, 0_M, 1_M, <_M \rangle,$$

where M is the universe of $\mathcal{M}$, S_M is a set of subsets of M, $+_M$ and $\cdot_M$ are binary operations on M, 0_M and 1_M are distinguished elements of M and $<_M$ is a binary relation on M.

The intended model for $\mathcal{L}_2$ is $\langle \mathbb{N}, \mathcal{P}(\mathbb{N}), +, \cdot, 0, 1, < \rangle$.

An ω-model $\mathcal{M}$ is a model of $\mathcal{L}_2$ with universe $\mathbb{N}$ and with the standard operations $+$ and $\cdot$, constants 0 and 1, and binary relation $<$, but with S_M merely a *subset* of $\mathcal{P}(\mathbb{N})$. In this case, we simply identify $\mathcal{M}$ with the family $S = S_M$. In addition to the intended model, we will be interested in the following.

(1) REC is the ω-model with S the set of *recursive* sets of natural numbers.

(2) $ARITH$ is the ω-model with S the set of *arithmetical* sets of natural numbers.

These definitions can be relativized to REC^B and $ARITH^B$ for any fixed $B \subset \mathbb{N}$.

The axioms of second order arithmetic include the eight axioms of Robinson Arithmetic (see Section IV.IV.4).

For the induction axiom, we can now discuss *sets* rather than *formulas*, so we have

IS $(\forall X)[(0 \in X \ \wedge \ (\forall n)(n \in X \to n+1 \in X)) \to (\forall n) n \in X]$

To ensure that some sets exist, we have a Comprehension Axiom for each formula $\phi(n)$ of $\mathcal{L}_2$:

C $(\exists X)(\forall n)[n \in X \iff \phi(n)]$.

Here we allow number and set parameters in the formula ϕ. These last two axioms imply the induction scheme IP of Peano Arithmetic and in fact a stronger, full second order induction scheme where the formula ϕ in IP may be any formula of $\mathcal{L}_2$. Note also that any ω-model also satisfies full second order induction.

These axioms compose the formal system Z_2 of second order arithmetic. By a subsystem of second order arithmetic, we mean a theory included in Z_2, generally obtained by weakening the axioms of induction and comprehension.

VII.1.1 Recursive Comprehension

The fundamental system RCA_0 consists of the basic axioms of Robinson Arithmetic together with Σ^0_1 induction and Δ^0_1 comprehension. Some definitions are required to define the hierarchy of formulas.

If t is a numerical term not containing n and ϕ is any formula of $\mathcal{L}_2$, then the *bounded quantifiers* $\forall n < t)$ and $(\exists n < t)$ are defined by

$(\forall n < t)\phi \equiv (\forall n)(n < t \to \phi)$ and

$(\exists n < t)\phi \equiv (\exists n)(n < t \ \& \ \phi)$

A formula ϕ of $\mathcal{L}_2$ is said to be a *bounded quantifier* (or Σ^0_0) formula if all of its quantifiers are bounded. ϕ is said to be Σ^0_1 if it is of the form $(\exists m)\psi$ where ψ is a bounded quantifier formula and ϕ is said to be Π^0_1 if it is of the form $(\forall m)\theta$ where θ is a bounded quantifier formula. More generally, ϕ is Σ^0_k if it is of the form $(\exists n_1)(\forall n_2)\dots n_k\theta$ where θ is a bounded quantifier formula, and simlarly for the Π^0_k formulas.

Definition VII.1.1. *A Σ^0_k (respectively Π^0_k) induction scheme has the form*

$$[\phi(0) \ \& \ (\forall n)(\phi(n) \to \phi(n+1))] \implies (\forall n)\phi(n),$$

where ϕ is any Σ^0_k (resp. Π^0_k) formula of $\mathcal{L}_2$. Here ϕ may have other number and set variables.

Definition VII.1.2. *1. A Σ^0_k (respectively Π^0_k) comprehension scheme has the form*

$$(\exists X)(\forall n)(n \in X \iff \phi(n)),$$

where ϕ is any Σ^0_k (resp. Π^0_k) formula of $\mathcal{L}_2$.

2. *A* bounded *Σ^0_k comprehension scheme has the form $(\forall n)(\exists X)(\forall i)(i \in X \iff (i < n \ \& \ \phi(n))$.*

3. *A Δ^0_k comprehension scheme has the form*

$$(\forall n)(\phi(n) \iff \psi(n)) \implies (\exists X)(\forall n)(n \in X \iff \phi(n)),$$

where ϕ is a Σ^0_k formula and ψ is a Π^0_k formula.

As above, ϕ and ψ may have other number and set variables.

The ω-models of RCA_0 may be characterized as follows.
S is an ω-model of RCA_0 if and only if

$S \neq \emptyset$;

$A \in S$ and $B \in S$ imply $A \oplus B \in S$;

$A \in S$ and $B \leq_T A$ imply $B \in S$.

It follows that RCA_0 has a minimum ω-model,

$$REC = \{A \in \mathcal{P}(\mathbb{N}) : A \ \text{ is recursive}\}$$

Simpson [192] outlines the development of ordinary mathematics within RCA_0. In particular the coding function $\langle n_1, \dots n_k\rangle$ are definable in RCA_0 and Gödel numbering of propositional and also first-order logic may be done there. Functions may be defined by primitive recursion and also by minimization.

Here are some other important results from [192].

Lemma VII.1.3. *The following are provable in RCA_0.*

1. *For any infinite set $X \subseteq \mathbb{N}$, there exists a strictly increasing function $\pi : \mathbb{N} \to \mathbb{N}$ such that X is the range of π.*

2. *Let $\phi(n)$ be a Σ^0_1 formula in which X and f do not occur freely. Then either there exists a finite set X such that $(\forall n)(n \in X \iff \phi(n))$, or there exists a one-to-one function $f : \mathbb{N} \to \mathbb{N}$ such that $(\forall n)(\phi(n) \iff (\exists m)(f(m) = n))$.* □

Proof. (1) Let $\pi(0)$ be the least $n \in X$ and for each k, let $\pi(k+1)$ be the least $k > \pi(n)$ such that $k \in X$.

(2) Suppose no finite set X exists as stated. Let θ be Σ^0_0 such that $\phi(n) \iff (\exists j)\theta(j,n)$ and let $Y = \{\langle j,n\rangle : \theta(j,n) \ \& \ (\forall i < j)\neg\theta(i,n)\}$. Then Y is infinite, so by part (1), there is a function π which enumerates Y in increasing order. Let $f(m) = \pi_2(\pi(m))$, where π_2 is the projection of $\langle x,y\rangle$ onto y. □

Theorem VII.1.4. *RCA_0 proves bounded Σ^0_1 comprehension.* □

Corollary VII.1.5. *RCA_0 proves Π^0_1 induction.*

Here is the basic results for trees.

Theorem VII.1.6. *The following is provable in RCA_0. If $T \subseteq \{0,1\}^*$ is a tree with no dead ends, then T has an infinite path.*

Proof. The leftmost path through T may be defined by primitive recursion. □

On the other hand, König's Lemma is not provable in RCA_0, since REC will contain a computable tree with no computable infinite path and therefore no path in REC.

VII.1.2 Weak König's Lemma

In this section we consider the stronger system WKL_0 and its relation to Π^0_1 classes.

Definition VII.1.7. 1. *Weak König's Lemma is the statement that every infinite subtree of $\{0,1\}^*$ has an infinite path.*

2. *WKL_0 is the subsystem of Z_2 consisting of RCA_0 plus Weak König's Lemma.*

It is clear that REC is not a model of WKL_0 so that WKL_0 is a proper extension of RCA_0. The formal system WKL_0 was first introduced by Friedman [71]. ω-models of WKL_0 are sometimes known as *Scott systems* in the literature, referring to [186]. The development of ordinary mathematics in WKL_0 is carried out in great detail by Simpson in [192].

The following equivalent forms of Weak König's Lemma are frequently used in the applications.

Theorem VII.1.8. *The following are equivalent are equivalent over RCA_0*

1. *WKL_0, i.e. every infinite tree $T \subset \{0,1\}^{<\mathbb{N}}$ has an infinite path.*

2. *(Σ^0_1 separation) Let $\phi_i(n)$, $i = 0,1$ be Σ^0_1 formulas in which X is does not occur freely. If $\neg\exists n(\phi_0(n) \wedge \phi_1(n))$, then*

$$\exists X \forall n((\phi_0(n) \to n \in X) \wedge (\phi_1(n) \to n \notin X)).$$

3. *If $f,g : \mathbb{N} \to \mathbb{N}$ are one-to-one with $(\forall m,n) f(m) \neq g(n)$, then there exists a set X such that, for all m, $f(m) \in X \ \wedge \ g(m) \notin X$.*

Proof. It is clear that (2) implies (1).

(1) $\implies$ (2). Assume (1) and let $T \subset \mathbb{N}^{\mathbb{N}}$ and g be given as stated and define $T^* \subset \{0,1\}^*$ as follows. For any $\tau \in T$ with $|\tau| = n$, let $\tau^* = 0^{\tau(0)}10^{\tau(1)} \ldots 0^{\tau(n-1}$. Then define T by Δ^0_1 Comprehension so that $\sigma \in T^*$ if and only if $\sigma \preceq \tau^*$ for some $\tau \in T$ with $|\tau| \leq g(0) + g(1) + \ldots g(|\sigma|)$.

Then T^* is an infinite subtree of $\{0,1\}^*$ and therefore possesses an infinite path f^*. Now define an infinite path $f \in [T]$ by primitive recursion so that $f(0) =$ the least $k < g(0)$ such that $f^*(k) = 0$ and for each n, $f(n+1)$ is the least $k < g(n+1)$ such that $f^*(k + g(n)) = 1$. □

Theorem VII.1.9. *The following are equivalent are equivalent over RCA_0*

1. *WKL_0, i.e. every infinite tree $T \subset \{0,1\}^{<\mathbb{N}}$ has an infinite path.*
2. *(Bounded Konig's Lemma) If $T \subset \mathbb{N}^{\mathbb{N}}$ is an infinite tree and there is a function g such that for all $\tau \in T$ and all $m < |\tau|$, $\tau(m) < g(m)$, then T has an infinite path.*

Proof. (1) $\implies$ (2). Assume (1) and let ϕ_0, ϕ_1 be given as stated and let θ_0, θ_1 be bounded quantifier formulas so that $\phi_i(n) \iff (\exists m)\theta_i(m,n)$. Now define $T \subseteq \{0,1\}^*$ by Δ^0_1 Comprehension so that

$$\sigma \in T \iff (\forall i < 2)(\forall m, n < |\sigma|)[\theta_i(m,n) \implies \sigma(n) \neq i].$$

T is an infinite tree and therefore has an infinite path X by Weak König's Lemma, which will satisfy the conclusion of (3).

(2) $\implies$ (1). Let $T \subseteq \{0,1\}^*$ be an infinite tree. Define the Σ^0_1 formulas ϕ_i so that

$$\phi_i(\sigma) \iff (\exists n)(\exists \tau \in \{0,1\}^n)[\sigma^\frown(i)^\frown\tau \in T \;\&\; (\forall \sigma \in \{0,1\}^n)\neg(\sigma^\frown(1-i)^\frown\sigma \in T)].$$

Then ϕ_0, ϕ_i satsify the hypothesis of (3), so there exists a set X such that for all σ, $\phi_0(\sigma) \to \sigma \in X$ and $\phi_1(\sigma) \to \sigma \notin X$. We can now define an infinite path through T as follows. Let $\sigma_0 = \emptyset$ and for each k, let $\sigma_{k+1} = \sigma_k{}^\frown 0$ if $\sigma_k \in X$ and otherwise $\sigma_{k+1} = \sigma_k{}^\frown 1$. Then $f = \cup_k \sigma_k$ belongs to $[T]$.

(2) $\implies$ (3). Assume (2) and let f and g be given as stated. Let $\phi_0(n) \iff (\exists m)f(m) = n$ and $\phi_1(n) \iff (\exists m)g(m) = n$. The hypothesis of (2) is satisfied by assumption and therefore there exists X as in the conclusion of (3), which will also satisfy the conclusion of (3).

(3) $\implies$ (2). Assume (3) and let ϕ_i be given as stated. Apply Lemma VII.1.3 to obtain two cases. First there may exist finite sets $X_i = \{n : \phi_i(n)\}$. If this holds for $i = 0$, let $X = X_0$ and if this holds for $i = 1$, let $X = \mathbb{N} - X_1$. If neither set exists, then there are one-to-one functions f_i such that $\phi_i(n) \iff (\exists m)(f_i(m) = n)$. It follows from the hypothesis of (3) that $(\forall m, n) f(m) \neq g(n)$. Hence by the conclusion of (2), we obtain a set X such that, for all m, $f(m) \in X$ and $g(m) \notin X$. This set X then satisfies the conclusion of (3). □

Scott [186] characterized the countable ω-models of WKL_0 as those $M \subseteq \mathcal{P}(\mathbb{N})$ such that there exists a complete extension Γ of Peano Arithmetic such that M is the family of subsets of $\mathbb{N}$ which are representable in Γ.

VII.1.3 Arithmetic Comprehension

In this section we consider the system ACA_0 and its relation to Π^0_1 classes. A formula is said to be *arithmetical* if it is Σ^0_k for some k.

Definition VII.1.10. *1.* **Arithmetical Comprehension** *The arithmetical comprehension scheme is* $(\exists X)[n \in X \iff \phi(n)]$ *where* ϕ *is an arithmetical formula of* $\mathcal{L}_2$ *in which* X *does not occur freely.*

2. ACA_0 *is the subsystem of* Z_2 *whose axioms are arithmetical comprehension, full induction and the basic axioms of Robinson arithemtic.*

The ω-models of ACA_0 may be characterized as follows.
S is an ω-model of ACA_0 if and only if

$S \neq \emptyset$;

$A \in S$ and $B \in S$ imply $A \oplus B \in S$;

$A \in S$ and $B \leq_T A$ imply $B \in S$.

$A \in S$ implies $A' \in S$.

It follows that $ARITH$ is the minimum ω-model for ACA_0.

Theorem VII.1.11. *The following are equivalent over* RCA_0.

1. ACA_0.

2. Σ^0_1 *comprehension.*

3. *If* $f : \mathbb{N} \to \mathbb{N}$ *is an injection, then the range of* f *is a set.*

Proof. The implications (1) $Implies$ (2) and (2) $\implies$ (3) are trivial. The implication (2) $\implies$ (3) follows easily from Lemma VII.1.3. For the implication (1) $\implies$ (2) we prove by induction that Σ^0_k comprehension implies Σ^0_{k+1} comprehension. Let $\phi(n) \iff (\exists j)\psi(n,j)$ where ψ is Π^0_k. By Σ^0_k comprehension, let $Y = \{(n,j) : \neg\psi(n,j)\}$. Then by Σ^0_1 comprehension let $X = \{n : (\exists j)(n,j) \notin Y\}$. □

Theorem VII.1.12. *The following are equivalent over* RCA_0.

1. ACA_0.

2. *(König's Lemma) If* T *is an infinite, finitely branching tree, then there is an infinite path through* T.

3. *König's Lemma restricted to trees* T *such that each* $\sigma \in T$ *has at most two immediate successors in* T.

Proof. (1) $\implies$ (2). Let T be an infinite, finite-branching tree. By arithmetic comprehension, there is a subtree T^* of T consisting of all $\sigma \in T$ such that σ has infinitely many extensions in T. Since T is finite branching, every $\sigma \in T^*$ has at least one immediate successor in T^*. Clearly $\emptyset \in T^*$ and for each n, we may define $g(k)$ to be the least n such that $(g(0), \dots, g(k-1), n) \in T^*$. Then $g \in [T]$ as desired.

(2) $\implies$ (3) is immediate, so it remains to prove (3) $\implies$ (1). Let $f : \mathbb{N} \to \mathbb{N}$ be one-to-one. Use Σ^0_0 comprehension to define the tree T by $\tau \in T$ if and only if

$$(\forall m, n < |\tau|)[f(m) = n \iff \tau(n) = m + 1]$$

and

$$(\forall n < |t|)[\tau(n) > 0 \implies f(\tau(n) - 1) = n].$$

Each $\sigma \in T$ has at most two possible immediate successors, $\sigma^\frown 0$ and $\sigma^\frown (m+1)$ where $f(m) = |\sigma|$. T is infinite by the following argument. Fix k and define Y by bounded Σ^0_1 comprehension to be $\{n < k : (\exists m) f(m) = n\}$. Now let $\sigma(n) = 0$ if $n \notin Y$ and $\sigma(n) = m + 1$ if $n \in Y \;\wedge\; f(m) = n$ for $n < k$. Then $|\sigma| = k$ and $\sigma \in T$. Hence by (3), there exists $g \in [T]$. By Δ^0_1 comprehension let $X = \{n : g(n) > 0\}$. Then X is the range of f as desired. □

VII.2 Mathematical Logic

In this section, we consider the connection between logical theories, infinite trees and subsystems of second order arithmetic.

A weak form of the completeness theorem can be proved even for first-order logic. Here is the propositional version.

Theorem VII.2.1. *[[192]] The following is provable in RCA_0. If Gamma is a consistent propositional theory, then there exists a countable model M for X such that $M \vdash \phi$ for all $\phi \in \Gamma$.*

Proof. Recall from Theorem III.III.9.3 that, for each finite sequence $\sigma = (\sigma(0), \dots, \sigma(n-1))$, we defined $P_\sigma = C_0 \wedge C_1 \wedge \cdots \wedge C_{n-1}$, where $C_i = A_i$ if $\sigma(i) = 1$ and $C_i = \neg A_i$ if $\sigma(i) = 0$. Given the theory Γ, define the tree T without dead ends by

$$\sigma \in T \iff \neg P_\sigma \notin \Gamma\}.$$

Since Γ is a theory, $\sigma \in T$ if and only if P_σ is consistent with Γ. T has no dead ends since Γ is consistent. That is, if $|\sigma| = n$ and P_σ is consistent with Γ, then either $P_\sigma \;\wedge\; A_i$ is consistent with Γ or $P_\sigma \;\wedge\; \neg A_i$ is consistent with Γ. It follows from Theorem VII.1.6 that T has an infinite path X and the model M is defined by letting $M \vdash A_i$ if $X(i) = 1$ and $M \vdash \neg A_i$ if $X(i) = 0$. □

Using Weak König's Lemma, we can prove the completeness theorem and the reverse is also true. (See [192].) We note first that the representation theorem III.III.9.1 for propositional logic can be proved in RCA_0, that is, for any countable set Γ of sentences, there exists a tree $T \subseteq \{0,1\}^*$ such that $[T]$ represents the set of complete consistent extensions of Γ. Likewise the reverse representation theorem III.III.9.3 can be proved in RCA_0. That is, given the tree T, define the set $\Gamma(T)$ to consist of all $P_\sigma \to A_n$ such that $\sigma \in T$ and $\sigma^\frown 0 \notin T$ and all $P_\sigma \to \neg A_n$ such that $\sigma \in T$ and $\sigma^\frown 1 \notin T$, where $|\sigma| = n$. Then there is a one-to-one correspondce between the complete consistent extensions of $\Gamma(T)$ and the infinite paths through T.

Theorem VII.2.2. *The following are equivalent are equivalent over RCA_0*

1. *WKL_0, i.e. every infinite tree $T \subset \{0,1\}^{<\mathbb{N}}$ has an infinite path.*
2. *Lindenbaum's Lemma: every countable consistent set of sentences has a complete consistent extension.*
3. *The completeness theorem for propositional logic with countably many variables.*
4. *The compactness theorem for propositional logic with countable many variables.*

Proof. (1) $\implies$ (2) follows from the representation theorem as discussed above. That is, given a consistent set Γ, we can build in RCA_0 an infinite tree representing the set of complete consistent extensions of Γ and then use Weak König's Lemma to find an infinite path X through T and hence a complete consistent extension of Γ.

(2) $\implies$ (3). For propositional logic, this is immediate. Just let Δ be a complete consistent extension of Γ and let $M(A_i) = 1$ if and only if $A_i \in \Delta$.

(3) $\implies$ (4). Suppose that every finite subset of Γ is satisfiable. Then Γ is consistent and hence has a model M by (3) and is therefore satisfiable.

(4) $\implies$ (1). Assume (4) and let $T \subseteq \{0,1\}^*$ be an infinite tree. Let $\Gamma(T)$ be constructed as above. Then $\Gamma(T)$ is finitely satisfiable and hence has a model (and therefore a complete consistent extension) by (4). But this implies that T has an infinite path. □

Exercises

VII.2.1. Show that a set of natural numbers is c. e. if and only if it is definable by a Σ^0_1 formula over the standard model of arithmetic and therefore is computable if and only if it is Δ^0_1 definable.

VII.2.2. Show that $S \subseteq \mathcal{P}(\mathbb{N})$ is a model of RCA_0 if and only if

(i) $S \neq \emptyset$;

$A \in S$ and $B \in S$ imply $A \oplus B \in S$;

$A \in S$ and $B \leq_T A$ imply $B \in S$.

Chapter VIII

Complexity Theory

In this chapter, we examine the notions of computable trees and effectively closed sets in a *resource-bounded* setting. We consider the complexity of the members of effectively closed sets as in Chapter IV from this point of view. We show that for any Π_1^0 class $P \subset \mathbb{N}^{\mathbb{N}}$, there is a polynomial time tree T such that $P = [T]$. Resource-bounded variations on the notions of boundedness for trees and classes are defined, such as *locally p-time*, highly p-time, and p-time bounded are defined and basis and antibasis results given. For example, every locally p-time tree possesses an infinite path which is computable in double exponential time and also, there is a p-time bounded Π_1^0 class with a unique element which is not p-time computable.

We also look at the representation problem and show that for essentially all of the problems from Part 2, polynomial time presented problems suffice to represent all Π_1^0 classes. This is based on the result that any computable relational structure is computably isomorphic to a polynomial time structure.

Let Σ be a (usually finite) alphabet. Then Σ^* denotes the set of finite strings of letters from Σ and Σ^ω denotes the set of infinite sequences. In particular, each natural number n may be represented in unary form by the string $tal(n) = 1^n$ if $n > 0$ and $tal(n) = 0$ if $n = 0$ and in (reverse) binary form by the string $bin(n) = i_0 \cdots i_k$, where $n = i_0 + i_1 \cdot 2 + \cdots + i_k \cdot 2^k$.

We let $Tal(\omega) = \{tal(n) : n \in \omega\}$ and $Bin(\omega) = \{bin(n) : n \in \omega\}$. Both sets are included in $\{0,1\}^*$. The tally and binary representation of the natural numbers will be essential for our study of feasible structures, problems and solutions. The main reason is due to the fact the feasibility of an algorithm is usually measured in terms of the computation time as a function of the length of the input to the algorithm. Note that since the tally representation of a number is of exponential length in comparison to the binary representation, it follows that a function which is polynomial time computable in the tally representation of the natural numbers is not necessarily polynomial time computable in the binary representation of the natural numbers. Indeed, we can only conclude that such a function is exponential time computable in the binary representation.

Thus it is essential that a definite representation be given for a feasible structure. A related reason is that two feasible sets need not be feasibly isomorphic. In particular, $Tal(\omega)$ and $Bin(\omega)$ are not p-time isomorphic. Thus we may have a p-time structure, say a graph, with universe $Tal(\omega)$, which is not isomorphic to a p-time structure with universe $Bin(\omega)$.

Our basic computation model is the standard multitape Turing machine of Hopcroft and Ullman [89]; see also Papadimitriou [167]. Note that there are different heads on each tape and that the heads are allowed to move independently. This implies that a string σ can be copied in linear time. An oracle machine is a multitape Turing machine M with a distinguished work tape, a query tape, and three distinguished states QUERY, YES, and NO. At some step of a computation on an input string σ, M may transfer into the state QUERY. In state QUERY, M transfers into the state YES if the string currently appearing on the query tape is in an oracle set A. Otherwise, M transfers into the state NO. In either case, the query tape is instantly erased. The set of strings accepted by M relative to the oracle set A is $L(M, A) = \{\sigma |$ there is an accepting computation of M on input σ when the oracle set is $A\}$. If $A = \emptyset$, we write $L(M)$ instead of $L(M, \emptyset)$.

Let $t(n)$ be a function on natural numbers. A Turing machine M is said to be $t(n)$ *time bounded* if each computation of M on inputs of length n where $n \geq 2$ requires at most $t(n)$ steps. A function $f(x)$ on strings is said to be in $DTIME(t)$ if there is a $t(n)$-time bounded deterministic Turing machine M which computes $f(x)$. For a function f of several variables, we let the length of $(x_1, \ldots, x_n)$ be $|x_1| + \cdots + |x_n|$. A set of strings or a relation on strings is in $DTIME(t)$ if its characteristic function is in $DTIME(t)$. A Turing machine M is said to be $t(n)$ *space bounded* if each computation of M on inputs of length n where $n \geq 2$ the work space required to carry out the computation is bounded by $t(n)$. A function $f(x)$ on strings is said to be in $DSPACE(t)$ if there is a $t(n)$-space bounded deterministic Turing machine M which computes $f(x)$. For a function f of several variables, we let the length of $(x_1, \ldots, x_n)$ be $|x_1| + \cdots + |x_n|$. A set of strings or a relation on strings is in $DSPACE(t)$ if its characteristic function is in $DSPACE(t)$.

We let

$LOGTIME = \bigcup_{c \geq 1} DTIME(c \cdot log_2(n))$,

$LOG = \bigcup_{c \geq 1} DSPACE(c \cdot log_2(n))$,

$LIN = \bigcup_{c > 0} DTIME(cn)$,

$P = \bigcup_{i \in \omega} DTIME(n^i)$,

$PSPACE = \bigcup_{i \in \omega} DSPACE(n^i)$,

$DEXT = \bigcup_{c \geq 0} DTIME(2^{c \cdot n})$,

$EXPSPACE = \bigcup_{c \geq 0} DSPACE(2^{c \cdot n})$,

$DOUBEXT = \bigcup_{c \geq 0} DTIME(2^{2^{c \cdot n}})$,

$DOUBEXPSPACE = \bigcup_{c \geq 0} DSPACE(2^{2^{c \cdot n}})$,

$EXPTIME = \bigcup_{c \geq 0} DTIME(2^{n^c})$,

$DOUBEXPTIME = \bigcup_{c \geq 0} DTIME(2^{2^{n^c}})$, and in general,

$DEX(S) = \bigcup_{t(n) \in S} DTIME(2^{t(n)})\}$.

We say that a function $f(x)$ is *polynomial time* if $f(x) \in P$, is *exponential time* if $f(x) \in DEXT$, and is *double exponential time* if $f(x) \in DDOUBEXT$.

A function $f(x)$ on strings is said to be in $NTIME(t)$ if there is a $t(n)$-time bounded nondeterministic Turing machine M which computes $f(x)$. A set of strings or a relation on strings is in $NTIME(t)$ if its characteristic function is in $NTIME(t)$. We let

$NLOG = \bigcup_{c \geq 1} NSPACE(C \cdot log_2(n))$,

$NP = \bigcup_{i \in \omega} NTIME(n^i)$,

$NEXT = \bigcup_{c \geq 0} \{NTIME(2^{c \cdot n})\}$,

$NEXPTIME = \bigcup_{c \geq 0} NTIME(2^{n^c})$,

A function f is said to be *non-deterministic polynomial time (NP)* if there is a finite alphabet Σ, a polynomial p, a p-time relation R and a p-time function g such that, for any σ and τ,

$$f(\sigma) = \tau \iff (\exists \rho \in \Sigma^{p(\sigma)})[R(\rho, \sigma) \ \& \ g(\rho, \sigma) = \tau].$$

Similar definitions apply to other complexity classes.

We fix enumerations $\{P_i\}_{i \in N}$ and $\{N_i\}_{i \in N}$ of the polynomial time bounded deterministic oracle Turing machines and the polynomial time bounded nondeterministic oracle Turing machines respectively. We may assume that $p_i(n) = max(2, n)^i$ is a strict upper bound on the length of any computation by P_i or N_i with any oracle X on inputs of length n. P_i^X and N_i^X denote the oracle Turing machine using oracle X.

For $A,\ B \subset \Sigma^*$, we shall write $A \leq_m^P B$ if there is a polynomial-time function f such that for all $x \in \Sigma^*$, $x \in A$ iff $f(x) \in B$. We shall write $A \leq_T^P B$ if A is polynomial time Turing reducible to B. For r equal to m or T, we write $A \equiv_r^P B$ if $A \leq_r^P B$ and $B \leq_r^P A$ and we write $A\ |_r^P B$ if not $A \leq_r^P B$ and not $B \leq_r^P A$.

VIII.1 Complexity of Trees

We think of a computable tree T as a set of finite sequences $(n_0, \dots, n_{k-1})$ of natural numbers and of an infinite path $(n_0, n_1, \dots)$ through T as a function from the natural numbers into the natural numbers which maps i to n_i. We shall define two natural representations of T which will be useful for the study of the complexity of trees and paths through trees. First we define the *binary representation* of T, $bin(T)$, as the set of finite strings $\{(bin(n_0), \dots, bin(n_{k-1})) : (n_0, \dots, n_{k-1}) \in T\}$. We also define the *tally representation* of T, $tal(T)$, to be the set of strings $\{(tal(n_0), \dots, tal(n_{k-1}) : (n_0, \dots, n_{k-1}) \in T\}$. The strings in $bin(T)$ and $tal(T)$ are over the finite alphabet $\{0, 1, ',', '(', ')'\}$ which has symbols for the comma and the left and right parentheses. We say that T is *p-time in binary* if $bin(T)$ is a polynomial time subset of Σ^*. Similarly we say T is *p-time in tally* if $tal(T)$ is p-time subset of Σ^*. Since $bin(n)$ can be computed in polynomial time from $tal(n)$, it follows that if $bin(T)$ is p-time, then $tal(T)$ is also p-time. Given an infinite path $x = (n_0, n_1, \dots)$ through T, the binary representation of x is the function $bin(x)$ from $Tal(\omega)$ to $Bin(\omega)$ defined by $bin(x)(tal(i)) = bin(n_i)$. The tally representation of x, $tal(x)$, is similarly defined by $tal(x)(tal(i)) = tal(n_i)$. Then we say that x is a *polynomial time path in binary* if the function $bin(x)$ is the restriction of p-time function from $\{0,1\}^*$ to $\{0,1\}^*$; we say that x is *p-time in tally* if $tal(x)$ is the restriction of a p-time function from $\{0,1\}^*$ to $\{0,1\}^*$. It is clear that if x is p-time in tally, then x is also p-time in binary, since $bin(x)(tal(i))$ can be computed from $tal(x)(tal(i))$ for each i. The reason for using $Tal(\omega)$ for the domain of $bin(x)$ is the following. For any path x, x is computable if and only if the initial segment function $\overline{x}$ is computable, where $\overline{x}(i) = (n_0, \dots, n_{i-1})$. We want to have a similar result for p-time paths, and this would be impossible if $\overline{x}$ had to map $bin(i)$, which has length roughly $log_2(i)$, to a string which must have length at least i. Similar definitions can be given for other notions of complexity, such as exponential time, non-deterministic polynomial time (NP), etc.

Recall that a tree $T \subset \omega^{<\omega}$ is highly computable if there is a recursive function f such that, for any node $\sigma \in T$, $f(\sigma)$ is the number of immediate successors $\sigma^\frown i$ of σ in T. Now given the number of successors of a node, we can search through all the possible immediate successors and find the largest one. Thus we can find a computable function g such that $g(\sigma)$ is the largest i such that if $\sigma = (\sigma_0, \dots, \sigma_n) \in T$, then $(\sigma_0, \dots, \sigma_n, i)$ is in T. Finally, we can also compute recursively the sequence $h(\sigma) = (i_1, \dots, i_d)$ which lists all i such that $(\sigma_0, \dots, \sigma_n, i)$ is in T in increasing order. It is clear that f is computable if and only if g is computable and if and only if h is computable. The situation is different for polynomial time complexity. Consider first the binary representation of T so that we identify a node $\sigma \in T$ with a sequence of numbers in $Bin(\omega)$. It is not hard to see that if h is p-time, then both f and g are p-time. However, these are the only relations which are guaranteed to hold between the three functions. To see this, consider the following three examples.

Example VIII.1.1. *Define the sequence $x_0, x_1, \dots$ of natural numbers by let-*

ting $x_0 = 1$ *and, for each* n, $x_{n+1} = 2^{x_n}$ *and let* $T = \{(x_0, \ldots, x_{i-1}) : i \in \omega\}$. *Then the tree* T *is p-time and* f *is p-time, since* $f(\sigma) = 1$ *for all* $\sigma \in T$. *However, the function* g *cannot be p-time since, for if* $\sigma = (x_0, \ldots, x_n)$, *then in the binary representation* $|\sigma| \leq 3|x_n|$ *whereas* $|x_{n+1}| = 2^{|x_n|}$.

Example VIII.1.2. *Define the tree* T_1 *computably by putting* $\emptyset \in T_1$ *and, for any* $\sigma = (x_0, \ldots, x_{n-1}) \in T_1$, *putting* $\sigma^\frown i \in T_1$ *if and only if* $i \leq 1 + x_0 + \cdots + x_{n-1}$. T_1 *is clearly p- time and the function* g *is also p-time since* $g((x_0, \ldots, x_{n-1})) = 1 + x_0 + \cdots + x_{n-1}$. *However, the function* h *which lists the immediate successors of any node is not p-time because, for any* n, *if* $\sigma = (1, 2, 4, \ldots, 2^n)$, *then* $h(\sigma) = (0, 1, \ldots, 2^{n+1})$, *so that in the binary representation* $|h(\sigma)| > 2^{n+1}$, *whereas* $|\sigma| = (n+2)(n+3)/2$.

Example VIII.1.3. *For this example, we will appeal to the intractability of the well-known* $P = NP$ *conjecture. That is, we will define a p-time tree* T_2 *for which the function* g *is p-time and such that if the associated function* f *were p-time, then the* $P = NP$ *conjecture would be true. The tree* T_2 *will be defined so that* $\sigma = (n_0, n_1, \ldots, n_{2k+1}) \in T_2$ *if and only if, for each* $i \leq k$, $bin(n_{2i})$ *codes a graph on* i *vertices and* $bin(n_{2i+1})$ *either codes a Hamiltonian path on the graph coded by* $bin(n_{2i})$ *or is a string of 1's of the appropriate length. Now a graph* G_i *on* i *vertices* $v_1, \ldots, v_i$ *is determined by a set of unordered pairs* (v_r, v_s) *of vertices (the edges of the graph). There are* $\binom{i}{2} = i(i-1)/2$ *possible edges in* G_i *and these may be lexicographically ordered so that a sequence* $e_1, e_2, \ldots, e_{\binom{i}{2}}$ *codes the graph* G_i *where, for all* $t \leq \binom{i}{2}$, $e_t = 1$ *if* G_i *has the* t*'th edge and* $e_t = 0$ *otherwise. Of course the (reverse) binary representation* $bin(n_i)$ *must end with a* 1, *so the graph* G_i *will actually be coded by the string* $(e_0, \ldots, e_{\binom{i}{2}}, 1)$. *Observe that this code for* G_i *will always be a string of length* $1 + \binom{i}{2}$ *and that any binary number* $bin(n)$ *of length* $1 + \binom{i}{2}$ *will code a graph on* i *vertices. Now a Hamiltonian path on* G_i *is a permutation* $(v_{r_0}, v_{r_1}, \ldots, v_{r_i})$ *of the vertices such that there is an edge joining* v_{r_t} *with* $v_{r_{t+1}}$ *for all* $t < i$. *Such a path will be coded by the binary sequence* $0^{r_0}10^{r_1}1 \ldots 0^{r_i}1$, *which will always be a binary number of length* $\binom{i+1}{2}$ *and binary number* $bin(n)$ *of length* $\binom{i+1}{2}$ *will code a possible Hamiltonian path on a graph of* i *vertices if and only if* $bin(n)$ *has exactly* i *1's. It is easy to see that there is a p-time algorithm which will decide, given two binary numbers* $bin(n)$ *and* $bin(m)$, *whether* $bin(n)$ *has length* $\binom{i}{2} + 1$ *for some* $i < |bin(n)|$ *and therefore codes a graph* G *on* i *vertices and whether* $bin(m)$ *codes a Hamiltonian path on that graph. The tree* T_2 *can now be defined by putting* $\sigma = (bin(n_0), \ldots, bin(n_{2k+1}) \in T_2$ *if and only if, for each* $i < k$, $bin(n_{2i})$ *codes a graph* G_i *on* i *vertices and* $bin(n_{2i+1})$ *either codes a Hamiltonian path on* G_i *or equals* $tal(\binom{i+1}{2})$). *It follows from the discussion above that* T_2 *is a p-time tree. Now the function* g *for this tree is p-time since, for any* $\sigma = (\sigma_0, \ldots, \sigma_t) \in T_2$, *we have* $g(\sigma) = 2^{\binom{i+1}{2}} - 1$ *if* $t = 2i$ *and* $g(\sigma) = 2^{1+\binom{i}{2}} - 1$ *if* $t = 2i - 1$. *(In each case,* $g(\sigma)$ *is just a string of* 1*s of the right length.) On the other hand, the function* f *associated with the tree* T_2 *has the property that for any* $\sigma = (\sigma_0, \ldots, \sigma_{2i}) \in T_2$, $f(\sigma) = 1$ *if and only if the graph* G_i *coded by* σ_{2i} *has no Hamiltonian path. Now suppose that* f *were p-time and let* $bin(n)$ *be a code* *for a finite graph on k vertices. For all $i < k$, let $n_{2i} = 2^{\binom{i+1}{2}} - 1$ and let $n_{2i+1} = 2^{1+\binom{i+1}{2}} - 1$. Finally, let $\sigma = (bin(n_0), \ldots, bin(n_{2k-1}), bin(n))$. Then the sequence σ can be computed from $bin(n_0)$ in polynomial time and G has a Hamiltonian path if and only if $f(\sigma) > 1$. It follows that if f were p-time, then the Hamiltonian path problem would be p-time. But it is well-known that the Hamiltonian path problem is NP-complete. (See Garey and Johnson [6] for an explanation of NP-completeness and the $P = NP$ problem.) Thus we have demonstrated that if the function f associated with the tree T_2 were p-time, then $P = NP$ would true.*

Now the situation is slightly different for tally representation of T where we identify a $\sigma \in T$ with a sequence of numbers in $Tal(\omega)$. Once again it is easy to see that if h is p-time, then f and g are p-time. Moreover, example (1) above will still show that f may be p-time without g being p-time. However in this case, if T is p-time in tally and g is p-time, then h is also p-time. To see this, suppose $\sigma = (\sigma_0, \ldots, \sigma_n)$. Note that to find $h(\sigma)$, we need only check whether $(\sigma_0, \ldots, \sigma_n, i) \in T$ for $i \leq g(\sigma)$. Now in the tally representation, $|(\sigma_0, \ldots, \sigma_n, 0)| < |(\sigma_0, \ldots, \sigma_n, 1)| < \ldots < |(\sigma_0, \ldots, \sigma_n, g(\sigma))|$. Then if it takes $q(|\sigma|)$ steps to check whether $\sigma \in tal(T)$ for each $\sigma = (tal(\sigma_0), \ldots, tal(\sigma_n))$, than it will take approximately

$$\sum_{i=0}^{|(\sigma_0,\ldots,\sigma_n,g(\sigma))|} q(i) \leq q(|(\sigma_0, \ldots, \sigma_n, g(\sigma))|)^2$$

steps to check whether $(\sigma_0, \ldots, \sigma_n, i) \in T$ for $i \leq g(\sigma)$. Thus it is easy to see that we can find $h(\sigma)$ in polynomial time in the tally representation of σ.

We say that a tree T is *locally p-time* in binary (respectively in tally) if all three of the functions defined above are p-time in binary (resp. tally). In the case that T is not itself p-time, then we will say that T is locally p-time if each of the functions is the restriction to T of a function which is p-time (in binary or tally).

Next we will show that if T is locally p-time, then T is also p-time. The same argument works for both binary and tally. Let Q be either $bin(T)$ or $tal(T)$ and suppose that the function h associated with Q is p-time. Given a sequence $\sigma = (\sigma_0, \ldots, \sigma_k)$, here is the procedure for testing whether $\sigma \in tal(T)$. Begin by computing $h(\emptyset) = (tau_1, \ldots, tau_d)$ and checking whether $\sigma_0 = \tau_i$ for some $i \leq d$. Then, for $j < k$ in turn, compute $h(\sigma_0, \ldots, \sigma_j)$ and check to see that σ_{j+1} is in this list. Suppose that $h(\tau)$ may be calculated in time $p(|\tau|)$, where p is some polynomial, then since each $(\sigma_0, \ldots, \sigma_j)$ is a substring of σ, we see that we can do each of the h computations in time no greater than $p(|\sigma|)$. To read the resulting list of possible successors of $(\sigma_0, \ldots, \sigma_j)$ and compare each one with σ_{j+1} can then be done in time at most $(c-1)p(|\sigma|)$ for some fixed constant c. Thus each step of the procedure takes time at most $cp(|\sigma|)$. Now there are k such steps and $k \leq |\sigma|$, so that the entire procedure takes time at most $c|\sigma|p(|\sigma|)$, which is again a polynomial function of $|\sigma|$.

The functions f, g and h describe the behavior of the tree at a particular node. Now sometimes we need to have a global bound as well. Note that for a computably bounded tree, there is a computable function p such that, for all natural numbers k and all $\sigma = (n_0, \ldots, n_k) \in T$, $n_k \leq p(k)$. We will say that a tree T is *p-time bounded in binary* if there is a p-time function p such that, for all natural numbers k and all $\sigma = (n_0, \ldots, n_k) \in T$, $|bin(n_k)| \leq p(1^k)$. A tree T is *p-time bounded in tally* if there is a p-time function p such that for any $\sigma = (n_0, \ldots, n_k) \in T$, we always have $n_k = |tal(n_k)| \leq p(1^k)$. Since we can compute $bin(n)$ from $tal(n)$ in polynomial time, it follows that any tree which is p-time bounded in tally is also p-time bounded in binary. Note that any tree $T \subset \{0,1\}^{<\omega}$ is p-time bounded, so that a tree may be p-time bounded without being p-time. One additional observation is worth making at this point. If T is p-time bounded in tally, then there will also be a p-time function q such that, for any $\tau = (n_0, \ldots, n_k) \in T$, $|tal(\tau)| \leq q(1^k)$. To see this, note that τ consists of the strings $tal(n_i)$ for $i \leq k$, separated by commas and with parentheses at the beginning and end. Thus

$$|\tau| = 2 + k + |n_0| + \cdots + |n_k| \leq 2 + k + p(1^0) + \cdots + p(1^k).$$

Thus we can define a p-time bound $q(1^k) = 2 + k + p(1^0) + \cdots + p(1^k)$, which is clearly p-time computable. The same observation holds for p-time bounded in binary.

Now suppose that T is p-time bounded in tally and that $Tal(T)$ is p-time. This implies that there are at most $p(1^k)$ possible choices for $tal(n_k)$, that is, the strings 1^e for $e < p(1^k)$. To compute $h(\sigma)$, where $\sigma = (tal(n_0), \ldots, tal(n_{k-1}))$, we simply use the p-time algorithm for membership in $tal(T)$ to test whether $\sigma * tal(i) \in tal(T)$ for all $i \leq p(1^k)$ and compile the list $(tal(i_1), \ldots, tal(i_d)) = h(\sigma)$ of all $tal(i)$ such that $\sigma * tal(i) \in T$. This shows that the function h is p-time in tally. It then follows by the discussion above that g and f are also p-time in tally. Hence in the tally representation, any p-time bounded, p-time tree is also locally p-time.

Let us say that a tree T is *highly p-time in binary* if T is p-time, locally p-time and also p-time bounded in binary. Similarly, T is highly p-time in tally if T is p-time, locally p-time and also p-time bounded in tally. Then we have shown that in tally, p-time plus p-time bounded implies highly p-time. On the other hand, we have also seen that these notions are distinct for the binary representation.

Similar definitions can be given for other notions of complexity. Our next theorem shows that any Π^0_1-class can be realized as the set of infinite paths through a p-time tree.

Theorem VIII.1.4. *Let T be a computable tree. Then there is a polynomial time tree P such that $[T] = [P]$. Furthermore, if T is computably bounded, then P is also computably bounded and if T is p-time bounded, then P is also p-time bounded.*

Proof. The same argument works for the binary and for the tally representation. We will give the binary argument for the first part and the tally argument for the second part, since these are the stronger results. Let ϕ be a computable function from $\omega^{<\omega}$ into $\{0,1\}$ such that $\sigma \in bin(T) \iff \phi(\sigma) = 1$. Let ϕ^s denote the partial computable function which results by computing ϕ for exactly s steps on any input and let T^s be the s'th approximation to T, given by

$$\sigma \in T^s \iff \phi^s(bin(\sigma)) = 1 \text{ or is undefined .}$$

Thus $T^0 \supset T^1 \supset \cdots$ and, for any σ, $\sigma \in T \iff (\forall s)(\sigma \in T^s)$.

Now define the p-time tree P by letting

$$\sigma \in P \iff (\forall \tau \prec \sigma)\tau \in T^{|bin(\sigma)|}.$$

Note that P is a p-time tree in binary since to compute whether $\tau \in T^{|bin(\sigma)|}$ requires $|bin(\sigma)|$ steps for all τ so that to compute whether $\sigma \in P$ requires roughly
$|bin(\sigma)|(|bin(\sigma)| + 1)$ steps.

It follows from the definition of P that $T \subset P$, so that $[T] \subset [P]$. Now suppose that $x \notin [T]$. Then there is some initial segment $\tau = x \upharpoonright n$ which is not in T. This means that, for some s, $\tau \notin T^s$. Since the sequence T^s is decreasing, we may assume that $s > n$. Now let $\sigma = x \upharpoonright s$, so that $|bin(\sigma)| \geq s$. It follows from the definition of P that $\sigma \notin P$. This implies that $x \notin [P]$. Thus $[T] = [P]$.

Now suppose that T is computably bounded in tally and let p be the computable function which computes, for each k, an upper bound $p(1^k)$ (in tally) for the possible value of n_k for any node $\sigma = (n_0, \dots, n_k) \in T$.

Suppose first that p is actually p-time. Then we can recursively define a tree Q such that $T \subset Q \subset P$ by putting $\sigma = (n_0, \dots, n_k) \in Q$ if and only if $\sigma \in P$ and, for all $i \leq k$, $n_i \leq p(1^i)$. It is clear that $[Q] = [T]$ and that Q is p-time since P and p are p-time.

Finally, suppose only that p is recursive and let p^s be the usual result of computing p for s steps. Once again we can define a highly recursive tree Q such that $T \subset Q \subset P$ by putting $\sigma = (n_0, \dots, n_k) \in Q$ if and only if $\sigma \in P$ and, for all $i \leq k$, either $p^k(1^i)$ is undefined or $n_i \leq p^k(1^i)$. Then again it is easy to check that Q is p-time in binary and that $[Q] = [T]$. □

Next we would like to consider conditions which might force the tree T to have a p-time (exponential time, etc.) path. Recall that a Π^0_1 class P is decidable if $P = [T]$ for a computable tree with no dead ends (or with $Ext(T)$ computable) and that a decidable Π^0_1 class always has a computable member. Recall also that any Π^0_1 singleton is necessarily computable. Next we show that the obvious p-time analogues of these results fail for p-time decidable trees.

Theorem VIII.1.5. *For any computable $x \in \{0,1\}^\omega$, there is a tree T which is polynomial time in binary and in tally and such that $[T] = \{x\}$.*

Proof. This follows from Theorem VIII.1.4, since for $x \in \{0,1\}^\omega$, the tree $T = \{(x(0), \dots, x(n-1) : n < \omega\}$ is computable □

Theorem VIII.1.5 shows that even if a polynomial time bounded p-time tree has a unique infinite path Π, Π may not be polynomial time. However there are some natural conditions which we can put on T which will ensure that in such situations we can at least get double exponential time paths or winning strategies and in some cases actually guarantee the existence of polynomial time paths or winning strategies.

Theorem VIII.1.6. *(a) Let $Ext(T)$ be a locally p-time tree in tally (respectively binary) and let $[T]$ be nonempty. Then $[T]$ contains an infinite path which is double exponential time computable in tally (resp. binary). Furthermore, if $Ext(T)$ is locally p-time in tally (resp. binary) and $[T]$ is finite, then every element of $[T]$ is computable in double exponential time in tally (resp. binary).*

(b) Let $Ext(T)$ be a locally p-time tree in tally (respectively binary) and let $[T]$ be nonempty. Moreover, assume that there is a linear time function h such that for all $\sigma = (n_0, \ldots, n_k) \in T$, $h(b(\sigma))$ lists all $b(n)$ such that $(n_0, \ldots, n_k, n) \in T$ where $b() = tal()$ if T is p-time in tally and $b() = bin()$ if T is p-time in binary. Then $[T]$ contains an infinite path which is exponential time computable in tally (resp. binary). Furthermore, if $[T]$ is finite, then every element of $[T]$ is computable in exponential time in tally (resp. binary)

(c) If $Ext(T)$ is a highly p-time tree in tally (resp. binary) and $[T]$ is nonempty, then $[T]$ contains an infinite path which is p-time time in tally (resp. binary). Furthermore, if $[T]$ is finite, then every element of $[T]$ is p-time in tally (resp. binary).

(d) If $Ext(T)$ is a p-time bounded, p-time tree in binary and $[T]$ is nonempty, then $[T]$ contains an infinite path which is $EXPTIME$ in binary. Furthermore, if $[T]$ is finite, then every element of $[T]$ is NP in binary.

Proof. To simplify the discussion, we will assume in all cases that $T = Ext(T)$, that is, that T has no dead ends. Thus the conditions set out for $Ext(T)$ will become the conditions for T.

(a) We give the proof for the binary representation of T. The proof for the tally representation is exactly the same except for replacing $bin(...)$ with $tal(...)$ at appropriate locations throughout. Let h be the p-time function such that for all $\sigma = (n_0, \ldots, n_k) \in T$, $h(bin(\sigma))$ lists all $bin(n)$ such that $(n_0, \ldots, n_k, n) \in T$. Then we can recursively define the p-time path x through T by letting $x(k)$ be the number n such that $bin(n)$ is the first entry of $h(bin(x \upharpoonright k))$. It remains to be checked that the computation of $bin(x(n))$ from 1^n can be done in double exponential time. Let c be a number such that $h(\tau)$ can be computed from τ in time bounded by $|\tau|^{c-1}$ for all $\tau \in Bin(Ext(T))$ with $|\tau| \geq 2$. For each k, let $\tau_k = (bin(x(0), \ldots, bin(x(k-1))$. Then $\tau_0 = \emptyset$ and, for each $k > 0$, τ_{k+1} can be computed from τ_k in time bounded by $|\tau_k|^c$. (Just start the computation of $h(\tau_k)$, stop it as soon as you have the first element $\rho = bin(x(k))$ in the list and then append ρ to the end of τ_k). Thus in particular $|\tau_{k+1}| \leq |\tau_k|^c$ for all

$k > 0$. Now choose c large enough so that $|\tau_1| \leq 2^c$. It then follows by induction that, for all k, $|\tau_k| \leq 2^{c^k}$. It follows that the computation of τ_{k+1} from τ_k can be done in time bounded by $(2^{c^k})^{c-1} \leq 2^{c^{k+1}}$. Thus the entire computation of $\tau_n = (bin(x(0)), \ldots, bin(x(n)))$ from 1^n takes time bounded by

$$\Sigma_{k<n} 2^{c^{k+1}} < 2^{c^n+1} < 2^{(c+1)^n},$$

which shows that x is computable in double exponential time in binary.

(b) Now suppose that $[T]$ is finite and let $x \in [T]$. By resticting T to the extensions of $x \upharpoonright n$ for sufficiently large n, we may assume that x is the unique infinite path through T. The result now follows immediately from the first part above.

(c) The proof is essentially the same as the proof of (a). Again we shall only give the proof in the case that T is p-time in binary. The point is that if h is linear time then it follows that for all $k \geq 1$, τ_{k+1} can be computed from τ_k in time $c \cdot |\tau_k|$ for some fixed constant c. If we pick c so that $c \geq |\tau_0|$, then it is easy to prove by induction that $|\tau_k| \leq c^{k+1}$ for all $k \geq 0$. Thus the entire computation of $\tau_n = (bin(x(0)), \ldots, bin(x(n)))$ from 1^n takes time bounded by

$$\Sigma_{k<n} c^{k+2} < (n+1)c^{n+1} < c^{2n+2},$$

which shows that x is computable in exponential time in binary.

Now if $[T]$ is finite and $x \in [T]$, then again by resticting T to the extensions of $x \upharpoonright n$ for sufficiently large n, we may assume that x is the unique infinite path through T. Then by the above argument if follows that $x \in DEXT$.

(d) The proof will be a minor modification of the proof of (a) above. Again the proofs are the same for tally and for binary so we will just give a binary version. Let q be a p-time function such that for any $\tau = (bin(n_0), \ldots, bin(n_k) \in bin(T)$, $|\tau| \leq q(1^k)$. Since $|1^k| = k$, it follows that for some constant b and all $k > 1$, we have $|\tau_k| \leq k^b$. Let c be a number such that $h(\tau)$ can be computed from τ in time bounded by $|\tau|^{c-1}$ for all $\tau \in Bin(Ext(T))$ with $|\tau| > 1$. Then the computation of τ_{k+1} from τ_k can be done in time bounded by $|\tau_k|^c \leq k^{bc}$ for all $k > 1$. Now let a be large enough so that $a > bc$ and also large enough so that τ_0 and τ_1 can both be computed in time bounded by a. Then the entire computation of $\tau_n = (bin(x(0)), \ldots, bin(x(n)))$ from 1^n takes time bounded by

$$a + 2^a + 3^a + \cdots + (n-1)^a \leq \Sigma_{k<n^a} k \leq n^{2a},$$

which shows that x is computable in polynomial time in binary.

If we assume further that $[T]$ is finite, then the same argument as given in (a) and (b) above shows that every element of $[T]$ is polynomial time computable.

(d) As in (c), we may assume that if $\tau = (bin(n_0), \ldots, n_k) \in T$ and $k > 1$, then $|\tau| \leq k^b$ so that in particular $n_k \leq k^b$. In this case, we are not assuming that T is locally p-time, so that we need a different algorithm for producing an infinite path x in $[T]$. We will define $x(k)$ recursively by making $x(k)$ be the least number n such that $(x(0), \ldots, x(k-1), n) \in T$. This means that we may have to check whether $(x(0), \ldots, x(k-1), x) \in T$ for all x with $|bin(x)| \leq k^b$. This is where the binary representation differs from the p-time representation, because there will now be 2^{k^b} different strings to check. Each check will require time at most $(k^b)^c$, so that the computation of $bin(x(k))$ from $(bin(x(0)), \ldots, bin(x(k-1))$ will require time less than $2^{k^{bc+b}}$ for $k > 1$. Now let a be large enough so that $a \geq bc+b$ and also large enough so that $bin(x(0))$ and $bin(x(1))$ can be computed in time $\leq a$. Then the entire computation of $\tau_n = (bin(x(0)), \ldots, bin(x(n)))$ from 1^n takes time bounded by

$$a + 2^{2^a} + 2^{3^a} + \cdots + 2^{n-1)^a} \leq \Sigma_{k<n^a} 2^k \leq 2^{n^{2a}},$$

which shows that $x \in EXPTIME$.

If we assume further that $[T]$ is finite, then the same argument as given in (a) above shows that every element of $[T]$ is $EXPTIME$. However, it is easy to show that the infinite paths through T are actually NP computable.

As above, we may assume that T has no dead ends and has a unique infinite path x. Thus for any k, $x(0), x(1), \ldots, x(k))$ is the unique finite path in T with $k+1$ entries. Furthermore, since T is p-time bounded, we know as above that $|(bin(x(0)), \ldots, bin(x(k)))| \leq k^b$ for some fixed b. Thus to compute $x(k)$ nondeterministically, we simply guess a string $\sigma = (bin(n_0), \ldots, bin(n_k))$ of length $\leq k^b$ and then use the p-time algorithm for T to test whether $\sigma \in T$. When the answer is yes, we read the value of $x(k)$ from the end of σ. Since there is only one possible correct guess for σ, this procedure will compute $x(k)$. □

Next we shall give two examples to show that the bounds given in parts (a) and (b) of Theorem VIII.1.6 can not be improved. Consider the following.

Example VIII.1.7. *A locally p-time tree T with a unique infinite path x such that $T = EXT(T)$ and x is double exponential time.*

Let $x(n) = 2^{2^n}$ for all n and let the tree T consist of all initial segments of x. Then $(n_0, \ldots, n_k) \in T$ if and only if $n_0 = 1$ and, for all $i < k$, $n_{i+1} = n_i^2$. It is clear that both $tal(T)$ and $bin(T)$ are p-time. Furthermore, for any $\sigma = (n_0, \ldots, n_k) \in T$, we have $h(\sigma) = n_k^2$, so that T is locally p-time in both binary and tally.

Example VIII.1.8. *A locally p-time tree T with a unique infinite path x such that $T = EXT(T)$, there is a linear time function h such that for all $\sigma = (n_0, \ldots, n_k) \in T$, $h(b(\sigma))$ lists all $b(n)$ such that $(n_0, \ldots, n_k, n) \in T$ where $b() = tal()$ if T is p-time in tally and $b() = bin()$ if T is p-time in binary, and x is exponential time.*

Let $x(n) = 2^n$ for all n and let the tree T consist of all initial segments of x. Then $(n_0, \ldots, n_k) \in T$ if and only if $n_0 = 1$ and, for all $i < k$, $n_{i+1} = 2n_i$. It is clear that both $tal(T)$ and $bin(T)$ are p-time and that the function h is linear time in both cases, so that T is locally p-time in both binary and tally.

For Π^0_1 classes in $\{0,1\}^{\mathbb{N}}$, boundedness conditions are not needed and tally and binary representations are identical. Here are the basis results for classes of various complexity. These will be applied later to logical theories and other mathematical examples.

Theorem VIII.1.9. *Let $P = [T]$ be a Π^0_1 class in $\{0,1\}^{\mathbb{N}}$, where T has no dead ends.*

(a) If T is computable in time $nlog(n)^{O(1)}$, then P has a member which is computable in time $nlog(n)^{O(1)}$.

(b) If T is LIN, then P has a member which is computable in time $O(n^2)$.

(c) If T is $PTIME$, then P has a $PTIME$ member.

(d) If T is $DEXT$, then P has a $DEXT$ member.

(e) If T is $EXPTIME$, then P has an $EXPTIME$ member.

Proof. Part (c) follows from Theorem VIII.1.6. We give the proof of (d) and leave the others as an exercise 2.

As in the proof of Theorem VIII.1.6, we compute the desired path recursively so that $x(k) = 1$ if $(x(0), \ldots, x(k-1), 0) \in T$ and $x(k) = 1$ otherwise. By assumption, there exists c so that this last step in the computation of $x(k)$ takes time $\leq 2^{c(k+1)}$ and thus the total computation requires time

$$\leq 2^c + 2^{2c} + \cdots + 2^{c(k+1)} \leq 2^{c(k+2)} = 2^{2c} \cdot 2^{ck},$$

plus a little time for bookkeeping. □

Theorem VIII.1.10. *Let $P = [T]$ be a Π^0_1 class in $\{0,1\}^{\mathbb{N}}$, where T has no dead ends.*

(a) If T is $LINSPACE$, then P has a member which is computable in $LINSPACE$.

(b) If T is $PSPACE$, then P has a $PSPACE$ member.

(c) If T is $EXPSPACE$, then P has an $EXPSPACE$ member.

Proof. We just sketch the proof of (a) and leave the others to the reader. To compute $x(k)$, we need to have stored $x(0), \ldots, x(k-1)$, whoch requires space k and then test whether $x(0), \ldots, x(k-1), 0) \in T$, which requires additional space ck, where c is fixed. This additional space may be reused, so that we can stay in $LINSPACE$. □

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Exercises

VIII.1.1. Show that there is a highly computable tree T with no dead ends such that there is not highly p-time tree wihout dead ends such that $[T] = [S]$.

VIII.1.2. Show that if T is decidable in time $nlog(n)^{O(1)}$, then T has an infinite path which is computable in time $nlog(n)^{O(1)}$, if T is decidable in linear space, then T has a infinite path which is computable in time $O(n^2)$ and if T is $EXPTIME$, then $[T]$ has an $EXPTIME$ member.

VIII.2 Complexity of Structures

Complexity theoretic or feasible model theory is the study of resource-bounded structures and isomorphisms and their relation to computable structures and computable isomorphisms. This subject has been developed during the 1990's by Cenzer, Nerode, Remmel and others. See the survey article [38] for an introduction. Complexity theoretic model theory is concerned with infinite models whose universe, functions, and relations are in some well known complexity class such as polynomial time, exponential time, polynomial space, etc. By far, the complexity class that has received the most attention is polynomial time. One immediate difference between computable model theory and complexity theoretic model theory is that it is not the case that all polynomial time structures are polynomial time equivalent. For example, there is no polynomial isomorphism f with a polynomial time inverse f^{-1} which maps the binary representation of the natural numbers $Bin(\mathbb{N}) = \{0\} \cup \{1\}\{0,1\}^*$ onto the tally representation of the natural numbers $Tal(\mathbb{N}) = \{1\}^*$. This is in contrast with computable model theory where all infinite computable sets are computably isomorphic so that one usually only considers computable structures whose universe is the set of natural numbers $\mathbb{N}$.

There are two basic types of questions which have been studied in polynomial time model theory. First, there is the basic existence problem, i.e. whether a given infinite computable structure $\mathcal{A}$ is isomorphic or computably isomorphic to a polynomial time model. That is, when we are given a class of structures $\mathcal{C}$ such as a linear orderings, Abelian groups, etc., the following natural questions arise.

(1) Is every computable structure in $\mathcal{C}$ isomorphic to a polynomial time structure?

(2) Is every computable structure in $\mathcal{C}$ computably isomorphic to a polynomial time structure?

For example, the authors showed in [34] that every computable relational structure is computably isomorphic to a polynomial time model and that the standard model of arithmetic $(\omega, +, -, \cdot, <, 2^x)$ with addition, subtraction, multiplication, order and the 1-place exponential function is isomorphic to a polynomial time model. The fundamental effective completeness theorem says that any decidable theory has a decidable model. It follows that any decidable relational theory has a polynomial time model. These results are examples of answers to questions (1) and (2) above. However, one can consider more refined existence questions. For example, we can ask whether a given computable structure $\mathcal{A}$ is isomorphic or computably isomorphic to a polynomial time model with a standard universe such as the binary representation of the natural numbers, $Bin(\mathbb{N})$, or the tally representation of the natural numbers, $Tal(\mathbb{N})$. That is, when we are given a class of structures $\mathcal{C}$, we can ask the following questions.

(3) Is every computable structure in $\mathcal{C}$ isomorphic to a polynomial time structure with universe $Bin(\mathbb{N})$ or $Tal(\mathbb{N})$?

(4) Is every computable structure in $\mathcal{C}$ computably isomorphic to a polynomial time structure with universe $Bin(\mathbb{N})$ or $Tal(\mathbb{N})$?

It is often the case that when one attempts to answer questions of type (3) and (4) that the contrasts between computable model theory and complexity theoretic model theory become more apparent. For example, Grigorieff [80] proved that every computable linear order is isomorphic to a Ptime linear order which has universe $Bin(\mathbb{N})$. However Grigorieff's result can not be improved to the result that every computable linear order is computably isomorphic to a Ptime linear order over $Bin(\mathbb{N})$. For example, Cenzer and Remmel [34] proved that for any infinite polynomial time set $A \subseteq \{0,1\}^*$, there exists a computable copy of the linear order $\omega + \omega^*$ which is not computably isomorphic to any polynomial time linear order which has universe A. Here $\omega + \omega^*$ is the order obtained by taking a copy of $\omega = \{0, 1, 2, \ldots\}$ under the usual ordering followed by a copy of the negative integers under the usual ordering.

The general problem of determining which computable models are isomorphic or computably isomorphic to feasible models has been studied by the authors in [34], [35], and [37]. For example, it was shown in [35] that any computable torsion Abelian group G is isomorphic to a polynomial time group A and that if the orders of the elements of G are bounded, then A may be taken to have a standard universe, i.e. either $Bin(\mathbb{N})$ or $Tal(\mathbb{N})$. It was also shown in [35] that there exists a computable torsion Abelian group which is not isomorphic, much less computably isomorphic, to any polynomial time (or even any primitive recursive) group with a standard universe. Feasible linear orderings were studied by Grigorieff [80], by Cenzer and Remmel [34], and by Remmel [175, 176]. Feasible vector spaces were studied by Nerode and Remmel in [160] and [161]. Feasible Boolean algebras were studied by Cenzer and Remmel in [34] and by Nerode and Remmel in [159]. Feasible permutation structures and feasible Abelian groups were studied by Cenzer and Remmel in [35] and [37]. By a *permutation structure* $\mathcal{A} = (A, f)$, we mean a set A together with a unary function f which maps A one-to-one and onto A.

General conditions were given in [38] which allow the construction of models with a standard universe such as $Tal(\mathbb{N})$ or $Bin(\mathbb{N})$ and these conditions were applied to graphs and to equivalence structures. For example, it was shown that any computable graph with all but finitely many vertices of finite degree is computably isomorphic to a polynomial time graph with standard universe. On the other hand, a computable graph was constructed with every vertex having either finite degree or finite co-degree (i.e. joined to all but finitely many vertices) which is not computably isomorphic to any polynomial time graph with a standard universe. An *equivalence structure* $\mathcal{A} = (A, R^{\mathcal{A}})$ consists of a set A together with an equivalence relation. It was also shown that any computable equivalence structure is computably isomorphic to a polynomial time structure with a standard universe.

In this section, we want to consider the connection between computable structures and resource-bounded structures and the corresponding connection between computable trees and resource-bounded trees as developed in section

VIII.1.

A relational structure is simply a structure which has no functions. We will present an improved version of the theorem (first due to Grigorieff [80]) from [34] that every computable relational structure is computably isomorphic to a polynomial time structure. This theorem will be our primary tool in the analysis of computable combinatorial structures. It is important to note that the polynomial time structure provided will have for its universe a polynomial-time set possibly different from $\{1\}^*$ or $\{0,1\}^*$. An example is constructed in [34] which shows that the theorem fails if any fixed polynomial time set A is specified in advance as the universe of the structure. The improved version of the theorem presented here applies to structures with two distinct types of objects, the first type being the normal universe of the structure, and with functions which map the first type into the second type. The type of example that we have in mind is a function from the vertices of a graph into the natural numbers which computes the degree of a vertex. The universe of the graph is now expanded by adding a p-time set which represents the natural numbers and the degree function now becomes part of the structure. Naturally, the new objects are not vertices and therefore are not joined to any other objects by edges.

Theorem VIII.2.1. *Let*

$$\mathcal{C} = (C, A, B, \{R_i^{\mathcal{C}}\}_{i\in S}, \{f_i^{\mathcal{C}}\}_{i\in T},),$$

be a computable structure such that

(i) *A and B are disjoint subsets of C with $C = A \cup B$ and B is a polynomial time set.*

(ii) *there is a computable isomorphism from $Bin(\omega)$ onto a subset of $Bin(\omega)\setminus B$ with a p-time inverse.*

(iii) *for each $i \in T$, f_i maps C into B.*

(iv) *for each $i \in S$, the relation R_i is independent of B, that is, for any $(x_1, \ldots, x_n) \in C^n$, where $n = s(i)$, any $j \leq n$ such that $x_i \in B$, and any $b \in B$, $R_i^{\mathcal{C}}(x_1, \ldots, x_n)$ if and only if $R_i^{\mathcal{C}}(x_1, \ldots, x_{j-1}, b, x_{j+1}, \ldots, x_n)$.*

(v) *for each $i \in T$, the function f_i is independent of B, that is, for any $(x_1, \ldots, x_n) \in C^n$, where $n = t(i)$, any $j \leq n$ such that $x_i \in B$, and any $b \in B$, $f_i^{\mathcal{C}}(x_1, \ldots, x_n) = f_i^{\mathcal{C}}(x_1, \ldots, x_{j-1}, b, x_{j+1}, \ldots, x_n)$.*

Then there is a computable isomorphism ϕ of $\mathcal{C}$ onto a p-time structure $\mathcal{M}$ such that $\phi(b) = b$ for all $b \in B$.

Proof. The idea of the proof is that we will replace each element x of A by a string y which codes x and is long enough to allow us to compute whether $x \in A$ in time $|y|$ and also to compute the relations and functions on A in time $|y|$ for all inputs which are less than or equal to x. These new strings may accidentally be in the set B, which must be kept disjoint from $A^{\mathcal{M}}$. This is the reason for the p-time mapping which takes an arbitrary string to one which is not in B. Let ψ be a p-time map from $Bin(\omega)$ into $Bin(\omega) \setminus B$ such that ψ^{-1} is also p-time. We can assume that A is an infinite set, since, if A is finite, then $\mathcal{C}$ is p-time itself. Let $\sigma_0, \sigma_1, \ldots$ be an effective enumeration of A in the usual order. Let b_0 be the shortest element of B. For any $x \in A$, we let $\nu(x)$ denote the number of steps needed to run the following algorithm.

> First start to list $\sigma_0, \sigma_1, \ldots$ until we find an s such that $\sigma_s = x$. Next for each $i \leq s$ such that $i \in S \cup T$, list all sequences $(x_1, \ldots, x_n)$ from $\{b_0, \sigma_0, \ldots, \sigma_s\}^n$ for $n = s(i)$ or $t(i)$ and then, for $i \in S$, compute whether $R_i(x_1, \ldots, x_n)$ holds and, for $i \in T$, compute $f_i^{\mathcal{C}}(x_1, \ldots, x_n)$.

Observe that the algorithm is completely uniform in x because our definition of computable structure ensures that there is a computable relation R such that $R(i, \langle x_1, \ldots, x_{t(i)} \rangle) \iff R_i(x_1, \ldots, x_{t(i)})$ and a computable function f such that $f(i, \langle x_1, \ldots, x_{t(i)} \rangle) = f_i(x_1, \ldots, x_{t(i)})$ Note that in order to obtain the list $\sigma_0, \ldots, \sigma_s$, we have to test whether $a \in A$ for all $a \leq x$. We then define a structure

$$\mathcal{M} = (M, \{R_i^{\mathcal{M}}\}_{i \in S}, \{f_i^{\mathcal{M}}\}_{i \in T})$$

as follows. For each $a \in A$, let $\phi(a) = \psi(a^\frown 0^\frown 1^{\nu(a)})$ and, for each $b \in B$, we let $\phi(b) = b$. It is clear that ϕ is a computable isomorphism from C onto a subset M of $Bin(\omega)$, that $\phi(B) = B$ and that $\phi(A)$ is disjoint from B. The structure $\mathcal{M}$ is the image of $\mathcal{C}$ under the isomorphism ϕ. This means that $A^{\mathcal{M}} = \{\phi(a) : a \in A\}$, $B^{\mathcal{M}} = B$, and $M = A^{\mathcal{M}} \cup B^{\mathcal{M}}$. For each $i \in S$ and $(x_1, \ldots, x_n) \in C$, $R_i^{\mathcal{M}}$ is defined by

$$R_i^{\mathcal{M}}(\phi(x_1, \ldots, \phi(x_n)) \iff R_i^A(x_1, \ldots, x_n),$$

where $s(i) = n$. For each $i \in T$, $f_i^{\mathcal{M}}$ is defined by

$$f_i^{\mathcal{M}}(\phi(x_1), \ldots, \phi(x_n)) = \phi(f_i^{\mathcal{A}}(x_1, \ldots, x_n)),$$

where $t(i) = n$.

It is clear that the function ϕ is a computable isomorphism from $\mathcal{A}$ onto $\mathcal{M}$. It remains to be seen that $\mathcal{M}$ is a polynomial time structure, that is, that M is a polynomial time set and that each relation R^M and function f^M is p-time.

We show that M is p-time as follows. It clearly suffices to show that $A^{\mathcal{M}}$ is p-time, since $B^{\mathcal{M}} = B$ is p-time. The procedure for testing whether an input y is in $A^{\mathcal{M}}$ is to compute $\psi^{-1}(y)$, check to make sure that it has a 0 in it, and then determine x and n such that $\psi^{-1}(y) = x^\frown 0^\frown 1^n$. Then we simply run the algorithm outlined above to input x for n steps. Then $y \in A^{\mathcal{M}}$ if and only if the algorithm terminates in exactly n steps and gives the answer that $x \in A$.

We show that the function $f_i^{\mathcal{M}}$ is p-time as follows. Fix i and let $f = f_i$, let $n = t(i)$ and let c be the maximum amount of time required to compute $f^C(x_1, \ldots, x_n)$ when $\{x_1, \ldots, x_n\} \subseteq \{b_0, \sigma_0, \sigma_1, \ldots, \sigma_{i-1}\}$. Now given input

$(y_1, \ldots, y_n)$, where each $y_i \in M$, the procedure for computing $f^{\mathcal{M}}(y_1, \ldots, y_n)$ is the following. First replace every $x_i \in B$ with $x'_i = b_0$ and let $x'_i = x_i$ for $x_i \in A$. Then compute $f^C(x'_1, \ldots, x'_n)$. We claim that this computation takes time at most $c + max\{|y_j| : 1 \leq j \leq n\}$. There are two cases of this claim to consider. First, if $\{x'_1, \ldots, x'_n\}$ is a subset of $\{b_0, \sigma_0, \ldots, \sigma_{i-1}\}$, then, by the definition of c, the computation takes at most c steps. On the other hand, if at least one of the $x'_j = x_j = \sigma_s$ for some $s \geq i$, then by the definition of ν, the computation takes less than $\nu(x_j)$ steps for some j; but of course $\nu(x_j) < |y_j| \leq max\{|y_j| : 1 \leq j \leq n\}$.

The argument for the relations is similar. This completes the proof of Theorem VIII.2.1. □

For an example, let $(\mathbb{N}, R, 0, f)$ be a computable structure where R is a binary relation defining a tree with root 0 on the set of even numbers and f is an injection mapping the even numbers onto the odd numbers so that $f(0) = 1$ and, for each n, $R(f^{-1}(2n+1), f^{-1}(2n+3))$. That is, f defines an infinite computable path through T. (We assume that $R(m, n)$ implies that both m and n are even.) Then the theorem provides a polynomial time tree with a polynomial time infinite path starting from the root.

VIII.3 Propositional Logic

In this section, we shall consider the complexity of theories in propositional logic and of the corresponding Π^0_1 class of complete consistent extensions of the theory.

It is first necessary to define the length $|\phi|$ of a formula ϕ. Suppose that the underlying set of propositional letters in our propositional language is $\{A_0, A_1, \ldots\}$. In the standard or binary representation of a sentence ϕ, the numeral i in a propositional letter A_i is written in binary representation $bin(i)$ so that the length $|A_i|$ in binary is $1 + |bin(i)|$. That is, $|bin(A_i)| = r + 2$ when $2^r \leq i < 2^{r+1}$. In the tally representation, the numeral i is written as 1^i so that $|tal(A_i)| = i + 1$. A complete consistent theory Γ is represented by a subset of ω, $S(\Gamma) = \{i : A_i \in \Delta\}$, or, equivalently, by the characteristic function in $\{0,1\}^\omega$ of $S(\Gamma)$. The set of all complete consistent extensions of a consistent set Δ of sentences is denoted as $CC(\Delta)$. We shall let a finite sequence $\sigma \in \{0,1\}^n$ represent the sentence $B(\sigma) = B_0 \wedge B_1 \wedge \ldots B_n$, where $B_i = A_i$ if $\sigma(i) = 1$ and $B_i = \neg A_i$ if $\sigma(i) = 0$.

We note that there is a lower limit on the complexity of non-trival propositional theories. To be more precise, the set SAT of consistent, or satisfiable, sentences is the classic NP complete set. Now a sentence ϕ is valid if and only if $\neg\phi$ is not satisfiable. Thus the smallest theory, the set of valid sentences is Co-NP complete. On the other hand, any complete propositional theory is determined by its underlying set of literals. That is, let $V = \{A_0, A_1, \ldots\}$ be a set of propositional variables, S be any subset of V and $\Gamma(S)$ be the consequences of $\{A_i : i \in S\} \cup \{\neg A_i : i \notin S\}$. Then S is computable from $\Gamma(S)$ in constant time. On the other hand, given any sentence ϕ containing variables $A_{i_1}, \ldots, A_{i_k}$, we can decide whether $\phi \in \Gamma(S)$ by first making each A_t true if it is in S and false if not, and then evaluating ϕ. That is, $\phi \in \Gamma(S)$ if and only if the value of ϕ is true. Thus $\Gamma(S)$ is computable from S in linear time and linear space. Thus there are complete propositional theories in any of the standard complexity classes such as linear time, linear space, polynomial time, polynomial space, etc.

Lemma VIII.3.1. *$tal(B(\sigma))$ has length $O(n^2)$ and may be computed in time $O(n^2)$ and $Bin(B(\sigma))$ has length $O(n \cdot log\ n)$ and may be computed in time $O(n\ log\ n)$.*

Proof. The sentence $B(\sigma)$ contains the atoms $A_0, A_1, \ldots, A_{n-1}$, $n-1$ conjunction symbols $\wedge$ and between 0 and n negation symbols $\neg$. The total length of the atoms in tally is

$$2 + 2 + 3 + 4 + \cdots + n = \frac{n^2}{2} + \frac{n}{2} + 1,$$

so that

$$\frac{n^2}{2} + \frac{3n}{2} \leq |tal(B(\sigma)| \leq \frac{n^2}{2} + \frac{5n}{2}$$

In binary, suppose first that $n = 2^{k+1} - 1$. Then the total length of the atoms is

$$2 \cdot 2 + \sum_{j=1}^{k} (j+1) 2^{j-1}$$

so that the total length of the atoms is strictly between $(k+1)2^{k-1} + 1$ and $(k+1)2^k$ and the length of $bin(B(\sigma))$ is between $(k+5)2^{k-1}$ and $(k+5)2^k$. Now suppose that $k \leq log(n) \leq k+1$, so that $2^k \leq n < 2^{k+1}$. It follows that $(5 + log(n))n/2 \leq (k+5)2^{k-1} \leq |bin(B(\sigma))| \leq (k+5)2^k \leq (5 + log(n)) \cdot 2n$. □

A set Δ of sentences is said to be *P-decidable* in binary (in tally) if there is a polynomial time Turing machine which given as input the binary (tally) representation of a formula ϕ, computes 1 if $\Delta \vdash \phi$ and computes 0 otherwise. We say that Δ is *weakly P-decidable in binary* (in tally) if there is a polynomial time Turing machine which given as input the binary (tally) representation of a conjunction ϕ of literals, computes 1 if $\phi \in SAT(\Delta)$ and computes 0 otherwise. One can define the notion of Δ being (weakly) $\mathcal{C}$-decidable in binary or tally for any complexity class $\mathcal{C}$ in a similar manner. A theory Γ is said to be P-axiomatizable if it possesses a polynomial time set Δ of axioms such that $\Gamma = Cn(\Delta)$. Again similar definitions apply to other notions of complexity.

Recall that the tree T represents $CC(\Delta)$, the set of complete consistent extensions of Δ, if the set $[T]$ of infinite paths through T equals the family of sets $S \subseteq \{A_0, A_1, \ldots\}$ such that $\Gamma(S)$ is a complete consistent extension of Δ. The canonical tree T which represents $CC(\Delta)$ is given by $\sigma \in T \iff B(\sigma) \in SAT(\Delta)$.

Theorem VIII.3.2. *Let Δ be a propositional theory.*

(a) If Δ is weakly $DTIME(n\ log(n)^{O(1)})$ decidable in binary, then $CC(\Delta)$ may be represented as the set of paths through a tree in $DTIME(n\ log(n)^{O(1)})$.

(b) If Δ is weakly P-decidable (respectively $PSPACE$ decidable) in either binary or tally, then $CC(\Delta)$ may be represented as the set of paths through a P-tree (resp. $PSPACE$-tree).

(c) If Δ is weakly $DEXT$-decidable (respectively, $EXPSPACE$-decidable) in tally or binary, then $CC(\Delta)$ may be represented as the set of paths through an $EXPTIME$-tree (resp. $\bigcup_{k\in\omega} DSPACE(2^{n^k})$-tree).

Proof. In each case, we shall let T be the canonical tree which represents $CC(\Delta)$. That is, $\sigma \in T \iff B(\sigma) \in SAT(\Delta)$.

(a) Suppose that Δ is weakly $DTIME(n\ log(n)^{O(1)})$ decidable in binary. By Lemma VIII.3.1, we can compute $bin(B(\sigma))$ from σ in time $O(n\ log\ n)$, so that T is in $DTIME(n\ log(n)^{O(1)})$.

(b) It easily follows from Lemma VIII.3.1 that we can compute $bin(B(\sigma))$ and $tal(B(\sigma))$ in polynomial time and space from σ. Thus if Δ is weakly P-decidable (weakly $PSPACE$-decidable), then T is a P-tree ($PSPACE$-tree).

(c)If Δ is weakly $DEXT$-decidable in tally ($EXPSACE$-decidable), it will require on the order of $2^{|\sigma|^2}$ time (space) to determine if $B(\sigma) \in SAT(\Delta)$ so that T is an $EXPTIME$-tree ($\bigcup_{k\in\omega} DSPACE(2^{n^k})$-tree). Similarly if Δ is weakly $DEXT$-decidable in binary ($EXPSACE$-decidable), it will require on the order of $2^{|\sigma| log(|\sigma|)}$ time (space) to determine if $B(\sigma) \in SAT(\Delta)$ so that again T is an $EXPTIME$-tree ($\bigcup_{k\in\omega} DSPACE(2^{n^k})$-tree). □

Thus we have the following corollary.

Corollary VIII.3.3. *Let Δ be a propositional theory.*

(a) If Δ is weakly P-decidable (respectively $PSPACE$ decidable) in tally or binary, then Δ has a P-decidable (resp. $PSPACE$-decidable) complete consistent extension in tally

(b) If Δ is weakly $DEXT$-decidable (respectively, $EXPSPACE$-decidable) in tally or binary, then Δ has a complete consistent extension which is $EXPTIME$ decidable (resp. $EXPSPACE$ decidable) in tally.

Proof. (a) Let Δ be weakly $PTIME$ decidable in tally or binary. By Theorem VIII.3.2, $CC(\Delta)$ may be represented as the set of paths through a P-decidable tree T. It follows from Theorem VIII.1.9 that T has an infinite $PTIME$ path $x \in \{0,1\}^{\mathbb{N}}$. The complete consistent extension Γ corresponding to x has axioms A_i for $x(i) = 1$ and $\neg A_i$ for $x(i) = 0$. Thus given an arbitrary sentence $\phi(A_0, \ldots, A_n)$ of length $geqn$, we can first compute $x(0), \ldots, x(n))$ in polynomial time and then use this to substitute true and false for the occurrences of the propositional variables in ϕ to compute the value of ϕ. This can certainly be done in polynomial time in tally. The proof for $PSPACE$ is similar.

(b) The proof is similar to (a). □

For the binary representation, note that $|A_n|$ is of order $logn$ and hence it may take exponential time to decide A_n from a polynomial time $x \in \{0,1\}^{\mathbb{N}}$. Thus we have

Corollary VIII.3.4. *Let Δ be a propositional theory.*

(a) If Δ is weakly P-decidable (respectively $PSPACE$ decidable) in either binary or tally, then Δ has a $DEXT$-decidable (resp. $EXPSPACE$-decidable) complete consistent extension in binary.

(b) If Δ is weakly $EXPTIME$-decidable (respectively, $EXPSPACE$-decidable) in tally or binary, then Δ has a complete consistent extension which is $DOUBEXT$ decidable (resp. $DOUBEXPSPACE$ decidable) in binary.

Since any $PTIME$ decidable theory is certainly weakly $PTIME$ decidable and likewise for other comlexity classes, these results hold with the "weakly" removed from the hypothesis.

It was shown in [44] that this difference in the complexity of the complete consistent extension between the tally and binary representations is necessary. That is,

(1) There is a propositional theory which is NP-decidable in binary but has no P-decidable complete consistent extension in binary.

(2) There is a propositional theory which is $DEXT$-decidable in binary but has no $EXPTIME$-decidable complete consistent extension in binary.

There are no nice basis results for *axiomatiable* theories. The corresponding representation results for axiomatizable theories do not require any restriction on the complexity of the set of axioms. In fact, our next result strengthens Theorem 4.1 of [36] which showed that any Π^0_1 class may be represented as the set of paths through a polynomial time tree.

A computable function f is said to be *time constructible* if and only if there is a Turing machine which on every input of size n halts in exactly $f(n)$ steps. In particular, the functions $log_2^k(n)$ are time constructible for $k \geq 1$ where we define $log_2^k(n)$ by induction as $log^1(n) = log(n)$ and for $k > 1$, $log_2^k(n) = log_2(log_2^{k-1}(n))$.

It was shown in [44] that for any time constructible function f which is nondecreasing and unbounded and any axiomatizable propositional theory Γ, Γ has a $DTIME(O(f))$ set of axioms and may be represented as the set of paths through a $DTIME(O(f))$-tree. Note that this is not necessarily a decidable tree.

The results for decidable trees are somewhat surprising. Let us first give a few definitions. Recall that SAT is the set of satisfiable, or consistent, propositional sentences and is the standard NP-complete set.

Theorem VIII.3.5. *The following are equivalent:*

(a) *$P = NP$;*

(b) *Every P-decidable tree represents the set of complete consistent extensions of some theory which is P-decidable in tally.*

(b) $\rightarrow$ *(a).* Let $T = \{0,1\}^*$ and suppose that Δ is a theory which is P-decidable in tally such that $\{0,1\}^\omega = [T]$ represents the set of complete consistent extensions of Δ. Then it is easy to see that $SAT(\Delta) = SAT$. But this means that

$$\phi \in SAT \iff \neg[\Delta \vdash \neg\phi]. \tag{VIII.1}$$

Since Δ is P-decidable in tally, (VIII.1) would imply that SAT is polynomial time and hence $P = NP$.

[(a) $\rightarrow$ (b)] Next suppose that $P = NP$ and let T be a P-decidable tree. Let $\phi(A_0, \ldots, A_n)$ be a propositional formula whose propositional letters are a a subset of $\{A_0, \ldots, A_n\}$ and which contains A_n. The canonical theory Δ such that $[T]$ represents the set of complete consistent extensions of Δ is defined by

$$\Delta \vdash \phi(A_0, \ldots, A_n) \iff (\forall \sigma \in T \cap \{0,1\}^n)(B(\sigma) \vdash \phi(A_0, \ldots, A_n)).$$

We will show that $SAT(\Delta)$ is NP and hence in P by our assumption. In tally, $n \leq |B(\sigma)| \leq 2n^2$, so that for each $\sigma \in \{0,1\}^n$, we can test whether $B(\sigma) \vdash \neg\phi(A_0, \ldots, A_n)$ in polynomial time in the length of $\phi(A_0, \ldots, A_n)$. Thus we can test $\phi(A_0, \ldots, A_n) \in SAT(\Delta)$ in the usual NP fashion, by guessing a string σ of length n and checking that $\sigma \in T$ and $B(\sigma)$ implies $\phi(A_0, \ldots, A_n)$. Thus $SAT(\Delta)$ is in P. $\square$

The corresponding result does not follow relative to the binary representation of theories. That is, the direction [(ii) $\rightarrow$ (i)] still holds, since the SAT problem is NP-complete in either tally or binary. However, the argument given for the reverse direction only shows that $SAT(\Delta)$ is $DTIME(2^{O(1)})$-decidable in binary. This is due to the fact that a short formula ϕ with a high numbered variable, such as a propositional variable A_{2^n-1} requires us to check whether $B(\sigma) \vdash \phi(A_0, \ldots, A_n)$ for $|\sigma| = 2^n - 1$ which would require time of order 2^n since $|B(\sigma)| \geq 2^n$. Thus since $|A_{2^n-1}| = n + 1$, such a check would require exponential time in $|\phi|$.

Part B

Applications of Π^0_1 Classes

Effectively closed sets arise naturally in the study of computable mathematics. In many problems associated with mathematical structures, such as the problem of finding a 4-coloring of a planar graph, the family of solutions may be viewed as a closed set under some natural topology. Thus for a computable structure, the set of solutions may be viewed as a Π^0_1 class. For another example, the set of zeroes of a continuous real function on the unit interval is a closed set and the set of zeroes of a computable real function will be a Π^0_1 class.

We will say that the Π^0_1 class P *represents* the set of solutions to a given problem if there is a one-to-one degree preserving correspondence between the elements of the class P and the solutions to the problem. It will be important whether the class P is bounded, or computably bounded. For example, the set of 4-colorings of a given computable graph G may be represented as a subclass of $\{0,1,2,3\}^{\mathbb{N}}$ and is therefore computably bounded. Then we may apply the basis results surveyed in Chapter IV, for example, Theorem III.2.15, and conclude that if G has a 4-coloring, then it has a 4-coloring of c.e. degree.

Now a fundamental observation is that computable problems, such as the graph-coloring problem, often do not have computable solutions. The results from Chapter IV on special Π^0_1 classes strengthen in various ways the basic result that a computably bounded Π^0_1 class may not have any computable members. In order to be able to transfer these results to results about the degrees of solutions to a given computable mathematical problem of a given type, one must establish that *every* computably bounded Π^0_1 class represents the set of solutions to some computable problem of that type. For example, Remmel [174] showed that, up to a permutation of the colors, every computably bounded Π^0_1 class represents the set of 3-colorings of some computable graph. It then follows from Theorem IV.2.4 that for any c.e. degree $\mathbf{c}$, there is a computable, 3-colorable graph G such that every 3-coloring of G has degree $\geq \mathbf{c}$.

It is very important to make the representation effective. For each type of mathematical problem, we shall establish a natural enumeration of the computable (and sometimes the c.e.) problems and then use the effective correspondence between solution sets and Π^0_1 classes to transfer results on index sets from Chapter VI. For example, we will show that the set of indices of computable graphs which have a computable 3-coloring is a Σ^0_3 complete set.

Many important problems in the history of Π^0_1 classes come from mathematical logic. The problem associated with a logical theory Γ is to find a complete consistent extension of Γ. For any effectively presented language $\mathcal{L}$, the set of sentences of $\mathcal{L}$ may be effectively listed as $\{\phi_n : n \in \mathbb{N}\}$. Then Γ is said to be *decidable* if $\{n : \phi_n \in \Gamma\}$ is a computable set and Γ is said to be *axiomatizable* if $\{n : \phi_n \in \Gamma\}$ is a c.e. set.

Shoenfield [189] showed in 1960 that the set of complete consistent extensions of a axiomatizable first theory can always be represented by a Π^0_1 class. The classical undecidability theorem of Turing and Church may be viewed as showing that the Π^0_1 class of complete consistent extensions of Peano Arithmetic has no computable element. The complete consistent extensions of a decidable theory Γ may be represented by a decidable Π^0_1 class.

Ehrenfeucht [69] showed in 1961 that, conversely, every computable bounded

Π_1^0 class P represents the set of complete consistent extensions of some axiomatizable theory Γ. In particular, Γ may be a theory of propositional logic or a theory for the language with a single, binary relation symbol. If P is decidable, then we may take Γ to be a decidable theory.

The chapters below include proofs of the theorems cited above, together with applications of results on members of Π_1^0 classes and on index sets for Π_1^0 classes.

Chapter IX

Algebra

Three types of computable and computably enumerable algebraic structures are considered: Boolean algebras, abelian groups, and commutative rings with unity. The associated problems are to find proper, prime and maximal ideals of rings and Boolean algebras and to find proper and maximal subgroups of abelian groups.

The set of prime ideals of a c. e. Boolean algebra or commutative ring with unity may always be represented by a c. b. Π^0_1 class, and the set of maximal ideals of a computable Boolean algebra may be represented by a *decidable* c. b. Π^0_1 class. The set of maximal ideals of a c. e. commutative ring with unity or of a c. e. Boolean algebra may always be represented by a Π^0_2 class.

For the reverse direction, any c. b. Π^0_1 class P may be represented by the set of maximal ideals of a c. e. Boolean algebra $\mathcal{B}$ and if P is decidable, then $\mathcal{B}$ may be taken to be computable. Finally, any Π^0_1 class of separating sets may be represented as the set of prime ideals of some c. e. commutative ring with unity [72].

A recursively presented group, ring, or field consists of a recursive subset U of ω, the universe of the structure, together with appropriate partial recursive functions over U for addition, subtraction, multiplication and/or division functions as required. Unless, explicitly stated otherwise, we will assume that all our structures are countably infinite so that there is no loss in generality in assuming that the underlying universe is ω. A *c. e. ring* is the quotient of a recursive ring modulo an *r.e.* ideal, a *c. e. group* is the quotient of a computable group modulo a c. e. normal subgroup and a *c. e. Boolean Algebra* is the quotient of a recursive Boolean Algebra modulo a c. e. ideal.

We will show that the set of prime ideals of c. e. commutative ring with unity and the set of prime ideals of a c. e. Boolean algebra can always be represented by a c. b. Π^0_1 class. We will show that the set of maximal ideals of a c. e. commutative ring with unity and the set of maximal subgroups of a c. e. group can always be represented by a Π^0_2 class. We shall also show that the set of all ideals or the set of all maximal ideals of a recursive Boolean algebra can be represented as the set of paths through a recursive tree with no dead ends.

Reversing such results, we will show that any c. b. bounded Π^0_1 class can be strongly represented by the set of maximal ideals of an c. e. Boolean algebra. We show that the set of paths through any recursive tree T with no dead ends can be represented as the set of maximal ideals of a recursive Boolean algebra. We shall also show that the set of separating set $S(A, B)$ of a pair of c. e. sets can be represented by the set of prime ideals or the set of maximal ideals of a c. e. commutative ring with identity.

We refer the reader to Downey [61] for a general survey of computable algebra.

Some definitions are needed. Recall that a subset H of an Abelian group $G = (G, +^G, -^G, 0^G)$ is a subgroup if it satisfies the following conditions:

(i) $0^G \in H$.

(ii) $a \in H$ and $b \in H$ implies $a -^G b \in H$.

H is a *maximal subgroup* if, in addition, there is no subgroup J of G such that $H \subset J \subset G$.

A subset I of a commutative ring with unity $R = (R, +^R, -^R, \cdot^R, 0^R, 1^R)$ is an ideal I is a subgroup of $R = (R, +^R, 0^R)$ and it satisfies the following additional conditions:

(iii) $a \in I$ and $r \in R$ implies $a \cdot^R b \in I$.

(iv) $1^R \notin I$.

I is a *prime ideal* if, in addition,

(v) $a \cdot^B b \in I$ implies $a \in I$ or $b \in I$.

I is a *maximal ideal* if, in addition, there is no ideal J such that $I \subset J$. It is easy to see that any maximal ideal is prime, but the converse is not always true.

The classical results that every proper subgroup of a group has an extension to a maximal (and therefore proper) subgroup and that every ideal in a ring has an extension to a maximal (and therefore prime) ideal follow easily from Zorn's Lemma. In particular, if the commutative ring R with unity is not a field, then R has, for each non-unit a a proper ideal $Ra = \{ra : r \in R\}$ and therefore has a maximal ideal.

Any Boolean algebra $(B, \vee^B, \wedge^B, \neg^B, 0^B, 1^B)$ may be viewed as a commutative ring with unity where $a \cdot^B b = a \wedge b$ and $a + b = (a \wedge^B \neg^B b) \vee^B (\neg^B a \wedge^B b)$. In a Boolean ring any prime ideal is maximal, so it follows from the Boolean algebra results that, for any Π^0_1 class P, there is a c. e. commutative ring with unity such that $Max(R) = Prime(R)$ is represented by P. However, there turns out to be a significant difference between Boolean rings and rings in general. The proof that any recursive Boolean ring has a recursive maximal ideal cannot be extended to arbitrary rings and in fact, a recursive ring need not have a recursive maximal ideal. This naturally led to the conjecture that any Π^0_1 class could be represented as the set of prime ideals of some commutative ring. By considering rings of polynomials, Friedman-Simpson-Smith obtained in [73] the partial result that any Π^0_1 class of separating sets can be represented as the set of prime ideals of some recursive commutative ring with unity.

IX.1 Boolean algebras

The Stone Representation Theorem implies that every Boolean algebra is isomorphic to the Boolean algebra of clopen sets of a topological space (indeed of a Boolean space). If the Boolean algebra is countable, the proof shows that it is isomorphic to the Boolean algebra $RC(P)$ of relatively clopen sets of a closed class P contained in $\{0,1\}^{\mathbb{N}}$, and of course $RC(P)$ is countable for every closed class P contained in $\{0,1\}^{\mathbb{N}}$. In this section we point out effectivized versions of this correspondence and use them to transfer some of our results on Π^0_1 classes to results on computable and c. e. Boolean algebras. In particular, we give an effective version of the Stone Representation Theorem, that every computable (c. e.) Boolean algebra is isomorphic to the set of its prime ideals. We determine the meaning of thinness and of the Cantor-Bendixson derivative in the setting of Boolean algebras. We also look at the connection between computable Boolean algebras and theories of propositional calculus, in particular with Martin–Pour-El theories. Finally, we interpret the results of the previous sections on Π^0_1 classes for computable and c. e. Boolean algebras. Here the Π^0_1 class represents the set of prime ideals of a c. e. Boolean algebra.

Some of the results are known as part of the folklore of the subject. For more on computable Boolean algebras, see Remmel [172].

A computable Boolean algebra B is given by a model $(\mathbb{N}, \preceq, \neg, \vee, \wedge)$ where $\preceq$ is a computable binary relation, $\neg$ is a computable unary operation, and $\vee$ and $\wedge$ are computable binary functions satisfying the usual properties of a Boolean algebra. In particular, there is a $\preceq$-least element 0 and a $\preceq$-greatest element 1, and we assume that $0 \in \mathbb{N}$ names the last and $1 \in \mathbb{N}$ names the greatest. We note that the *complement* $\neg a$ may be computed by searching for the element $b \in B$ such that $a \wedge b = 0$ and $a \vee b = 1$, and thus we do not need to assume that it is computable. The partial ordering $\preceq$ may be defined (and in fact computed) from the two binary operations in that $a \preceq b \iff a \vee b = b \iff a \wedge b = a$. (See the exercises.) We will also use the operation $a + b = (a \wedge \neg b) \vee (b \wedge \neg a)$, which will be computable for any computable Boolean algebra.

An element a of a Boolean algebra $\mathcal{A}$ is an *atom* if there does not exists $b \in \mathcal{A}$ such that $0 < b < a$. $\mathcal{A}$ is said to be *atomless* if it has no atoms. Alternatively, we may say that $\mathcal{A}$ is *dense* if the ordering $\preceq$ is dense, that is, whenever $a < b$ in $\mathcal{A}$, then there exists $c \in \mathcal{A}$ such that $a < c < b$. $\mathcal{A}$ is said to be *atomic* if for every $b \in \mathcal{A}$, there exists an atom $a \in \mathcal{A}$ such that $a \preceq b$.

The fundamental computable atomless Boolean algebra $\mathcal{Q}$ may be thought of as the family of clopen subsets of $\{0,1\}^{\omega}$. Each clopen set has a unique representation as a finite union of disjoint intervals $I(\sigma_1) \cup \cdots \cup I(\sigma_k)$, where each σ_i has the same length and k is taken to be as small as possible. Then the join ($\vee$) and meet ($\wedge$) operations are clearly computable, as well as the complement operation and the partial ordering relation on $\mathcal{Q}$.

Alternatively, we may consider the fundamental Boolean algebra $\mathcal{Q}(\omega)$ to be the Lindenbaum algebra of propositional calculus over an infinite set $\{A_0, A_1, \dots\}$ of propositional variables. Here two propositions p and q are equal in $\mathcal{Q}(\omega)$ if they have the same truth table, so that this is a computable equivalence relation.

A c. e. Boolean algebra is given by a model $(\mathbb{N}, \preceq, \vee, \wedge)$ such that $\preceq$ is a c.e. relation which is a pre-ordering, $\vee, \wedge$ are total computable binary functions, and the quotient structure $B = (\mathbb{N}, \preceq, \vee, \wedge)/\equiv$ is a Boolean algebra (where $n \equiv m \iff n \preceq m \;\&\; m \preceq n$). We can suppose that $0 \in \mathbb{N}$ names the least and $1 \in \mathbb{N}$ the greatest element of $\mathcal{B}$. Note here that $\equiv$ is preserved under the operations.

A subset I of a Boolean algebra $\mathcal{B}$ is said to be an *ideal* if for all $a, b \in B$,

(i) If $a \in I$ and $b \in I$, then $a \vee b \in I$;

(ii) If $b \in I$ and $a \leq^B b$, then $a \in I$.

An ideal I is *proper* if $1 \notin I$ and is *prime* if, for all a, b: (iii) $a \vee b \in I \rightarrow a \in I$ or $b \in I$.

An ideal I is *principal* if, for some $b \in \mathcal{B}$, $I = I(b) = \{a \in \mathcal{B} : a \leq b\}$.

Finally, an ideal I is *maximal* if I is proper and there is no proper ideal J with $I \subset J$.

For any ideal I, the equivalence relation $\equiv^I$ is defined by

$$a \equiv^I b \iff a + b \in I.$$

It is clear that $\equiv^I$ is c. e. if I is c. e. and is computable if I is computable.

Conversely, given an operation-preserving equivalence relation $\equiv$ on $\mathcal{B}$, the corresponding ideal I may be defined as $\{a : a \equiv 0\}$. Then I will be computable (c. e.) if $\equiv$ is computable (c. e.).

The dual notion of an ideal is a *filter*. A subset M of a Boolean algebra $\mathcal{B}$ is a filter if it is closed uner $\wedge$ and is closed upwards. It is easy to see that M is a filter if and only if the set $M^d = \{\neg b : b \in M\}$ is an ideal; similarly for any ideal I, we may define the dual filter $I^d = \{\neg b : b \in I\}$. Downey [59] develops the theory of c. e. Boolean algebras from the point of view of c. e. filters.

Let us define a *computable quotient Boolean algebra* to be the quotient $\mathcal{B}/\equiv^B$, where $\mathcal{B} = (B, \equiv^B, \neg^B, \wedge^B, \vee^B)$ is a computable structure such that $B \subset \omega$, such that $\equiv^B$ is an equivalence relation on B, such that the unary operation $\neg^B$ and the two binary operations $\vee^B$ and $\wedge^B$ preserve the equivalence classes, and hence the set of equivalence classes forms a Boolean algebra.

Lemma IX.1.1. *Any computable quotient Boolean algebra $\mathcal{B}$ is isomorphic to a computable Boolean algebra $\mathcal{A}$.*

Proof. Define the universe A of $\mathcal{A}$ by
$A = \{b \in B : (\forall a < b)\neg(a \equiv^B b)\}$.

For any $b \in B$, let $\psi(b)$ be the least a such that $a \equiv^B b$. Then define the operations on A by
$\neg^A(a) = \psi(\neg^B(a))$,
$a \vee^A b = \psi(a \vee^B b)$, and
$a \wedge^A b = \psi(a \wedge^B b)$.

It is clear that the set A together with these operations forms a Boolean algebra which is isomorphic to the Boolean algebra on the equivalence classes of $\mathcal{B}$ and that the set A and each of the Boolean operations is computable. □

For any Boolean algebra $\mathcal{B}$ with universe $B = \omega$, let $P(\mathcal{B})$ be the class of maximal ideals $\mathcal{B}$. It is easy to see that $P(\mathcal{B})$ is a closed subclass of 2^ω, where an ideal J is represented as by its characteristic function.

Theorem IX.1.2. *If $\mathcal{A}$ is a c. e. quotient Boolean algebra, then $P(\mathcal{A})$ is a Π^0_1 class and if $\mathcal{A}$ is a computable Boolean algebra, then $P(\mathcal{A})$ is a decidable Π^0_1 class.*

Proof. Suppose that $\mathcal{A} = \mathcal{B}/\equiv$ is a c. e. quotient Boolean algebra. We can represent the class $P(\mathcal{A})$ of prime ideals on $\mathcal{A}$ as follows.

$x \in P(\mathcal{A}) \iff$

(1) $(\forall a)(\forall b)[a \equiv b \to x(a) = x(b)]$ and

(2) $(\forall a)(\forall b)[x(a) = x(b) = 1 \to x(a \vee^B b) = 1]$ and

(3) $(\forall a)(\forall b)[x(a) = 1 \to x(a \wedge^B b) = 1]$ and

(4) $(\forall a)[x(a) = 1 \iff x(\neg^B a) = 0]$.

This clearly defines a Π^0_1 class. Observe that either $x(0^B) = 1$ or $x(1^B) = 1$ by (4) and hence $x(0^B) = 1$ by (3), so that $x(1^B) = 0$. Thus any $x \in P(\mathcal{A})$ represents a proper prime ideal. If $\mathcal{B}$ is actually a computable Boolean algebra, then we can omit clause (1) and define a computable tree T with no dead ends such that $P(\mathcal{B}) = [T]$, as follows. T is defined to be the set of finite sequences $x = (x(0), \ldots, x(n-1))$ which satisfy the following, where $lh(x) = n$.

(2)' $(\forall a < n)(\forall b < n)[(x(a) = x(b) = 1 a \vee^B b < n) \to x(a \vee^B b) = 1]$ and

(3)' $(\forall a < n)(\forall b < n)[(x(a) = 1 a \wedge^B b < n) \to x(a \wedge^B b) = 1]$ and

(4)' $(\forall a < n)(foralli < 2)[(x(a) = i \neg^B a < n) \to x(\neg^B a) = 1 - i]$.

$(5)'_k$ $(\forall a_1 < a_2 < \cdots < a_k < n)(x(a_1) = \cdots = x(a_k) = 0 \to a_1 \wedge^B a_2 \cdots \wedge^B a_k \neq 1^B)$.

Clause (5) is needed to establish the finite intersection property for $\{a < n : x(a) = 0\}$ which will ensure that any $\sigma \in T$ can be extended to a prime ideal in $P(\mathcal{A})$. This then implies that T has no dead ends. □

We can now apply our general results about Π^0_1 classes to Boolean algebras. The following is a consequence of Theorems III.2.15 and IV.1.4.

Theorem IX.1.3. *(i) For any c. e. Boolean algebra $\mathcal{B}$, $\mathcal{B}$ has a prime ideal J of low c. e. degree (so that J is computable in $\mathbf{0}'$).*

(ii) For any computable Boolean algebra $\mathcal{B}$, $\mathcal{B}$ has a computable prime ideal.

Theorem IX.1.4. *For any c. e. Boolean algebra $\mathcal{B}$ with no computable prime ideal, there exists a c. e. degree* $\mathbf{a}$ *such that $\mathcal{B}$ has no prime ideals of degree* $\leq \mathbf{a}$.

Proof. This follows from Theorem IV.thm:nla. □

Theorem IX.1.5. *For any For any c. e. Boolean algebra $\mathcal{B}$ with no computable prime ideals, there exists two prime ideals, I and J, of $\mathcal{B}$ such that any set computable from I and computable from J is in fact computable.*

Proof. This follows from Theorem IV.IV.2.12. □

The following theorem is a corollary of Theorems V.V.2.3,V.4.4 and V.2.2.

Theorem IX.1.6. *Let $\mathcal{B}$ be a c. e. Boolean algebra only countably many prime ideals. Then*

(a) $\mathcal{B}$ has a computable prime ideal.

(b) If $\mathcal{B}$ has only finitely many prime ideals, then every prime ideal is computable.

(c) Every prime ideal of $\mathcal{B}$ is hyperarithmetic.

Now we will briefly consider the notion of rank for ideals. It is an exercise below 5 that for any Π^0_1 class P, an element U of $RC(P)$ is an atom if and only if $U \cap P$ is a singleton.

Proposition IX.1.7. *For any prime ideal J of a c. e. Boolean algebra $\mathcal{B}$, J is isolated in $P(\mathcal{B})$ if and only if J is principal.*

Proof. Suppose that J is a principal and prime ideal. Then for some b, we have $J = \{a : b \leq a\}$. It follows that J is isolated in the interval $I(J \upharpoonright b+1)$.

Suppose that J is isolated in the interval $I(\sigma)$ where $lh(\sigma) = n$. Let $b_\sigma = b_0 \wedge b_1 \wedge \dots b_{n-1}$ where $b_i = i$ if $x(i) = 1$ and $b_i = \neg i$ if $x(i) = 0$. The isolation means that J is the only prime ideal of $\mathcal{B}$ that contains b_σ. It follows that J is generated by b_σ. □

Note that a principal ideal is prime if and only if it is generated by an atom.

The following is a corollary of Theorem V.V.4.3.

Theorem IX.1.8. *Let $\mathcal{B}$ be a c. e. Boolean algebra which has a unique non-principal prime ideal J. Then $J \leq_T \mathbf{0}''$ and if $\mathcal{B}$ is computable, then $J \leq_T \mathbf{0}'$.*

Next we consider the reverse direction of the correspondence between Π^0_1 classes and c. e. quotient Boolean algebras.

For an arbitrary Π^0_1 class P, let $RC(P)$ be the Boolean algebra of relatively clopen subsets of P, that is, $\{U \cap P : U \in \mathcal{Q}\}$ under the standard set operations. Let $\mathcal{B}(P)$ denote the c. e. Boolean algebra resulting from $\mathcal{Q}$ by taking the equivalence relation

$$U \equiv^P V \iff U \cap P = V \cap P.$$

(Note that the corresponding ideal $I = \{U \in \mathcal{Q}(\{0,1\}^{\mathbb{N}}) : U \cap P = \emptyset\}$.)

The notion of a perfect closed set corresponds to the notion of an atomless Boolean algebra in the following sense.

Proposition IX.1.9. *For any closed set P, P is perfect if and only if $RC(P)$ is atomless.*

Proof. Assume first that P is perfect and let $U \cap P \neq \emptyset$. Then $U \cap P$ contains at least two elements and hence U can be partitioned into two clopen sets U_1 and U_2 such that $U_1 \cap P$ and $U_2 \cap P$ are distinct nonempty sets in $RC(P)$. It follows that $RC(P)$ is atomless.

Next assume that P is not perfect and let x be isolated in P. This means that there is a clopen set U such that $U \cap P = \{x\}$. Clearly $U \cap P$ is an atom in $RC(P)$. □

Theorem IX.1.10. *Let $P \subseteq \{0,1\}^{\mathbb{N}}$ be a Π^0_1 class. Then the quotient algebra $\mathcal{B}(P)$ is isomorphic to the Boolean algebra $RC(P)$, the equivalence relation $\equiv_P$ is computably enumerable and hence $RC(P)$ is a c. e. Boolean algebra. Furthermore, if P is decidable, then $\equiv^P$ is computable and $\mathcal{B}(P$ is a computable Boolean algebra.*

Proof. The isomorphism mapping $U/equiv^P$ to $U \cap P$ is clearly a computable isomorphism from $\mathcal{B}(P)$ to $RC(P)$.

To see that $\equiv^P$ is a c. e. relation, let $P = [T]$ where T is a computable tree and suppose that $U = I(\sigma_1) \cup \dots I(\sigma_k)$ and $V = I(\tau_1) \cup \dots I(\tau_m)$. Then

$$U \cap P \subseteq V \cap P \iff (\forall i \leq k) I(\sigma_i) \cap P \subseteq V \cap P.$$

But for any σ, we have

$$I(\sigma) \cap P \subset V \cap P \iff (\exists n)(\forall \tau)[(lh(\tau) = n \,\& \sigma \prec \tau \,\&\, \tau \in T) \to (\exists i \leq m)(\tau_i \prec \tau)].$$

Finally, $\equiv^P$ is c. e. , since
$U \equiv^P V \iff [U \cap P) \subset V \cap PV \cap P \subset U \cap P]$.

If P is decidable, then T has no dead ends, so we can take k to be the maximum of $\{lh(\tau_i) : i \leq m\}$, so that $\equiv^P$ will be computable and hence each operation of $\mathcal{B}(P)$ also computable. □

Theorem IX.1.11. *(a) For any Π^0_1 class $P \subseteq \{0,1\}^{\mathbb{N}}$, P is computably homeomorphic to the set of prime ideals of $RC(P)$.*

(b) For any Boolean algebra $\mathcal{B}$ with universe $B = \omega$, $RC(P(\mathcal{B}))$ is isomorphic to $\mathcal{B}$. For a c. e. Boolean algebra, this isomorphism is effective.

Proof. (a) Map the element x of P to the prime ideal $J(x) = \{U \cap P : x \notin U\}$. If $x \neq y$, then there must be a clopen set P such that $x \in U$ and $y \notin U$, so that the map is injective. Given a prime ideal J of $RC(P)$, we claim that there must be a unique element x_J of P which belongs to every $U \cap P$ in J. Every finite subset of J has nonempty intersection since $\emptyset \notin J$, hence by compactness $\bigcap J$ is nonempty. Let x be an element of $\bigcap J$, suppose that $y \neq x$ and let U be a clopen set such that $x \in U$ and $y \notin U$. Then $U \cap P \in J$ but $y \notin U$ and hence $y \notin \bigcap J$. Thus $\bigcap J$ is a singleton. We leave it to the exercises to show that x_J may be computed effectively from J.

(b) Let $\mathcal{B}$ be a Boolean algebra with universe $B = \omega$. The isomorphism from $\mathcal{B}$ to $RC(P(\mathcal{B}))$ is given by mapping the element b to $\{J : J$ is a prime ideal of $\mathcal{B} \,\&\, b \notin J\}$, that is, to $P(\mathcal{B}) \cap U(b)$, where $U(b)$ is the clopen set defined by $x \in U(b) \iff x(b) = 0$. □

We now have the following corollaries.

Theorem IX.1.12. *For any c. e. degree* $\mathbf{c}$*, there is a c. e. Boolean algebra* $\mathcal{B}$ *such that the c. e. degrees of prime ideals of* $\mathcal{B}$ *are exactly the c. e. degrees above* $\mathbf{c}$*.*

Proof. This is an immediate consequence of Theorems IV.IV.2.4 and IX.1.10. □

Theorem IX.1.13. *There is a c. e. Boolean algebra* $\mathcal{B}$ *such that any two prime ideals of* Γ *are Turing incomparable.*

Proof. This follows from Theorem IV.IV.2.10. □

Next we consider the translation of the notion of a thin Π^0_1 class to the corresponding notion for Boolean algebras. Let us say that a c. e. Boolean algebra is *thin* if every c. e. ideal of $\mathcal{B}$ is principal. Downey [59] defined a c. e. filter M in $\mathcal{Q}$ to be *superthick* if, for every c. e. filter W such that $M \subset W \subset \mathcal{Q}$, there exists $b \in \mathcal{Q}$ such that $W = \langle M, b\rangle$. Here $a \in \langle M, b\rangle$ if and only if there exists $x \in M$ such that $b \wedge x \leq a$.

Lemma IX.1.14. *Let* $\mathcal{A}$ *be the c. e. Boolean algebra defined as the quotient of* $\mathcal{Q}$ *modulo the ideal* I*. Then the following are equivalent.*

(i) The Π^0_1 *class* $P(\mathcal{A})$ *is thin.*

(ii) The filter I^d *is superthick.*

(iii) $\mathcal{B}$ *is thin.*

Proof. We will show that (i) and (iii) are equivalent and leave the rest as an exercise. We may assume by Theorem IX.1.10 that $P \subseteq \{0,1\}^{\mathbb{N}}$ is a Π^0_1 class, that $I = \{V \in \mathcal{Q} : V \cap P = \emptyset\}$ is a c. e. ideal in $\mathcal{Q}$ and that $\mathcal{A} = \{[U] : U \in \mathcal{Q}\}$, where $[U]$ is the equivalence class in $\mathcal{Q}$ of U under the equivalence relation defined by $U \equiv_P V \iff U \cap P = V \cap P$. Then we have, for any $x \in \{0,1\}^{\mathbb{N}}$,

$$x \in P \iff (\forall U \in I) x \notin U.$$

Suppose first that P is thin and let J be a c. e. ideal in $\mathcal{A}$. Define the Π^0_1 class $Q \subset P$ by

$$x \in Q \iff (\forall V \in \mathcal{Q})([V] \in J \implies x \notin V).$$

By assumption, $Q = P \cap U$ for some $U \in \mathcal{Q}$ and it follows that $J = \{[V] : [V] \subseteq [U]\}$ and is principal.

For the converse, suppose that $\mathcal{A}$ is thin and let $Q \subset P$ be a Π^0_1 class. Then $J = \{[V] : V \cap Q = \emptyset\}$ is a c. e. ideal in $\mathcal{A}$. By assumption, there exists U such that, for all $V \in \mathcal{Q}$, $[V] \in J \iff [V] \subseteq [U]$. It is then easy to see that $Q = P \cap \overline{U}$. □

Here is an existence result concerning Π_1^0 classes of rank one. This follows from Theorem V.V.4.5 and Corollary V.V.5.2.

Theorem IX.1.15. *(a) For any degree* $\mathbf{b} \leq 0'$*, there is a computable Boolean algebra* $\mathcal{B}$ *with a unique non-principal prime ideal* J *such that* J *has degree* $\mathbf{b}$.

(b) For any degree $\mathbf{b}$ *such that* $\mathbf{0}' \leq \mathbf{b} \leq \mathbf{0}'$*, there is a c. e. Boolean algebra* $\mathcal{B}$ *with unique non-principal prime ideal* J *of degree* $\mathbf{b}$.

Exercises

IX.1.1. Show how to compute the partial ordering $\leq^B$ of a Boolean algebra B from the $\vee^B$ and $\wedge^B$ operations.

IX.1.2. Show that the $\approx$ relation in a c. e. Boolean algebra is preserved under the operations. That is, if $a \approx b$ then $\neg a \approx \neg b$ and similarly for the binary operations.

IX.1.3. Show how to carefully define the Boolean algebra of clopen sets to see that it is in fact computable.

IX.1.4. Show how to compute x_J in the proof of Theorem IX.1.11. (Hint: for any clopen U, exactly one of U and $\{0,1\}^{\mathbb{N}} - U$ belongs to J; to find $x(0)$, check which one of $I((0))$ and $I((1))$ belongs to J.)

IX.1.5. Show that for any closed set P, U is an atom in the Boolean algebra $RC(P)$ if and only if $U \cap P$ is a singleton.

IX.1.6. Complete the proof of Lemma IX.1.14.

Chapter X

Computer Science

Non-monotonic logics arose in attempts to formalize several notions of "common-sense" reasoning. These systems include the default logic of Reiter [171] and the stable semantics of general logic programs [78] due to Gelfond and Lifschitz. Classical logic is monotonic in that a deduction from a set of premises remains valid for any larger set of premises. Minsky [153] suggested that there is another form of reasoning which is not monotonic. That is, common sense and even scientific reasoning forces one to make assumptions in the absence of complete information. Thus new information may naturally lead to the rejection of previous beliefs. The set of stable models of a logic program is in some sense a non-monotonic generalization of the set of complete consistent extensions of a set of premises. Marek, Nerode and Remmel [137] showed that different versions of a logic program may be used to represent c. b., bounded and unbounded Π^0_1 classes.

Another area of theoretical computer science where Π^0_1 classes have application is the study of ω-languages, that is, sets of infinite words. Here an ω-language is the set of infinite words which are *accepted*, in some fashion, by a program. In particular, a Π^0_1 class may be viewed as the set of infinite words which are accepted by a deterministic automata M in the sense that an infinite sequence $x = (x(0), x(1), \dots$ is accepted by M if M is always in an accepting state after reading each initial segment $(x(0), \dots, x(k))$ of x. L. Staiger and K. Wagner [207, 204, 205, 206] have examined several other widely studied notions of acceptance which produce different classes. The relation between these notions and Π^0_1 classes is developed in [43].

X.1 Non-monotonic Logic

Chapter XI

Graphs

There are several combinatorial problems associated with computable graphs. These include the graph coloring problem, the problems of Hamiltonian and Euler circuits, the vertex partition problem, and various matching or marriage problems. In each case, the set of solutions to any such problem may be represented by a Π^0_1 class. To obtain a bounded Π^0_1 class, it is sometimes necessary to assume that each vertex of the computable graph has finite degree and to obtain a c. b. Π^0_1 class, it is sometimes necessary to assume that the graph is *highly computable*, that is, the set of vertices joined to vertex v can be computed from v.

For the reverse direction, there are a variety of results. In each case, the set of solutions can represent an arbitrary Π^0_1 class of separating sets. For the graph-coloring problem, Remmel [174] showed that the 3-coloring problem for highly computable graphs can represent an arbitrary c. b. Π^0_1 class. Manaster and Rosenstein [133] showed that the set of surjective marriages in a symmetically highly computable society can likewise represent an arbitrary c. b. Π^0_1 class. On the other hand, Remmel [173] showed that this last result does not hold when each person knows at most two other people; this problem is related to the the Schroder-Bernstein theorem, where one tries to construct an isomorphism between two sets given injections in each direction.

For each section, we begin by giving a list of the problems and the required definitions together with some of the history of each problem. Next we explain (in varying detail) how to prove that the set of solutions to any such problem can be represented by a computably bounded Π^0_1 class. Then we apply the results of Chapters IV and V to obtain corollaries which apply to the set of solutions of any such problem. Conversely we also consider for each problem, whether the set of solutions to such a problem can represent any *c.b.* Π^0_1 class. In each case, we show that the set of solutions to such a problem can represent the class of separating sets of any two disjoint *c.e.* sets. Then we apply the results of Chapters IV and V to obtain corollaries which give the existence of "pathological" problems of each type. Next we consider index sets for such problems using the methods of Chapter VI. Then we examine the reverse mathematics of such problems as in

Chapter VII. Finally, we look at complexity-theoretic versions of some of the problems.

XI.1 Matching problems

A *computable society* $S = (B, G, K)$ consists of disjoint computable sets B, the set of boys, and G, the set of girls, and a computable binary relation $K \subseteq B \times G$. Here $K(b, g)$ means b knows g. The solutions in this case are the set of *marriages*, or *matchings*, that is, 1:1 maps $f : B \to G$ such that $K(b, f(b))$ holds for all b. For any subset B' of B, let $K(B') = \{g : (\exists b \in B')K(b, g)\}$. Marshall Hall [82] extended the classical Philip Hall Theorem to infinite societies and proved that, for any countable society $S = (B, G, K)$, if every boy knows only finitely many girls and, for any finite subset $B' \subseteq B$, $|B'| \leq |K(B')|$, then there is a marriage for S. We say that a computable society $S = (B, G, K)$ is *highly computable* if there is a partial computable function $k : B \to \omega$ such that, for each $b \in B$, $k(b)$ equals the cardinality of $K(b)$. We say that S is *symmetrically highly computable* if there is also a partial computable function $\overline{k}$ such that, for each $g \in G$, $\overline{k}(g)$ is the cardinality of the set of boys which know g.

The problems which we consider are:

(i) The general problem of finding a marriage in a highly computable society S,

(ii) the surjective matching problem, that is, finding a marriage $f : B \to G$ which is both 1:1 and onto in a symmetrically highly computable society S, and

(iii) the surjective matching problem, where each person knows at most two other people in a symmetrically highly computable society S.

Problems (i) and (ii) were analyzed by Manaster and Rosenstein in [133, 134], who showed that the set of marriages in case (i) and (ii) is always a *c.b.* Π^0_1 class, but does not always contain a computable element. Moreover, Manaster, Rosenstein showed that in case (ii), the set of surjective marriages can represent an arbitrary *c.b.* Π^0_1 class. We note that problem (iii) contains a computable version of Banach's strengthening of the Schroder-Bernstein theorem, which was shown to be noneffective by Remmel [173]. That is, suppose we take 1:1 computable functions with computable ranges $f : B \to G$ and $g : G \to B$ where B and G are computable sets. Then we can form a highly computable society $S = (B, G, K)$, where $K(x, y)$ holds if and only if $f(x) = y$ or $g(y) = x$. For such a society S, the only surjective marriages h arise from some partition $B = B_1 \cup B_2$, where $h = f\lceil B_1 \cup g^{-1}\lceil B_2$, and the existence of such marriages are guaranteed by Banach's result. (See [173] for details.) It was shown by Remmel in [174] that the set of surjective marriages in case (iii) cannot represent an arbitrary *c.b.* Π^0_1 class in contrast to the Manaster-Rosenstein result for case (ii).

In each case, the set of solutions to such a problem can be represented by a Π^0_1 class [133, 134].

Theorem XI.1.1. *For any computable instance of each of the three matching problems described above, the set of solutions can be represented by a Π^0_1 class. If the given graph is highly computable, then the class is computably bounded.*

Proof. We may assume that B is the set of even numbers and G is the set of odd numbers. In (1), a marriage is simply a 1:1 map $g : B \to G$ such that $(b, g(b)) \in K$ for all $b \in B$. We can represent g by a map $x_g : \mathbb{N} \to \mathbb{N}$ by letting $x_g(i) = 2g(2i) + 1$. Thus the Π^0_1 class $P \subset \mathbb{N}^{\mathbb{N}}$ which represents the set of solutions is given by

$$x \in P \iff (\forall i)(2i, 2x(i) + 1) \in K \;\&\; (\forall i, k)(x(i) = x(k) \to i = k).$$

If S is highly computable, then given i, we can compute the finite set $G_i = \{j : (2i, 2j + 1) \in K\}$. Since $x_g(i) \in G_i$, this shows that P is computably bounded.

For problems (ii) and (iii), the solution is a pair of functions, one from B into G and one from G into B, which are inverses of each other. This matching can be represented by a single function from $\mathbb{N}$ to $\mathbb{N}$ and the set of solutions will again be a Π^0_1 class, and will be c. b. if S is highly computable. □

We can derive a number of immediate corollaries to Theorem XI.1.1.

Theorem XI.1.2. *For each highly computable society S and matching problem of type (i), (ii), or (iii), the following hold.*

(a) *If S has a solution, then S has a solution in some c. e. degree.*

(b) *If S has a solution, then S has solutions s_1 and s_2 such that any function computable in both s_1 and s_2 is recursive.*

(c) *If S has a solution but only has countably many solutions, then S has a computable solution.*

(d) *If S has only finitely many solutions, then each solution is computable.*

(e) *If S has a solution but has no computable solution, then for any countable sequence of nonzero degrees $\{\mathbf{a}_i\}$, S has continuum many solutions s which are mutually Turing incomparable and such that the degree of s is incomparable with each $\mathbf{a}_i$.*

Next we consider the reverse direction of this correspondence. That is, given an arbitrary c. b. Π^0_1 class P, is there a matching problem of a given type such that P represents the set of matchings?

Theorem XI.1.3. *[[133]] The problem of finding a surjective marriage in a computable society can represent an arbitrary bounded Π^0_1 class and the problem of finding a surjective marriage in a symmetrically highly computable society can represent an arbitrary c. b. Π^0_1 class.*

Proof. Let P be the set of infinite paths through a computable tree T. Let $B = \{2 < \sigma >: \sigma \in T \ \{\emptyset\}\}$ and $G = \{2 < \sigma > +1 : \sigma \in T\}$. K consists of all pairs $(2 < \sigma >, 2 < \sigma > +1)$ for $\sigma \in T$ as well as the pairs $2 < \sigma >, 2 < \sigma \upharpoonright n > +1)$ for any $\sigma \in T$ with length $n+1$. An infinite path $x \in T$ corresponds to the matching which assigns girl $2 < x \upharpoonright n > +1$ to boy $2 < x \upharpoonright n+1 >$ and assigns girl $2 < \sigma > +1$ to boy $2 < \sigma >$ if σ is not an initial segment of x. It is clear that if T is highly computable, then K will also be highly computable. □

Theorem XI.1.4. *The following problems can represent the c. b. Π^0_1 class of separating sets for any pair of disjoint infinite c. e. sets.*

(i) The problem of finding a marriage in a highly recursive society.

(ii) The problem of finding a surjective marriage in a symmetrically highly recursive society where each person knows at most two other people.

Proof. (i) For each $i \in \omega$, we will specify a boy b_i and two girls $g_{0,i}$ and $g_{1,i}$ so that b_i knows both $g_{0,i}$ and $g_{1,i}$ and no other. Our highly computable society $S = (B, G, K)$ will be such that $G = \{g_{0,i}, g_{1,i} : i \in \omega\}$ and $B = R \cup \{b_i : i \in \omega\}$, where $R = \{r_s : (A^s \cup B^s) - (A^{s-1} \cup B^{s-1}) \neq \emptyset\}$ is some infinite set of boys held in reserve. A marriage f for S will code a set C_f by specifying that $i \in C_f$ if and only if $f(b_i) = g_{1,i}$. We then determine who the boys in R know in stages in such a way that

(a) if $i \in A$, then one boy in R knows $g_{1,i}$ and no others and no boy in R knows $g_{0,i}$;

(b) if $i \in B$, then one boy in R knows $g_{0,i}$ and no others and no boy in R knows $g_{1,i}$;

(c) if $i \notin A \cup B$, then no boy in R knows $g_{0,i}$ or $g_{1,i}$.

Then if i enters $A \cup B$ at stage s, we put $r_s \in B$ and we put $(r_s, g_{1,i})$ in K if $i \in A$ and $(r_s, g_{0,i})$ in K if $i \in B$. It is clear that this defines a highly computable society S and that there is a one-to-one degree-preserving correspondence between the marriages f for S and the separating sets C of A and B, given by mapping f to C_f.

(ii) Fix a pair A and B of infinite disjoint c. e. sets and recursive enumerations $\{A^s\}_{s\in\omega}$ and $\{B^s\}_{s\in\omega}$ such that, for all s, $A^s, B^s \subseteq \{0, 1, \ldots, s\}$ and there is at most one element of $A \cup B$ which comes into $A \cup B$ at stage s.

We first partition ω into a computable sequence $(G_0, B_0, G_1, B_1, \ldots)$ of infinite computable sets . For any fixed i, let $g^0_i < g^1_i < \ldots$ and $b^0_i < b^1_i < \ldots$ list the elements of G_i and B_i in increasing order. Our symmetrically highly computable society $S = (B, G, K)$ will be thought of as a bipartite graph with $B = \cup_i B_i$ and $G = \cup_i G_i$. The idea is to construct a connected component of S with vertex set $G_i \cup B_i$ for each i. We construct the i-th component in stages, so that at stage s, we determine the edges out of g^k_i and b^k_i for $k \leq 2s$. We begin as if we are going to construct the two-way infinite chain in which b^0_i is joined to g_i^0 and g_i^1 and such that, for each $n > 0$, b_i^{2n} is joined to g_i^{2n-2} and g_i^{2n} and b_i^{2n-1} is joined to g_i^{2n-1} and g_i^{2n+1}. See Figure XI.1

Observe that there are exactly two possible surjective marriages f for such a component depending on whether $f(b_i^0) = g_i^0$ or $f(b_i^0) = g_i^1$. A marriage $f : B \to G$ for S will code a separating set C_f for A and B by letting $i \in C_f$ if and only if $f(b_i^0) = g_i^1$. Then it is easy to see that all we need to do to ensure that each marriage f of S corresponds to a separating set C_f for A and B is to construct the i-th component so that it is a one-way chain starting in B_i if $i \in A$, a one-way chain starting in G_i if $i \in B$, and the full two-way infinite chain if $i \notin A^s \cup B^s$. Thus we build the chain until we see that $i \in A \cup B$ at some stage s. That is, at each stage t, we add b_i^k and g_i^k for $k \in \{2t, 2t+1\}$ as pictured in Figure XI.1. Then if $i \in B^s$ omit b_i^{2n} and g_i^{2n} from the chain for all $n \geq s$ so that the chain will be a one-way infinite starting a girl g_i^{2s-2}. If $i \in A^s$, then add b_i^{2s} and we omit g_i^{2s} plus all boys and girls of the form b_i^{2n} and g_i^{2n} for $n > s$ from the chain so that the chain will be a one-way infinite chain starting at b_i^{2s}.

We note that we can consider this example as a computable version of problem (ii) by simply directing the edges of the graph down the left hand side of the graph and up the right hand side of the graph. That is, we can define the function $f : B \to G$ by saying that $f(b^*) = g^*$ is there is a directed edge from b^* to g^* in some component and define the function $g : G \to B$ by saying that $g(g^*) = b^*$ if there is a directed edge from g^* to b^* in some component. □

Given these representation results, we have the usual corollaries.

Theorem XI.1.5. *(a) For each one of the three matching problems,*

(1) There is a computable society S which has a matching but has no computable matching.

(2) There is a computable society S such that that any two distinct matchings are Turing incomparable.

(3) If $\mathbf{a}$ is a Turing degree and $\mathbf{0} <_T \mathbf{a} \leq_T \mathbf{0}'$, then there is a computable society s which has a matching of degree $\mathbf{a}$ but has no computable matching.

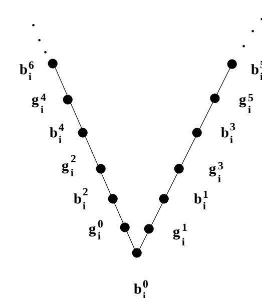
(b) For the surjective matching problem, the following also hold.

(4) There is a computable society S such that if $\mathbf{a}$ is the degree of any matching and $\mathbf{b}$ is a c. e. degree with $\mathbf{a} \leq_T \mathbf{b}$, then $\mathbf{b} = \mathbf{0}'$.

(5) If $\mathbf{c}$ is any c. e. degree, then there exists a computable society S such that the set of c. e. degrees which contain matchings equals the set of c. e. degrees $\geq_T \mathbf{c}$.

XI.2 Graph-coloring problems

A countable infinite graph $G = (V, E)$ consists of a subset V of the natural numbers called *vertices* together with a symmetric subset E of $V \times V$, called


b_i^6
g_i^4
b_i^4
g_i^2
b_i^2
g_i^0
b_i^0
g_i^1
b_i^1
g_i^3
b_i^3
g_i^5
b_i^5

the *edges*. G is said to be computable if the sets V and E are computable. We say that vertices u, v are joined by an edge (u, v). The *degree* of a vertex u of G is the cardinality of the set of vertices joined to u. A *k-coloring* of the graph G is a map g from V into $\{1, 2, \ldots, k\}$ such that $g(u) \neq g(v)$ whenever $(u, v) \in E$. The *k-coloring problem* for a graph G is to determine whether G has any k-colorings. The set of solutions to this problem is the set of k-colorings of G. We make the convention that, unless stated otherwise, the graphs we shall discuss are assumed to be connected, have no loops or multiple edges, and have the property that each vertex v of G is of finite degree.

The graph coloring problem has been studied in combinatorics for over a century. Two classical results for finite graphs are Brooks' Theorem [19] that every graph with all vertices of degree $\leq k$ and with no $k+1$-cliques is k-colorable and the Four Color Theorem of Haaken and Appel [3] that every planar graph is 4-colorable. These results are easily extended to infinite graphs by a compactness argument. A natural question is whether such results can be effectivized. The answer to this question is yes for Brooks' Theorem, that is, Schmerl showed in [183] that every computable graph with all vertices of degree $\leq k$ and with no $k+1$-cliques has a computable k-coloring. On the other hand, the Four Color Theorem cannot be effectivized. Bean constructed in [12] a 3-colorable, computable, planar connected graph which has no computable k-coloring for any k.

A computable graph $G = (V, E)$ is said to be *highly computable* if there is a partial computable function $f : V \to \omega$ such that, for each $v \in V$, $f(v)$ is the degree of v. Highly computable graphs are of interest for several reasons. One reason is the result of Bean [12] that any highly computable k-colorable graph has a computable $2k$-coloring, in contrast to the result cited above for arbitrary computable graphs. This result was improved by Schmerl [182] from $2k$ to $2k-1$, who also showed that $2k-1$ is the best possible result. It follows from the work of Bean and Schmerl that every highly computable planar graph has a computable 6-coloring. This result was improved by Carstens [20] from 6 to 5, but the highly computable four color problem remains open.

Bean showed in [12] that the set of k-colorings of a highly computable graph is always a computably bounded Π^0_1 class. (See Exercise 1.)

Conversely, Remmel [174] showed that every c. b. Π^0_1 class can actually be strongly represented by a highly computable k-coloring problem.

The problem of feasible graphs and colorings has been studied by Cenzer and Remmel in [37].

Exercises

XI.2.1. Show that for any highly computable graph $G = (V, E)$ and any finite k, the set of k-colorings of G may be represented by a computably bounded Π^0_1 class $P \subseteq \{0, 1, \ldots, k-1\}^{\mathbb{N}}$.

XI.2.2. Show that if G is a planar graph, then the set of 5-colorings of G always has cardinality $2^{\aleph_0}$ and hence not every c.b. Π^0_1 class may be represented as the set of 5-colorings of a planar graph. (Hint: every planar graph G is 4-colorable, by the theorem of Appel and Haken [3].)

XI.3 The Hamiltonian circuit problem

Let $G = (V, E)$ be a countably infinite graph. Two vertices u, v of G are *adjacent* if $(u, v) \in E$ and two edges (u_1, v_1) and (u_2, v_2) are *adjacent* if either $v_1 = u_2$ or $u_1 = v_2$. A *one-way* (respectively *two-way*) *Hamiltonian circuit* (or *Hamiltonian path*) for G is a one-to-one correspondence f between the natural numbers ω (resp. the integers Z) and V such that consecutive vertices are adjacent, i.e. $(f(i), f(i+1)) \in E$ for all i. The dual concepts are the *one-way* (respectively *two-way*) *Euler path*, which is a one-to-one correspondence between the natural numbers ω (resp. the integers Z) and E such that consecutive edges are adjacent. For each of these four notions, let us also define the associated notion of being such a path for a subgraph. That is, a one-way Hamiltonian sub-path for G will be a one-to-one embedding of the natural numbers into V such that consecutive vertices are adjacent. The other three definitions are similar.

In each case, the problem here is whether a given graph has such a path. We will focus on the sub-path problems.

Theorem XI.3.1. *For each of the following problems, the set of solutions can be represented as a Π^0_1 class. In cases (a) and (b), the class is bounded if the each vertex has finite degree and is c. b. if the graph is highly computable.*

(a) The one-way Hamiltonian (Euler) sub-paths starting from a fixed vertex in a recursive graph.

(b) The two-way Hamiltonian (Euler) sub-paths through a fixed vertex in a recursive graph.

(c) The one-way Hamiltonian (Euler) paths starting from a fixed vertex in a recursive graph.

(d) The two-way Euler paths through a fixed vertex in a recursive graph.

Proof. (a) Let the computable graph $G = (V, E)$ with fixed vertex v_0 be given. Then a one-way Hamiltonian (Euler) sub-path is a function f from ω into V with $f(0) = v_0$ such that $(f(n), f(n+1) \in E$ for all n and such that, for the Hamiltonian path, $m \neq n$ implies that $f(m) \neq f(n)$ and, for the Euler path, $m \neq n$ implies that the edges $(f(m), f(m+1))$ and $(f(n), f(n+1))$ are different. In each case, this clearly defines a Π^0_1 class P. If each vertex v has finite degree, then there is a function g such that all vertices joined to vertex v are $\leq g(v)$. It follows that we can compute a bound $h(m)$ for the possible value of $f(m)$ by letting $h(0) = v_0$ and in general $h(m+1) = sup\{g(v) : v \leq h(m)\}$. This shows that P is bounded. If G is highly computablee, then the function g may be taken to be computable, so that P is computably bounded.

(b) Again let the computable graph $G = (V, E)$ with fixed vertex v_0 be given. Then a two-way Hamiltonian (Euler) sub-path

$$\ldots, \pi(-1), \pi(0) = v_0, \pi(1), \ldots$$

can be represented as a function f from ω into V with $f(0) = v_0$ such that $(v_0, f(1)) \in E$, such that $(f(n), f(n+2)) \in E$ for all n and such that, for the Hamiltonian path, the function f is one-to-one, and, for the Euler path, no edge occurs twice in the list

$$\ldots, (f(3), f(1)), (f(1), f(0)), ((f(0), f(2)), (f(2), f(4)), \ldots.$$

It follows as in (a) that the class P of two-way Hamiltonian (Euler) sub-paths is a Π^0_1 class, is bounded if each vertex of G has finite degree, and is c. b. if G is highly computable.

(c) We first give the proof for one-way Hamiltonian paths. Recall that $V = \omega$ and represent a one-way Hamiltonian path

$$\pi = (\pi(0) = v_0, \pi(1), \pi(2), \ldots)$$

by a function f such that $f(2n) = \pi(n)$ and $f(2v+1) = n$ such that $v = \pi(n)$. This is clearly a one-to-one degree-preserving correspondence between the one-way Hamiltonian paths of G and the Π^0_1 class P. Then the Π^0_1 class P of solutions is the set of functions f such that $f(0) = v_0$, such that $(f(2n), f(2n+2)) \in E$ for all n, and such that, for all v and n, $f(2n) = v$ if and only if $f(2v+1) = n$. For the one-way Euler paths π, we take $f(2n) = \pi(n)$ and let $f(2[u, v] + 1) = n + 1$ such that $\pi(n) = u$ and $\pi(n+1) = v$ if $(u, v) \in E$ and otherwise $f(2[u, v]+1) = 0$. In either case, the assumption that G is highly computable does not necessarily imply that P is even bounded.

(d) Represent a two-way Hamiltonian path by a function f so that the path is given by $\ldots, f(4), f(1), f(0) = v_0, f(3), f(6), \ldots$ and such that $f(3v+2) = n$ such that $n \neq 2 \ mod\ 3$ and $f(n) = v$. Represent a two-way Euler path π again by a function f so that $\pi = \ldots, f(4), f(1), f(0) = v_0, f(3), f(6), \ldots$ and now such that $f(3[u, v] + 2) = n$ such that $n \neq 2 \ mod\ 3$ and $f(n) = u$ and $f(n+3) = v$. □

It follows that if each vertex of G has finite degree and G has a one-way or two-way Hamiltonian (Euler) sub-path, then it has such a sub-path which is computable in $\mathbf{0}''$. In the cases (c) and (d) of the Hamiltonian and Euler paths, we can only conclude, even for a highly computable graph G, that G has a solution recursive in some Σ^1_1 set. We leave the other usual corollaries for the reader.

Bean [13] showed that if G is highly computable and has an Euler path, then G will actually have a computable Euler path. This is not the case for Hamiltonian paths, by the following reasoning. If every highly computable graph G with a Hamiltonian path had a hyperarithmetic Hamiltonian path, then the set of highly computable graphs with Hamiltonian paths would be Π^1_1, by the Spector-Gandy theorem II.10.5. However, Harel [84] showed that the problem of the existence of (one-way or two-way) Hamiltonian paths in a highly computable graph is Σ^1_1-complete and therefore not Π^1_1. It follows that the set of Hamiltonian paths of a highly computable graph is not always a c. b. Π^0_1 class. That is, if it were always a c. b. class, then by Theorem every highly computable graph with a Hamiltonian path would have a Hamiltonian path computable in $\mathbf{0}'$ and hence a hyperarithmetic Hamiltonian path. This applies to one-way and two-way paths.

For the reverse direction, we have the following result of Bean [13].

Theorem XI.3.2. *For any c. b. Π^0_1 class P, there is a highly computable planar graph G and a one-to-one computable isomorphism between P and the set of Hamiltonian paths for G.*

Proof. Let P be the set of infinite paths through the highly computable tree T. G is constructed in stages, beginning with vertices 0 and 0^{out} and edge $(0, 0^{out})$. At stage n, let $\sigma_0, \sigma_1, \ldots, \sigma_m$ be the nodes of T at level n and introduce a circuit of $3(m+1)$ vertice in G given by

$$(\sigma_0^{out}, \sigma_0^{in}, \sigma_{0,1}, \sigma_1^{out}, \sigma_{1,2}, \ldots, \sigma_m^{out}, \sigma_m^{in}, \sigma_{m,0}.\sigma_0^{out}).$$

For every node τ_i at level $n-1$ and every successor σ_j at level n of τ_i, also add an edge $(\tau_i^{out}, \sigma_j^{in})$ to G. (For the two-way circuit, also add a vertex v_n and edge (v_{n-1}, v_n), where $v_0 = 0$. It is clear that G is a highly computable planar graph. The desired correspondence between P and the Hamiltonian paths of G is given as follows. The node σ_j of T follows the node τ_i on the infinite path through T if and only if the vertex σ_j^{in} immediately follows the vertex τ_j^{out} on the Hamiltonian path. □

It follows that there is a highly computable graph G which has Hamiltonian paths but has no computable Hamiltonian paths. Other corollaries are left to the reader.

This problem, posed by S. Ulam, is to show that for each partition of the vertex set V of a graph $G = (V, E)$ into sets of uniformly bounded cardinality, there is at least one set of the partition which is adjacent to m (or more) other sets of the partition. Here we say that two sets S_1 and S_2 are adjacent if there exist vertices $v_1 \in S_1$ and $v_2 \in S_2$ such that $(v_1, v_2) \in E$. The partition number m of a graph G is the least number m for which the statement is true. The vertex partition problem was studied by Cenzer and E. Howorka [27], who computed the vertex partition numbers of various well-known graphs, including the m-regular trees T_m and the planar mosaic graphs M_3, M_4 and M_6. The tree T_m may be viewed as $\{1, 2, \ldots, m\}^*$. The graphs M_3, M_4 and M_6 may be viewed as tilings of the plane by regular hexagons, squares and equilateral triangles. In each case, the partition number of the graph turns out to be the degree of the graph. In this situation, the Π^0_1 class arises from the dual problem. That is, given the graph G and numbers k and m, to find a k-partition P of the graph such that no set has m neighbors. Here a k-partition is a partition of V into sets of cardinality $\leq k$. The solution to such a problem may be represented as a function f from $V \times V$ into $\{0, 1\}$ which is to be the characteristic function of the equivalence relation with equivalence classes being the sets of the partition.

Theorem XI.3.3. *For any highly computable graph $G = (V, E)$, and any finite k and m, the set C of k-partitions of V such that no set in the partition is adjacent to m other sets may be represented by a Π^0_1 class in $\{0,1\}^{\mathbb{N}}$.*

Proof. Let $G = (V, E)$ be a highly recursive graph and let k, m be positive integers. Let C be the set of k-partitions of V such that no set in the partition is adjacent to m other sets. As indicated above, we may represent a partition by the characteristic function f of the corresponding equivalence relation. Let us assume that $V = \omega$ for simplicity and let C be the class of all such functions for which there is no set in the partition represented by f which has m neighbors. Now a function $f \in \{0,1\}^\omega$ will be in the class C if it satisfies the following conditions:

(i) $(\forall u)[f(u,u) = 1]$.

(ii) $(\forall u, v)[f(u,v) = f(v,u)]$.

(iii) $(\forall u, v, w)[f(u,v) = f(v,w) = 1 \rightarrow f(u,w) = 1]$.

(iv) $(\forall u_1, u_2, \ldots, u_{k+1})(\exists i, j \leq k+1)[f(u_i, u_j) = 0]$.

(v) $(\forall u_1, v_1, u_2, v_2, \ldots, u_m, v_m)[((\forall i, j \leq m)[f(u_i, u_j) = 1]$ & $(\forall i \leq m)[E(u_i, v_i)]) \rightarrow (\exists i, j \leq m)[f(v_i, v_j) = 1]]$.

The first three clauses are the requirement that f is the characteristic function of an equivalence relation. The fourth clause is the requirement that each set in the corresponding partition has cardinality $\leq k$ and the final clause is the requirement that no set in the partition is adjacent to m other sets. □

Chapter XII

Orderings

There are several problems associated with partially ordered sets (posets) and also with computable linear orderings and ordered structures.

Problems to be considered include the decomposition of a poset into chains and into antichains, as well as the problem of expressing a partial ordering as the intersection of finitely many linear orderings. For each computable instance of these problems, the set of solutions can be represented as a c. b. Π^0_1 class and can represent an arbitrary Π^0_1 class of separating sets.

For a computable linear ordering $\mathcal{A}$, we consider the problem of finding suborderings of type ω or ω^*, the problem of finding an ω-successivity or ω^*-successivity, and the problem of finding a self-embedding of $\mathcal{A}$.

Finally, we consider the problem of finding an ordering of a computable Abelian group or formally real field. As usual, the set of orderings can always be represented by a c. b. Π^0_1 class and Metakides and Nerode [151] showed that any c. b. class. On the other hand, Solomon [199] showed that not every c. b. Π^0_1 class can be represented as the set of orderings of a computable abelian group.

XII.1 Partial orderings

In this section we consider three problems associated with partially ordered sets (posets). Two of these are the dual problems of covering a poset with chains or with antichains. The third problem is the dimension problem, that is, expressing a poset as the intersection of linear orderings.

We first describe the problems and show that the solution set to a computable problem always forms a c. b. Π^0_1 class, and then apply the results of Part One to obtain corollaries which apply to the set of solutions of any such problem. We also consider for each problem, whether, conversely, the set of solutions to such a problem can represent any c. b. Π^0_1 class. For each problem, we show that the set of solutions to such a problem can represent the class of separating sets of any two disjoint c. e. sets and we apply the results of Part One to obtain corollaries which give the existence of "pathological" problems of each type.

Decomposition problems for posets

Here we start with a computable poset $\mathcal{A} = (A, \leq^A)$, which consists of a computable subset A of $\mathbb{N}$ and a computable partial ordering $\leq^A$. The *width* of $\mathcal{A}$ is the maximum cardinality of an antichain in $\mathcal{A}$ and the *height* of $\mathcal{A}$ is the maximum cardinality of a chain in $\mathcal{A}$.

(a) The first decomposition theorem we consider is Dilworth's theorem [58], which states that any poset $\mathcal{A}$ of width n can be covered by n chains. The problem here is to find such a covering of $\mathcal{A}$ by n chains and the set of solutions corresponds to the various coverings of $\mathcal{A}$ by n chains. The effective version of Dilworth's theorem has been analyzed by Kierstead in [102], where he showed that every computable poset $\mathcal{A}$ of width n can be covered by $(5^n - 1)/4$ computable chains, while for each $n \geq 2$, there are computable posets of width n which cannot be covered by $4(n-1)$ chains. See Kierstead's article [101] in this volume for details and more results.

Thus, the set of solutions of this problem for a computable poset $\mathcal{A}$ can be represented as the set of maps $f : A \to \{1, 2, \ldots, n\}$ such that $f^{-1}(\{i\}) = \{x \in A : f(x) = i\}$ is a chain for each i, which is clearly a c. b. Π^0_1 class.

(b) There is a natural dual to Dilworth's theorem which says that every poset of height n can be covered by n antichains. The problem again is to find such a covering. The effective version of the latter theorem was analyzed by Schmerl, who showed that every computable poset of height n can be covered by $(n^2 + n)/2$ computable antichains while for each $n \geq 2$, there is a computable poset of height n which cannot be covered by $(n^2 + n)/2 - 1$ computable antichains. Furthermore, Szeméredi and Trotter showed that there exist computabl partial orders of height n and computable dimension 2 which still cannot be covered by $(n^2 + n)/2 - 1$ computable antichains. These results are reported by Kierstead in [102].

(2) Dimension of posets problem The poset $\mathcal{A} = (A, R)$ is defined to be n-*dimensional* if there are n linear orderings of A, $(A, L_1), \ldots, (A, L_n)$, such that $R = L_1 \cap \cdots \cap L_n$. The notion of the dimensionality of posets is due to Dushnik and Miller, who showed in [68] that a countable poset (A, R) is n-dimensional if and only if it can be embedded as a subordering in the product ordering $\mathbb{Q}^n$, where $\mathbb{Q}$ is the set of rational numbers under the usual ordering. A (computable) poset (A, R) has *(computable) dimension* equal to d, for d finite, if there are d (computable) linear orderings $(A, L_1), \ldots, (A, L_d)$ such that $R = L_1 \cap \cdots \cap L_d$, but there are not $d-1$ (computable) linear orderings $(A, L'_1), \ldots, (A, L'_{d-1})$ such that $R = L'_1 \cap \cdots \cap L'_{d-1}$. Kierstead, McNulty and Trotter have analyzed in [105], the computable dimension of computable posets and have shown that in general, the computable dimension of a poset is not equal to its computable dimension.

Given a countable poset (A, R) with $A \subseteq \omega$, we can code a set of d linear orderings of A, $(A, L_1), \ldots, (A, L_d)$ as follows. Let $a_0 < a_1 < \cdots$ be an increasing enumeration of A. Then given d linear orderings of $\{a_0, \ldots, a_{n-1}\}$, there clearly are $(n+1)^d$ ways to extend the d linear orderings to d linear orderings on $\{a_0, \ldots, a_n\}$. One can fix some effective enumeration of these extensions for each n, so that it then becomes possible to code each d-tuple of linear orderings by a function $f : A \to \omega$ where $f(a_n) \leq (n+1)^d - 1$ for all n. Thus the set of solutions for the n-dimensionality problem of a computable poset (A, R) can be represented as the set of all $f : A \to \omega$ such that f codes an n-tuple, $(A, L_1), \ldots, (A, L_n)$, of linear orderings on A such that $R = L_1 \cap \cdots L_n$, which is a c. b. Π^0_1 class.

We state the first theorem and leave the details of the representation to the reader.

Theorem XII.1.1. *For each specific computably presented instance of one of the poset problems P listed above, the set of solutions can be represented as a c. b. Π^0_1 class.*

As usual, we can now derive a number of immediate corollaries from the results of Part One. We state only a few of these and leave the rest to the reader. For example, the following is true.

Theorem XII.1.2. *(a) If a computable poset $\mathcal{A}$ has a covering by n chains, then $\mathcal{A}$ can be covered by n chains $C_1, \ldots C_n$ such that $C_1 \oplus \cdots \oplus C_n$ has c. e. degree.*

(b) If $\mathcal{A} = (A, R)$ is a computable poset such that the family of sets $\{(A, L_1), (A, L_2), \ldots, (A, L_n)\}$ of n linear orderings such that $R = L_1 \cap L_2 \cap \cdots \cap L_n$ is countably infinite, then $\mathcal{A}$ has computable dimension $\leq n$.

(c) If a computable poset $\mathcal{A}$ has a covering by n antichains, but has no covering by n computable antichains, then for any countable sequence of nonzero degrees $\{\mathbf{a}_i\}$, $\mathcal{A}$ has a continuum of coverings $\{A_1, A_2, \ldots, A_n\}$ by n antichains, which are pairwise Turing incomparable and such that the degree of $\{A_1, A_2, \ldots, A_n\}$ is incomparable with each $\mathbf{a_i}$.

Next we consider the reverse direction of this correspondence.

Theorem XII.1.3. *Each of the three problems described above can strongly represent the c. b. Π^0_1 class of separating sets for any pair of disjoint infinite c. e. sets.*

Proof. Fix a pair A and B of infinite disjoint c. e. sets and computable enumerations $\{A^s\}_{s\in\omega}$ and $\{B^s\}_{s\in\omega}$ such that, for all s, $A^s, B^s \subseteq \{0, 1, \ldots, s\}$ and there is at most one element of $A \cup B$ which comes into $A \cup B$ at stage s.

(1) *The problem of covering a computable poset of width k by k chains.*
First consider the case $k = 2$. We begin with the poset $\mathcal{D}_0$ consisting of two one-way chains $\{a_{i,j} : i = 0, 1 \wedge j \in \omega\}$ and $\{b_{i,j} : i = 0, 1 \wedge j \in \omega\}$ where we have $a_{i,j} \leq a_{i,k}$ whenever $j < k$ and $a_{0,j} \leq a_{1,j}$ as well, and similarly for the

$b_{i,j}$. The two chains are linked by having $a_{0,j} \leq b_{1,j}$ and similarly $b_{0,j} \leq a_{1,j}$. Let us call the posets $\{a_{0,i}, a_{1,i}, b_{0,i}, b_{1,i}\}$ the *i-th block* of the poset $\mathcal{D}_0$. The i-th block of $\mathcal{D}_0$ is pictured in Figure XII.1(A).

Our final poset $\mathcal{D} = (D, \leq_D)$ will consist of the poset $\mathcal{D}_0$ together with an infinite computable set E whose relations to the elements of $\mathcal{D}_0$ and among themselves is to be specified in stages. Now it is clear that a decomposition of this poset, up to renaming the chains, is completely determined by the choice, for each i, of either

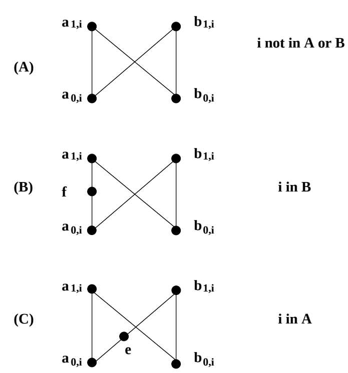
(a) putting $a_{0,i}$ and $a_{1,i}$ in one chain and $b_{0,i}$ and $b_{1,i}$ in the other, or

(b) putting $a_{0,i}$ and $b_{1,i}$ in one chain and $a_{1,i}$ and $b_{0,i}$ in the other.

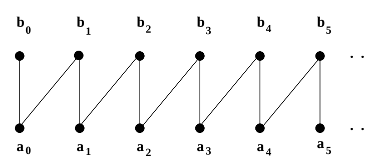
Thus we can think of a chain decomposition $f : D \to \{1,2\}$ as coding up a set C_f, where $i \in C_f$ if and only if we use choice (b) for the i-th component, that is, if and only if $f(a_{0,i}) = f(b_{1,i})$. Now the idea is to define the relations between the remaining computable set E so that we introduce an element e in the i-th component between $a_{0,i}$ and $a_{1,i}$ if $i \in B$, see Figure XII.1(B). This will force e, $a_{0,i}$ and $a_{1,i}$ to be in the same chain. We introduce an element f in the i-th component between $b_{0,i}$ and $a_{1,i}$ if $i \in A$, see Figure XII.1(C). This will force f, $b_{0,i}$ and $a_{1,i}$ to be in the same chain. Finally we have no new element in the i-th component if $i \notin A \cup B$. It is not difficult to see that this can be accomplished so as to ensure that $\mathcal{D}$ is a computable poset of width 2 and that such actions will ensure that the correspondence $f \to C_f$ will be a one-to-one degree-preserving correspondence between the decompositions of $\mathcal{D}$ into two chains and the separating sets of $A \cup B$. We leave the details to the reader. For the case where $k > 2$, one simply adds to the poset described a set of $k-2$ computable infinite one-way chains, all of whose elements are incomparable with $\mathcal{D}$ and so that elements from different chains are also incomparable.

(2) *The problem of covering a computable poset of width k by k antichains.* Again we shall initially consider the case $k = 2$. The poset $\mathcal{D} = (D, \leq_D)$ will consist of two parts. The first part of the poset will consist of a computable antichain $c_0, c_1, \ldots$, and the second part will consist of two antichains $a_0, a_1, \ldots$ and $b_0, b_1, \ldots$ where $a_0 \leq b_0$ and, for each i, $a_{i+1} \leq b_i$ and $a_{i+1} \leq b_{i+1}$, see Figure XII.1.

We will complete the partial ordering on $\mathcal{D}$ by specifying the relations between the two parts in stages. Clearly, up to renaming the antichains, there is a unique decomposition of the second part of the poset into two antichains. We think of a decomposition of $\mathcal{D}$ into two antichains as coding up a set C_f by specifying $i \in C_f$ if and only if f assigns c_i to the same antichain as the a's. Then, for each i, we define c_i to be greater than a_s if $i \in A^{s+1} \setminus A^s$ and incomparable to a_s otherwise, and define c_i to be less than b_s if $i \in B^{s+1} \setminus B^s$ and incomparable to b_s otherwise. It is then easy to check that $\mathcal{D}$ is a computable poset of height two and that, up to renaming the antichains, the correspondence $f \to C_f$ is a one-to-one degree preserving correspondence between decompositions of $\mathcal{P}$ into two antichains and separating sets of $A \cup B$. For the case where



(A)
a 1,i
b 1,i
a 0,i
b 0,i
i not in A or B
(B)
f
i in B
(C)
e
i in A


b0
b1
b2
b3
b4
b5
...
...
a0
a1
a2
a3
a4
a5

$k > 2$, one simply adds to the poset described a set of $k-2$ computable infinite antichains, all of whose elements are comparable with every element of $\mathcal{D}$ and so that elements from different antichains are also comparable.

(3) *The problem of expressing a computable poset $\mathcal{P} = (P, \leq P)$ of dimension d as the intersection of d linear orderings.*

We consider the case oftwo dimensional partial orderings. First we partition $\mathbb{N}$ into two infinite computable sets $C = \{c_0 < c_1 < \cdots\}$ and $D = \{d_0 < d_1 < \cdots\}$. For each i, we let $C_i = \{c_{5i}, c_{5i+1}, c_{5i+2}, c_{5i+3}, c_{5i+4}\}$. We shall define a computable partial ordering $<_P$ on ω in stages. Given any two sets E and F, $E <_P F$ will denote that, for any $e \in E$ and $f \in F$, $e <_P f$. We start by defining $<_P$ so that $C_0 <_P C_1 <_P C_2 <_P \cdots$. This means that if $<_1$ and $<_2$ are two linear orderings such that $<_1 \cap <_2 = <_P$, then the only difference between $<_1$ and $<_2$ on C is how $<_1$ and $<_2$ order the elements within the blocks C_i. For each block C_i, $<_P$ is defined so that we have the Hasse diagram in Figure XII.1(A).

It is then easy to check that, up to a permutation of the indices of the linear orderings $<_1$ and $<_2$, there are precisely two ways to define $<_1$ and $<_2$ on C_i so that $<_1 \cap <_2$ equals $<_P$ restricted to A_i, namely,

(I) $c_{5i} <_1 c_{5i+1} <_1 c_{5i+2} <_1 c_{5i+3} <_1 c_{5i+4}$ and
$c_{5i+2} <_2 c_{5i+4} <_2 c_{5i+3} <_2 c_{5i} <_2 c_{5i+1}$, or

(II) $c_{5i} <_1 c_{5i+1} <_1 c_{5i+2} <_1 c_{5i+4} <_1 c_{5i+3}$ and
$c_{5i+2} <_2 c_{5i+3} <_2 c_{5i+4} <_2 c_{5i} <_2 c_{5i+1}$.

Note that the difference between (I) and (II) is that in the ordering where the elements c_{5i}, c_{5i+1} precede the elements $c_{5i+2}, c_{5i+3}, c_{5i+4}$, we have c_{5i+3} preceding c_{5i+4} in (I), while in (II) c_{5i+4} precedes c_{5i+3}.

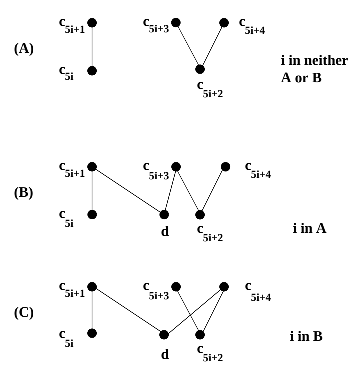
We can thus use a pair of linear orderings $<_1$ and $<_2$ such that $<_1 \cap <_2 = <_P$ is defined within the blocks C_i to code a set $S(<_1, <_2) \subseteq \omega$ by declaring $i \in S$ if and only if $<_1$ and $<_2$ are of type (I) on C_i.

The key to our ability to code up a tree of separating sets for a pair of disjoint c. e. sets A and B is the following. If we add an element d to the Hasse diagram as pictured in Figure XII.1(B), then only linear orderings $<_1$ and $<_2$ of type (I) can be extended to $C_i \cup \{d\}$ so that $<_1 \cap <_2 = <_P$ and if we add an element d to the Hasse diagram as pictured in Figure XII.1(C), then only linear orderings $<_1$ and $<_2$ of type (II) can be extended to $C_i \cup \{d\}$ so that $<_1 \cap <_2 = <_P$.

That is, it is easy to check that, up to a permutation of indices there is only one way to define linear orderings $<_1$ and $<_2$ on $C_i \cup \{d\}$ so that $<_1 \cap <_2 = <_P$ if $<_P$ has the Hasse diagram as pictured in Figure XII.1(B), namely

(I'): $c_{5i} <_1 d <_1 c_{5i+1} <_1 c_{5i+2} <_1 c_{5i+3} <_1 c_{5i+4}$ and
$c_{5i+2} <_2 c_{5i+4} <_2 d <_2 c_{5i+3} <_2 c_{5i} <_2 c_{5i+1}$.

Similarly, up to a permutation of indices, there is only one way to define linear orderings $<_1$ and $<_2$ on $C_i \cup \{d\}$ so that $<_1 \cap <_2 = <_P$ if $<_P$ has the Hasse diagram as pictured in Figure 5 (C), namely


(A)
c_{5i+1}
c_{5i}
c_{5i+3}
c_{5i+4}
c_{5i+2}
i in neither A or B
(B)
c_{5i+1}
c_{5i}
c_{5i+3}
c_{5i+4}
d
c_{5i+2}
i in A
(C)
c_{5i+1}
c_{5i}
c_{5i+3}
c_{5i+4}
d
c_{5i+2}
i in B

(II'): $c_{5i} <_1 d <_1 c_{5i+1} <_1 c_{5i+2} <_1 c_{5i+4} <_1 c_{5i+3}$ and
$c_{5i+2} <_2 c_{5i+3} <_2 d <_2 c_{5i+4} <_2 c_{5i} <_2 c_{5i+1}$.

Now to complete our definition of $<_P$ on ω, we proceed in stages as follows.

Stage 0 If $i \in A^0$, let $C_{i-1} <_P \{d_0\} <_P C_{i+1}$, and define $<_P$ on $C_i \cup \{d_0\}$ so that we have a Hasse diagram as in Figure XII.1(B). If $i \in B^0$, let $C_{i-1} <_P \{d_0\} <_P C_{i+1}$ and define $<_P$ on $C_i \cup \{d_0\}$ so that we have a Hasse diagram as in Figure XII.1(C). If $A^0 \cup B^0 = \emptyset$, define $\{d_0\} <_P C$. Note this defines $<_P$ on all of $C \cup \{d_0\}$ by transitivity.

Stage $s > 0$. Assume we have defined $<_P$ on $C \cup \{d_0, \ldots, d_{s-1}\}$ so that for all $j < s$, $C_{i-1} <_P \{d_j\} <_P C_{i+1}$ if $i \in (A^j \cup B^j) \setminus (A^{j-1} \cup B^{j-1})$ and $\{d_j\} <_P C \cup \{d_0, \ldots, d_{j-1}\}$ otherwise. Then if $i \in A^s \setminus A^{s-1}$, let $C_{i-1} <_P \{d_s\} <_P C_{i+1}$ and define $<_P$ on $C_i \cup \{d_s\}$ so that we have a Hasse diagram as pictured in Figure XII.1(B). If $i \in B^s \setminus B^{s-1}$, let $C_{i-1} <_P \{d_s\} <_P C_{i+1}$ and define $<_P$ on $C_i \cup \{b_s\}$ so that we have a Hasse diagram as pictured in Figure XII.1(C). If $(A^s \cup B^s) \setminus (A^{s-1} \cup B^{s-1}) = \emptyset$, define $\{d_s\} <_P C \cup \{d_0, \ldots, d_{s-1}\}$. Again this defines $<_P$ on all of $C \cup \{d_0, \ldots, d_s\}$ by transitivity.

This completes the proof of Theorem XII.1.3. □

As usual, there are a number of immediate corollaries and we state only a few.

Theorem XII.1.4. *(a) There is a computable poset of width k which has no covering by k chains.*

(b) There is a computable poset $\mathcal{A}$ of height k such that any two distinct coverings of $\mathcal{A}$ by k antichains are Turing incomparable, where distinct means not obtainable from the other by a permutation of the antichains in combination with the shifting of a finite number of elements.

(c) If $\mathbf{a}$ is a Turing degree and $\mathbf{0} <_T \mathbf{a} \leq_T \mathbf{0}'$, then there is a computable poset $\mathcal{A} = (A, R)$ of dimension d, but not of computable dimension d such that there exists a set $\{(A, L_1), \ldots, (A, L_d)\}$ of degree $\mathbf{a}$ of linear orderings such that $R = L_1 \cap \cdots \cap L_d$.

XII.2 Linear orderings

There are three problems discussed in this subsection related to a given computable linear ordering $\mathcal{A} = (A, \leq^A)$.

(1) The problem of finding a subordering of $\mathcal{A}$ of type ω or of type ω^*.

(2) The problem of finding an ω-successivity or an ω^*-successivity in $\mathcal{A}$.

(3) The problem of find a self-embedding of $\mathcal{A}$.

(1) Suborderings of type ω or ω^*

A standard classical result is that any infinite linear ordering has a subordering $\{f(0), f(1), \ldots\}$ of order type either ω or ω^* (the order type of the negative integers). Tennenbaum and independently Denisov showed that there is an infinite computable linear ordering of order type $\omega + \omega^*$ which has no computably enumerable subordering of either type (see Rosenstein [179] or Downey [62]). The suborderings of type ω (respectively ω^*) are simply the functions $f : \omega \to A$ such that $f(n) \leq^A f(n+1)$ (resp. $f(n+1) \leq^A f(n)$) for all n. Thus in each case the set of solutions to the problem of finding such a subordering is a Π^0_1 class, but is clearly not bounded. For example, if $\mathcal{A}$ is the standard ordering $(\omega, \leq)$, then the class of suborderings of $\mathcal{A}$ of type ω is just the class of all increasing sequences of natural numbers, which is homeomorphic to ω^ω and not even compact. We observe that the class of suborderings of type ω is always a perfect set, since for any such subordering f and any n, there is another subordering of type ω given by $(f(0), f(1), \ldots, f(n), f(n+2), f(n+4), \ldots)$.

Theorem XII.2.1. *For any computable linear ordering $\mathcal{A} = (A, \leq^A)$, the class of suborderings of $\mathcal{A}$ of type ω (respectively, of type ω^*) is a perfect Π^0_1 class.*

Thus all we can say is that if a computable linear ordering has a subordering of type ω (respectively, type ω^*), then it has such a subordering which is computable in some Σ^1_1 set. It was shown by Manaster that any computable linear ordering has a Π^0_1 subordering of type ω or of type ω^* (see Downey [62]).

(2) Successivities

An element b of $\mathcal{A}$ is said to be the *successor* of an element a if $a <^A b$ and there is no c such that $a <^A c <^A b$; in such a case, we write $b = S_A(a)$. We say that a subordering f of type ω in $\mathcal{A}$ is *an ω-successivity* if $f(n+1)$ is the successor of $f(n)$ in the linear ordering for each n, and similarly define an $\omega^* -$ *successivity*. Then the family P of ω-successivities is a Π^0_1 class and likewise the family of ω^*-successivities.

Observe that the class of ω-successivities of the standard ordering on ω consists of all sequences $(n, n+1, n+2, \ldots)$ and is thus a countable set in which all elements are isolated. As for the suborderings above, this class is not necessarily compact.

In general, there is at most one ω-successivity f for each starting element $f(0) = a$, so that every member of the class P of ω-successivities is isolated; a class with this property is said to be *scattered*. Clearly P is also countable. Furthermore, we can define a bounded computable tree T with $P = [T]$ by $(a_0, a_1, \ldots, a_n) \in T$ if and only if

$$(\forall i < n)[a_i <_A a_{i+1} \;\&\; (\forall m < a_{i+1})\neg(a_i <_A m <_A a_{i+1})].$$

A similar argument applies for ω^*-successivities.

Theorem XII.2.2. *For any computable linear ordering $\mathcal{A} = (A, \leq^A)$, the class of ω-successivities (respectively ω^*-successivities) of $\mathcal{A}$ is a scattered, bounded Π^0_1 class.*

As an immediate application, we have the following.

Corollary XII.2.3. *Every ω-successivity (respectively ω^*-successivity) of a computable linear ordering $\mathcal{A}$ is computable in $\mathbf{0}'$.*

This of course may also be proven directly from the definition of a successivity. It follows from the result of Tennenbaum and Denisov that there is a computable linear ordering of type $\omega + \omega^*$ which has no computable ω-successivity.

(3) Self-embeddings

Another classical result is due to Dushnik and Miller [68], who showed that an infinite countable linear ordering always has a non-trivial self-embedding. Hay, Manaster and Rosenstein [86] constructed a computable linear ordering of type ω with no non-trivial computable self-embedding. A map $f : A \to A$ is a self-embedding of $\mathcal{A}$ if, for all a and b, $f(a) \leq^A f(b)$ if and only if $a \leq b$. The family of self-embeddings of a computable linear ordering is again seen to be a Π^0_1 class. For the standard ordering on ω, it is clear that a self-embedding is the same thing as a subordering of type ω. Thus the class of self-embeddings need not be compact.

Now $\mathcal{A}$ always has a computable self-embedding, namely the identity function. If $\mathcal{A}$ has a non-trivial self-embedding, then we can fix an element a and consider the Π^0_1 class of self-embeddings f such that $f(a) \neq a$. It follows as usual that $\mathcal{A}$ at least has a non-trivial self-embedding which is computable in some Σ^1_1 set.

Theorem XII.2.4. *For any computable linear ordering $\mathcal{A} = (A, \leq^A)$, the class of self-embeddings of $\mathcal{A}$ is a Π^0_1 class.*

Theorem XII.2.5. *For any computable linear ordering $\mathcal{A} = (A, \leq^A)$, if $\mathcal{A}$ has a non-trivial self-embedding, then $\mathcal{A}$ has a self-embedding computable in a Σ^1_1 set.*

Downey and Lempp [67] showed that the proof-theoretical stregnth of the Dushnik-Miller theorem is ACA, which implies that every computable linear ordering has a self-embedding which is computable in $\mathbf{0}'$.

XII.3 Ordered algebraic structures

In this section, we consider two problems:

(1) The problem of finding an ordering of an Abelian group.

(2) The problem of finding an ordering of a formally real field.

In each case, the set of solutions to a given effective problem can always be represented by a *r.b.* Π^0_1 class and in case (2), any *r.b.* Π^0_1 class can be represented by such a set.

In this section, we will assume that a computably presented group, ring, or field is given by computable addition, subtraction, multiplication and division functions on the set ω, as appropriate. A *c. e. ring* is the quotient of a computable ring modulo a c. e. ideal and a *c. e. group* is the quotient of a computable group modulo a c. e. normal subgroup. An ordering will be represented by the cone of positive elements.

A formally real field is a field F such that no sum of (non-zero) squares equals zero. A field $(F, +^F, \cdot^F)$ is said to be *ordered* by the relation $\leq$ provided that $\leq$ is a linear ordering such that for all $a, b, c \in F$,

(i) $a \leq b \rightarrow a +^F c \leq b +^F c$.

(ii) $(0 \leq a \;\&\; 0 \leq b) \rightarrow 0 \leq a \cdot^F b$.

An ordering for a commutative group $(G, +^G, 0^G)$ is defined similarly except in this case the ordering $\leq$ need only satisfy condition (i).

The set $C = C_{\leq} = \{a \in F : 0 \leq a\}$ clearly satisfies the following for any $a, b \in F$:

(i) $a, b \in C \rightarrow a +^F b \in C$.

(ii) $a, b \in C \rightarrow a \cdot^F b \in C$.

(iii) $(a \in C \;\&\; 0^F -^F a \in C) \iff a = 0^F$.

(iv) $a \in C \;\vee\; 0^F -^F a \in C$.

A subset C of F satisfying (i) to (iv) is said to be a *positive cone* of F. Thus any linear ordering of F defines a positive cone and conversely any positive cone C of F defines a linear ordering by

$$a \leq b \iff b -^F a \in C.$$

Thus we will identify the set of linear orderings of a field F with the set of positive cones of F.

For a commutative group $(G, +^G, 0^G)$, a cone C need only satisfy (i), (iii) and (iv).

The classical result of Artin-Schreier [5] is that any formally real field can be ordered. Craven showed in [55] that any closed subset C of the Cantor space can be represented as the set of orderings of some formally real field F. Metakides and Nerode [150] made this proof effective by showing that if C is a Π^0_1 class, then F may be taken to be a computable field. Downey and Kurtz observed that the field F may have additional orderings which are compatible with the group structure although not compatible with the field structure. The classical result for groups is due to Levi [128], who showed that an Abelian group can be ordered if and only if it is torsion-free. Downey and Kurtz constructed in [66] a computable group isomorphic to $\oplus_\omega Z$ which has no computable ordering.

Theorem XII.3.1. *For each specific c. e. instance of the problems (1) and (2) listed above, the set of solutions can be represented as a c. b.* Π^0_1 *class.*

Proof. For computable structures, this is immediate from the discussion above. For a c. e. structure, say, $F = R/I$, observe that a positive cone C on R/I corresponds to a subset C' of R satisfying clauses (i), (ii) and (iv) along with the following modified version of clause (iii).

(iii) $(a \in C' \ \& \ 0^F -^F a \in C') \iff a \in I.$

We leave it to the reader to translate these four clauses into a definition of a computable tree T such that $[T]$ represents the set of positive cones on F. The proof for ordered groups is similar. □

We can as usual derive a number of immediate corollaries from the results of Part One. For example,

Theorem XII.3.2. (a) *Any c. e. presented group which has an ordering has an ordering of c. e. degree.*

(b) *If the set of orderings of the c. e. presented group G is countably infinite and nonempty, then G has a computable ordering.*

(c) *If the c. e. presented field F has only finitely many orderings, then every ordering of F is computable.*

Next we turn to the other direction of our correspondence, that is, representing an arbitrary Π^0_1 class by the set of solutions to certain of these problems. The problem of orderings of formally real fields was solved by Metakides and Nerode in [151].

Theorem XII.3.3. *Any c. b. Π^0_1 class P can be represented by the set of orderings of a formally real field.*

Proof. Let the computable tree $T \subseteq \{0,1\}^{<\omega}$ be given so that $P = [T]$. The construction begins with the underlying ring $R = \mathbf{Q}[x_i : i \in \omega]$ (the ring of polynomials with rational coefficients in infinitely many variables). We define a computable maximal ideal of R such that the set of orderings of the field R/I represents $[T]$. We sketch a proof is which is somewhat different from that in [151].

The first step of our construction is to adjoin to $\mathbf{Q}$ the radicals $\sqrt{p_i}$, where p_i is the i-th prime. That is, we put $x_i^2 - p_i$ into I for each i. Thus we have initially a continuum of possible orderings on $Q[\sqrt{p_i} : i < \omega]$, where to each $\Pi \in \{0,1\}^\omega$ there corresponds the ordering $R(\Pi)$ determined by taking $x_i > 0$ if $\Pi(i) = 0$ and $x_i < 0$ if $\Pi(i) = 1$. Now for any $\sigma \notin T$, we use an auxiliary variable y_σ to eliminate the ordering corresponding to σ in the following manner. We uniformly and effectively define, for σ of length n, a polynomial $f_\sigma(x_0, \dots, x_{n-1})$ such that for $(e_0, \dots, e_{n-1}) \in \{0,1\}^n$, $f_\sigma((-1)^{e_0}\sqrt{2}, (-1)^{e_1}\sqrt{3}, \cdots, (-1)^{e_{n-1}}\sqrt{p_{n-1}}) < 0$ if and only if $(e_0, \dots, e_{n-1}) = \sigma$. Then we add to I the polynomial $y_\sigma^2 = f_\sigma(x_0, \dots, x_{n-1})$, thus adjoining to our field a square root for $f_\sigma(x_0, \dots, x_{n-1})$. It follows that if $\sigma \prec \Pi$, then the ordering $R(\Pi)$ is not compatible with the field, since it forces a negative element to have a square root. The function f_σ is defined to be $f_\sigma(x_0, \ldots, x_{n-1}) = c_\sigma - (-1)^{\sigma(0)}x_0 - \cdots - (-1)^{\sigma(n-1)}x_{n-1}$, where c_σ is the least integer c such that $\sqrt{2} + \sqrt{3} \cdots + \sqrt{p_{n-1}} > c$.

(For example, let $\sigma = (0,1)$. Then we want $f_\sigma(\sqrt{2}, -\sqrt{3}) < 0$, $f_\sigma(\sqrt{2}, \sqrt{3}) > 0$, $f_\sigma(-\sqrt{2}, -\sqrt{3}) > 0$, and $f_\sigma(-\sqrt{2}, \sqrt{3}) > 0$. We compute that $3 < \sqrt{2} + \sqrt{3} < 4$ and define $f_\sigma(x_0, x_1) = 3 - x_0 + x_1$.)

Finally, to prevent any additional orderings from arising due to the new roots in the field, we add a sequence of roots $y_{i,j}$ to the field such that $y_{i,0} = y_i$ and $y_{i,j+1}^2 = y_{i,j}$. Thus each y_i and each $y_{i,j}$ is forced to be positive. □

This representation theorem has, as usual, a number of immediate corollaries of which we state only a few.

Theorem XII.3.4. *(a) There is a computable formally real field which has no computable ordering.*

(b) There is a computable formally real field which has continuum many orderings and such that any two distinct orderings are Turing incomparable.

(c) There is a computable formally real field F such that if $\mathbf{a}$ is the degree of any ordering of F and $\mathbf{b}$ is a r.e. degree with $\mathbf{a} \leq_T \mathbf{b}$, then $\mathbf{b} =_T \mathbf{0}'$.

(d) There is a computable formally real field F which has a unique non-computable ordering $\leq_0$, such that this ordering $\leq_0$ has degree $\mathbf{0}'$, and such that for any other ordering $\leq$ of F, there is some finite subset A of F such that for any ordering $\leq'$ of R, if $\leq$ agrees with $\leq$ on A, then $\leq = \leq'$.

D. R. Solomon [199] showed that the analogue of the Metakides-Nerode theorem fails for Abelian groups, that is, every abelian group has either two or has infinitely many orderings and therefore not every Π^0_1 class may be represented as the set of orderings of a computable abelian group.

Chapter XIII

Infinite Games

The set of winning strategies for an effective, closed $\{0,1\}$-game of perfect information was shown in [36] to strongly represent any c. b. Π^0_1 class. We will consider more general closed games here.

For any subset C of $\mathbb{N}^{\mathbb{N}}$, the infinite game $G(C)$ of perfect information is defined as follows. Two players, I and II, alternately play an infinite sequence $z = (x(0), y(0), x(1), y(1), \ldots)$ and player II wins this play if $z \in C$. A *strategy* for Player II is a (partial) function Θ from $\omega^{<\omega}$ into ω. For any play $x = (x(0), x(1), \ldots)$ of the game by Player I, the play $\Theta(x)$ of the game when Θ is applied to x is given by $(x(0), y(0), x(1), \ldots)$, where, for each n, $y(n) = \Theta((x(0), y(0), \ldots, y(n-1), x(n))$. The strategy Θ is said to be a *winning strategy* for Player II in the game $G(C)$ if, for any play x of the game by Player I, $\Theta(x) \in C$. The notion of a strategy and a winning strategy for Player I is similarly defined. The game $G(C)$ is said to be *determined* if one of the two players has a winning strategy. Gale and Stewart showed in [75] that the game $G(C)$ is determined if C is either closed or open. For a closed set C, we have $C = [T]$ for some tree T, and we will sometimes refer to $G(C)$ as $G(T)$. We say that $G(T)$ is a *computably presented Gale-Stewart game* if T is a recursive tree and that $G(T)$ is *bounded (respectively, computably bounded)* if the set $[T]$ is bounded (resp. c. b.).

As pointed out in [36], strategies need to be coded to avoid always having a perfect set of winning strategies.

Let $\tau_0, \tau_1, \ldots$ effectively enumerate the nonempty elements of $\omega^{<\omega}$ in increasing order where we order the sequences by the sum of the sequence plus the length and then lexicographically. Thus $\tau_0 = (0), \tau_1 = (00), \tau_2 = (1), \ldots$. For each $\tau \in \omega^{<\omega}$, let $n(\tau)$ be the unique n such that $\tau = \tau_n$. Then an arbitrary sequence $z = (z(0), z(1), \ldots) \in \omega^\omega$ codes a strategy Θ_z for Player II in the following manner. For any play $x = (x(0), x(1), \ldots)$ of Player I, the strategy Θ_z produces the following response $y = (y(0), y(1), \ldots)$ by Player II. First, $y(0) = z(n((x(0)))$ and for any k, $y(k+1) = z(n)$, where $\tau_n = (x(0), \ldots, x(k))$, that is, $\Theta_z((x(0), y(0), \ldots, y(k-1), x(k)) = z(n)$. Thus $z(0) = \Theta_z((0))$, $z(1) = \Theta_z(0, \Theta_z(0), 0)$, $z(2) = \Theta_z((1))$, and so on. It is clear that the result $\Theta_z(x)$ of applying this strategy to a play $x = (x(0), x(1), \ldots)$ of the game by Player I can be computed from x and z by a computable functional. For a finite sequence $z\lceil n = (z(0), \ldots, z(n-1))$, $\theta_{z\lceil n}$ is a *partial strategy* which, applied to any partial play $x\lceil m+1 = (x(0), \ldots, x(m))$ of Player I with $n(x\lceil m) < n$, gives a partial response $\theta_{z\lceil n}((x(0), y(0), \ldots, y(m-1), x(m))) = y(m)$ where for all $r \leq m$, $y(r) = z(k_r)$ if $n((x(0), \ldots, x(r)) = k_r$.

Now, for any tree $T \subseteq \omega^{<\omega}$, let $WS(T)$ be the set of codes

$$z = (z(0), z(1), \ldots) \in \omega^\omega$$

for winning strategies of Player II in the game $G(T)$.

Theorem XIII.0.5. *For any computable tree T:*

(a) $WS(T)$ is a Π^0_1 class.

(b) If T is finitely branching, then $WS(T)$ is bounded.

(c) If T is highly computable, then $WS(T)$ is computably bounded.

Proof. We will define a computable tree Q such that $WS(T) = [Q]$, as follows. First $\emptyset$ is in Q and then for any $\sigma = (z(0), \ldots, z(n-1))$, $\sigma \in Q$ if and only if, for all sequences $\nu = (x(0), \ldots, x(r-1))$ where $n(\nu) < n$, the result of applying the partial strategy θ_σ coded by σ to the partial play ν is in T. It follows from the discussion above that there is a computable function g such that, for each n, the value $z(n)$ of a coded strategy gives the play $y(g(n))$ of player II at step $g(n)$. If T is finitely branching, then there are only finitely many possible choices for $y(g(n))$ which allow player II to win the game, so that only finitely many values are possible for $z(n)$. This makes $WS(T)$ bounded. If T is highly computabl, then we can actually compute a list of these possible values from $g(n)$. Thus $WS(T)$ will be computably bounded. □

As usual, we can derive a number of immediate corollaries. We state the following and leave the rest to the reader.

Theorem XIII.0.6. *Let T be a computable tree such that player II has a winning strategy for the Gale-Stewart game $G(T)$.*

(a) There is a winning strategy which is computable in some Σ^1_1 set and, if there are only finitely many winning strategies, then each winning strategy is hyperarithmetic.

(b) If T is finitely branching, then there is a winning strategy which is computable in $0''$.

(c) If T is highly computable, then there is a winning strategy of c. e. degree and, if there are only countably many winning strategies, then there is a computable winning strategy.

(d) If T is highly computable and there is no computable winning strategy, then there is a continuum of pairwise Turing incomparable winning strategies.

Next we consider the set of winning strategies for Player I (who is trying to get the play into the open set). Let $WS'(T)$ be the set of codes for winning strategies of Player I. Note that for a Gale-Stewart game $G(C)$, the set of winning strategies of Player I is in general not a closed set or an open set.

Theorem XIII.0.7. *For any computable tree T:*

(a) $WS'(T)$ is a Π^1_1 class.

(b) If T is finitely branching, then $WS(T)$ is an open set.

(c) If T is highly computable, then $WS(T)$ is a Σ^0_1 class.

Proof. We describe the class of actual strategies Θ and leave it to the reader to translate this into the coded strategies as in the proof of Theorem XIII.0.5. In general, Θ is a winning strategy for Player I if and only if, for all plays y of Player II, the result $\Theta(y)$ of the game when Player I uses the strategy Θ is not in the set $[T]$, that is,

$$(\forall y)(\exists n)[(x(0), y(0), x(1), y(1), \ldots, x(n), y(n)) \notin T]$$

where $x(i+1) = \Theta((x(0), y(0), \ldots, x(i), y(i)))$ for all i.

If T is finitely branching, let $f(n)$ give an upper bound for the possible values of $\sigma(n)$ for any $\sigma \in T$. Then we can use König's Infinity Lemma as usual to express this in the form

$$(*)\ (\exists n)(\forall (y(0), y(1), \ldots, y(n)))[(x(0), y(0), x(1), y(1), \ldots, x(n), y(n)) \notin T],$$

where each $y(i) \leq f(2i)$, so that the $(\forall)$ quantifier is bounded, which shows that $WS'(T)$ is an open set.

Finally, if T is highly computable, then we may take the function f to be computable, so that the characterization (*) above makes $WS'(T)$ a Σ^0_1 class. □

Theorem XIII.0.8. *Let T be a computable tree such that Player I has a winning strategy for the Gale-Stewart game $G(T)$.*

(a) *There is a Δ^1_2 winning strategy and, if there are only finitely many winning strategies, then each winning strategy is Δ^1_2.*

(b) *If T is finitely branching, then there is a computable winning strategy.*

Proof. (a) This follows from the theorem that Δ^1_2 is a basis for Π^1_1, which is a corollary of the Novikov-Kondo-Addison Uniformization Theorem (see Hinman [87], pp. 196-198) for details).

(b) Since $WS'(T)$ is open and nonempty, there must be an interval of coded winning strategies, which of course will contain a computable strategy. □

Now we consider the reverse direction of the correspondences given in Theorems XIII.0.5 and XIII.0.7.

Theorem XIII.0.9. *For any computable tree Q, there is a computable tree T and an effective one-to-one degree preserving correspondence between the Π^0_1 class $[Q]$ of infinite paths through Q and the class $WS(T)$ of winning strategies for the effectively closed game $G(T)$. If Q is finitely branching (respectively highly computable), then T may be taken to be finitely branching (resp. highly computable).*

Proof. Let the computable tree Q be given. Our basic idea is that each path $\Pi = (\pi(0), \pi(1), \ldots) \in [Q]$ should correspond to a strategy Θ_Π which acts as follows. Given any partial play, $((x(0), \ldots, x(m))$ of Player I, Θ_Π will respond with

$$\Theta_\Pi((x(0), y(0)), \ldots, y(m-1), x(m))) = y(m)$$

where $y(m) = 0$ if $x(i) > 0$ for any $i \leq m$ and $y(m) = \pi(m)$ if $x(i) = 0$ for all $i \leq m$. Thus whenever Player I plays a value $x(i) > 0$, then ever after Θ_Π will respond with a 0 and if Player I plays all 0's, then Θ_Π will respond by reproducing the path Π. It is easy to see that when we code the strategy Θ_Π via a sequence $z = (z(0), z(1), \ldots)$ that z will have the same Turing degree as Π. Thus the correspondence $\Pi \to \Theta_\Pi$ will be an effective 1:1 degree preserving correspondence. Thus all we need to do is recursively define a computable tree $T \subseteq \omega^{<\omega}$ so that $WS(T) = \{z : z \text{ is a code of } \Theta_\Pi \text{ for some } \Pi \in Q\}$. We begin with sequences (a, b) of length 2 by putting $(a, b) \in T$ if and only if, either $a > 0$ and $b = 0$ or $a = 0$ and $(b) \in Q$. (This ensures that if Player I starts with an $x > 0$, then any winning strategy Θ for Player II must respond with a 0, whereas if Player I starts with a 0, then Player II must respond by starting a sequence in Q. Similar remarks will apply to the subsequent nodes we put in T.) Then, for each n and each $\tau = (x(0), y(0), \ldots, x(n), y(n)) \in T$, do the following:

(1) If $x(k) > 0$ for some $k \leq n$, then put $\tau^\frown a^\frown 0 \in T$ and leave $\tau^\frown a^\frown b$ out of T for all a and for all $b > 0$.

(2) If $x(k) = 0$ for all $k \leq n$, then put $\tau^\frown a^\frown b \in T$ if and only if, either $a > 0$ and $b = 0$ or $a = 0$ and $(y(0), \ldots, y(n), b) \in Q$.

It easily follows from the definition of T that for any $\Pi = (\pi(0), \pi(1), \ldots) \in [Q]$, Θ_Π is a winning strategy for Player II for the game $G(T)$. Now suppose that Θ is a winning strategy for Player II for $G(T)$. Then we can define a $\Pi = (\pi(0), \pi(1), \ldots) \in [Q]$ such that $\Theta = \Theta_\Pi$ by recursion as follows. For each n, let $\pi(n) = \Theta((0, \pi(0), 0, \pi(1), \ldots, 0, \pi(n-1), 0))$. It is easy to see from our definition of T that $\Pi \in Q$ and that $\Theta = \Theta_\Pi$. Thus the correspondence $\Pi \to \Theta_\Pi$ is our desired effective 1:1 degree preserving correspondence between $[Q]$ and $WS(T)$.

Suppose now that Q is finitely branching (respectively, highly computable). Let $f(\pi)$ be an upper bound on $\{s : \pi^\frown s \in Q\}$; if Q is highly computablee, then f is computable. Now given a partial code $\sigma = (z(0), \ldots, z(n-1)) \in WS(T)$ for a strategy for the game $G(T)$, we will indicate how to compute an upper bound $g(\sigma)$ for $\{t : \sigma^\frown t \in WS(T)\}$. First compute the n-th finite sequence

$\tau_n = (\tau(0), \ldots, \tau(k-1))$ in the enumeration described above, and use σ to compute the partial play $\pi = (\tau(0), y(0), \ldots, \tau(k-2), y(k-2), \tau(k-1))$–this can be done since for any $i < k$, $\tau \lceil i$ appears before τ in the enumeration. Now there are two cases in the computation of $g(\sigma)$. If $\tau(k-1) > 0$, then $g(\sigma) = 0$ and if $\tau(k-1) = 0$, then $g(\sigma) = f(\pi)$. Thus $WS(T)$ is finitely branching and if Q is highly computable, then g is recursive so that $WS(T)$ is highly computable. □

As usual, there are a number of immediate corollaries and we state only a few. Note that all of the examples below are games in which player II (who is trying to force the play into the closed set) has the winning strategy.

Corollary XIII.0.10. *(a) There is a computably presented Gale-Stewart game such that Player II has a winning strategy but has no hyperarithmetic winning strategy.*

(b) There is a computably presented, bounded Gale-Stewart game $G(C)$ such that Player II has a winning strategy and for any winning strategy Θ with $\mathbf{0}' <_T \Theta \leq_T \mathbf{0}''$, there is a Σ^0_2 set A such that $\mathbf{0}' <_T A \leq_T \Theta$.

(c) For any c. e. degree $\mathbf{c}$, there is a computably presented, computably bounded Gale-Stewart game $G(C)$ such that Player II has a winning strategy and the set of c. e. degrees which contain winning strategies for $G(C)$ equals the set of c. e. degrees $\geq_T \mathbf{c}$.

Next we consider the reverse direction for games in which Player I has a winning strategy. Here the bounded games all have computable winning strategies and nothing more can be said. For the unbounded games, the reverse direction demonstrates the connection between Π^1_1 classes and the game quantifier of Moschovakis. Recall that the Π^0_1 class with index e is the set $[T_e]$ of infinite paths through the e-th primitive recursive tree T_e. A theorem of Moschovakis states that the set of indexes e such that Player I has a winning strategy for the game $G(T_e)$ is a universal Π^1_1 set. See [155] for a discussion of the game quantifier and this theorem.

Note that every winning strategy for Player I is a limit point of the set of winning strategies for Player I, since once the play of the game has gotten into the open set, Player I may play anything at all from that point on. Thus we cannot hope to represent even every Π^0_1 class with a one-to-one correspondence.

Theorem XIII.0.11. *For any Π^1_1 class $Q \subseteq \omega^\omega$, there is a recursively presented Gale-Stewart game $G(C)$ and a recursive function F such that $y \in V \iff F(y) \in WS'(C)$.*

Proof. Suppose that $y \in Q \iff (\forall x)(\exists n)R(x\lceil n, y\lceil n)$. Define the closed set C to be $\{(x, y) : (\forall n)\neg R(x\lceil n, y\lceil n)\}$. For each $y \in \omega^\omega$, let $F(y)$ code the strategy which simply plays y in response to any play x of Player I. Then it is clear that $F(y)$ codes a winning strategy if and only if $y \in Q$. □

Theorem XIII.0.12. *(a) There is a computably presented Gale-Stewart game $G(C)$ such that the set $WS'(T)$ of winning strategies for Player I is not Σ^1_1.*

(b) There is a computably presented Gale-Stewart game $G(C)$ for which Player I has a winning strategy but has no hyperarithmetic winning strategy.

Proof. (a) This is immediate from Theorem XIII.0.11.

(b) Let $Q = \{z\}$ be a Π^1_1 singleton such that z is not hyperarithmetic and let the game $G(C)$ and the recursive function F be given by Theorem XIII.0.11. Then it is clear that Player I has a unique winning strategy which consists of playing $z(n)$ at his n-th turn, and that this strategy has the same degree as z. □

Chapter XIV

The Rado Selection Principle

In this section, we summarize the results of Jockusch, Lewis and Remmel from [93]. A Rado Family consists of collection of finite subsets $\{A_i : i \in I\}$ of $A = \cup_{i \in I} A_i$ and a collection of finite partial functions $\{\phi_F :\in A^F$: F is a finite subset of $I\}$ such that for each finite subset F of I, $\phi_F(i) \in A_i$ for all $i \in F$. The Rado selection problem is to find a choice function $f : I \to A$ such that for any finite subset F of I, there is a finite extension $E \supseteq F$ such that $f(i) = \phi_E(i)$ for all $i \in F$. We call such a choice function a *Rado selector*. Rado proved in [170] that any such family has a Rado selector. A finite set $F = \{x_1 < \ldots < x_n\}$ of natural numbers may be coded by $k = 2^{x_1} + 2^{x_2} + \cdots + 2^{x_n}$. In this case, we write $F = D_k$. We let 0 code the empty set. Then a family $\{A_i : i < \omega\}$ of finite sets may be coded by a function f such that $A_i = D_{f(i)}$ for each i. Similarly a family of finite partial choice functions ϕ_F may be coded by a single function g such that $g(i) = j$ if and only if $D_j = \{2^{x+1}3^{y+1} : x \in D_i \ \& \ \phi_{D_i}(x) = y\}$. A Rado family together with the coding described above is an *effective Rado family* $A = I = \omega$ and if the coding functions f and g are both computable.

Given an effective Rado family $\mathcal{F}$ as above, let $Ch(\mathcal{F})$ be the set of functions $h : \omega \to \omega$ such that

(i) $h(i) \in A_i$ for each i and

(ii) for each finite $F \subseteq \omega$, there is a finite extension E such that $\phi_E(i) = h(i)$ for all $i \in F$.

The following is Theorem 3 of [93].

Theorem XIV.0.13. *For any effective Rado family $\mathcal{F}$, there is a bounded strong Π^0_2 class P and an effective, degree-preserving correspondence between P and $Ch(\mathcal{F})$.*

Proof. We can define a tree T which is computable in $\mathbf{0}'$ such that $[T] = Ch(\mathcal{F})$ as follows. A finite path $(y_0, y_1, \ldots, y_n)$ is in T if and only if

(i) $y_i \in A_i$ for all $i \leq n$ and

(ii) there exists a finite set M such that $\{0, \ldots, n\} \subseteq M$ and $\phi_M(i) = y_i$ for all $i \leq n$. □

Applying Theorems III.2.15, and V.2.2, we obtain the following.

Corollary XIV.0.14. *Let $\mathcal{F}$ be an effective Rado family. Then*

(a) $\mathcal{F}$ has a Rado selector of Σ^0_2 degree.

(b) If $\mathcal{F}$ has only countably many Rado selectors, then $\mathcal{F}$ has a Rado selector which is computable in $\mathbf{0}'$.

The following is Theorem 2 of [93].

Theorem XIV.0.15. *For any nonempty bounded strong Π^0_2 class P, there exists an effective Rado family $\mathcal{F}$ and an effective, degree-preserving correspondence between P and $Ch(\mathcal{F})$.*

We can now prove the following.

Corollary XIV.0.16. *(i) There is an effective Rado family such that, for any degree $\mathbf{a}$ of a Rado selector for $\mathcal{F}$ and any Σ^0_2 degree $\mathbf{b} \geq_T \mathbf{a}$, $\mathbf{b} = \mathbf{0}''$.*

(ii) There is an effective Rado family such that, for any two degrees $\mathbf{a}$, $\mathbf{b}$ of Rado selectors for $\mathcal{F}$, $\mathbf{a} \not\leq \mathbf{b} \vee \mathbf{0}'$.

(iii) There is an effective Rado family such that, for any degree $\mathbf{a} \leq_T \mathbf{0}''$ of a Rado selector for $\mathcal{F}$, there is a Σ^0_2 degree $\mathbf{b}$ with $\mathbf{0}' \leq_T \mathbf{b} \leq_T \mathbf{a}$.

Chapter XV

Analysis

Computable functions on real numbers are just effectively continuous functions and Π^0_1 classes of reals are just effectively closed sets. Computable real functions may be represented by computable functions on natural numbers, by enumerating a countable basis of rational intervals. Effectively closed, compact sets of reals may be represented by Π^0_1 classes in $\{0,1\}^{\mathbb{N}}$. Weihrauch [217] has provided a comprehensive foundation for computability theory on various spaces, including the space of compact sets and the space of continuous real functions.

The basic example of a Π^0_1 class of reals is the sete of zeros of a computable function. Nerode and Huang [158] showed that any Π^0_1 class in $\{0,1\}^{\mathbb{N}}$ may be represented as the set of zeroes of a computable real function and Ko [112] showed that this can be done by polynomial time computable functions. This leads easily to the set of points where extreme values occur and to the set of fixed points of a computable function. Effective real dynamical systems have been studied by Cenzer [22], [113] and more recently by Cenzer, Dashti, King, Toska and Wyman [24, **?**] and by S. Simpson.

Index sets for effectively closed sets of reals were studied by Cenzer and Remmel in [41, 42, 43].

Results from Chapter VI are used to obtain the complexity of index sets related to the cardinality, computable cardinality, measure and category of effectively closed sets of reals. Index sets for computable real functions are defined and lead to complexity results for index sets corresponding to the zeroes, extrema, and fixed points of such functions.

Brattka and Weihrauch [16, 217] identify three different types of "effectively closed" sets in Euclidean space $\Re^n$. These are determined from an enumeration I_m of the basic open sets (or intervals) and considering whether the set of m such that I_m (or $\overline{I_n}$ meets K (or not) is a computably enumerable set. Of course, there are four possible notions here, and these can be refined further by asking whether the c.e. sets are in fact computable. These notions are developed in [42] and applied to the graphs of computable functions. In particular, we examine the question of whether a function with an effectively closed graph is "necessarily" continuous.

Polynomial time and NP versions of effectively closed sets are studied and a version of the "P=NP" problem is given. Here the choice of the basic open sets is crucial.

The fundamental problems for which the solution sets may be represented as Π^0_1 classes are the following.

(1) Zeroes of continuous functions

The classical problem here is to find a zero for a continuous function. The intermediate value theorem can be used to show the existence of a zero for a continuous function which is negative at one point and positive at another point. The effective version of this theorem also holds, that is, any computable function on the reals which is negative at one point and positive at another point has a computable zero, which can be computed by repeatedly splitting the interval between the two initial points. (See Pour-El–Richards [169] for a proof.) However, Lacombe [124, 125] showed that there are computable functions which have zeroes but have no computable zeros. We will give the improvement of this result due to Nerode and Huang [158] by showing that every Π^0_1 class is the set of zeroes of some computable function.

(2) The Extreme Value Theorem

The classical result here is that any function which is continuous on a compact set takes on a maximum and a minimum on that set. The problem here is to find a point where the maximum or minimum is attained. Lacombe showed that the extreme values of a computable function on $[0, 1]$ are themselves computable and also constructed a computable function F on $[0, 1]$ which does not attain its maximum at any computable point. We will present the result of Nerode and Huang [158] that any Π^0_1 class may be represented as the set of points where some effectively continuous function attains its maximum.

(3) Fixed points of continuous functions

The problem here is to find a fixed point for a given continuous function. A simple application of the intermediate value theorem shows that any continuous function F on [0,1] has a fixed point. It is well known that if F is effectively continuous, then F will have a computable fixed point. The Brouwer Fixed Point Theorem says that a continuous function on $[0, 1] \times [0, 1]$ will also have a fixed point, but Orevkov [164] showed that there need not be a computable fixed point. J. Miller [152] defined the notion of a *fixable* set as a Π^0_1 class $Q \subseteq [0, 1] \times [0, 1]$ for which there exists a computably continuous function F such that $Q = \{z : F(z) = z\}$. He gave a beautiful result which characterizes the fixable sets. Results for other spaces are different. On the real line, the continuous function $F(x) = x + 1$ has no fixed point. On ω^ω, the function $F((x(0), x(1), \dots) = (1 + x(0), 1 + x(1), \dots)$ has no fixed point. On the Cantor space the function $F(x(0), x(1), \dots) = (1-x(0), 1-x(1), \dots)$ has no fixed point.

(4) Dynamical systems

We will give a few results on effective real dynamical systems from Cenzer [22] and from Ko [113]. We shall view a dynamical system as determined by a continuous function F on a space X. The associated problem is to determine the behavior of the sequence $x, F(x), F(F(x)), \ldots$ for a given x. In particular, we want to find those points x for which this sequence is bounded or converges to some finite number and those x for which the sequence is unbounded or diverges to infinity where X is either the real line or the Baire space. If F is a polynomial, then it is always possible to compute a bound c such that $\{F^{(n)}(x) : n < \omega\}$ is bounded if and only if $|F^{(n)}(x)| < c$ for all n. In fact, we can take c large enough so that $F(x) > x + 1$ for all $x > c$, so that $lim_{n\to\infty} F^{(n)}(x) = \infty$ for all $x > c$. In this situation, we say that ∞ is an *attracting point* for F. Then $\{x : |F^{(n)}(x)| \leq c \text{ for all } n\}$ is called the *Julia set* of F. (See Blum, Shub and Smale [15].) It is then easy to see that the Julia set of any continuous function must be a compact set and we will show that for a computably continuous function, the Julia set is a Π^0_1 class. The first problem for dynamical systems is to find a member of the Julia set.

A point x is said to be a *periodic point* of a continuous function F if $F^{(n)}(x) = x$ for some finite n. The *basin of attraction B(x)* of x is defined to be $\{u : lim_n F^{(n)}(u) = x\}$. The periodic point x is said to be *attracting* if there is some open neighborhood U about x such that $U \subseteq B(x)$. The basin of attraction of infinity may also be defined as $\{u : lim_n F^{(n)}(u) = \infty\}$. Thus the basin of attraction is an open set. We will show that for a computably continuous function, the complement of a basin of attraction is a Π^0_1 class. If 1 is an attracting periodic point of a function F on $\{0,1\}^\omega$ or $[0,1]$, then we will refer to the complement of $B(1)$ as the Julia set of F. The problem here is to find a point not in the basin of attraction.

Before turning to the problems mentioned above, we give a brief introduction to computable analysis, including the problem of characterizing the computable image of the interval and the related concept of a real as a Dedekind cut of rationals, which was studied by Soare in [196, 197].

A basic principle of computable analysis is that a computable function on the real numbers is an effectively continuous function and a Π^0_1 class is an effectively closed set. We will consider the real line $\Re$, as well as three subspaces: the space of irrationals, which is homeomorphic to the Baire space ω^ω and two compact subspaces, the interval [0,1] and the Cantor space, which is computably homeomorphic to $\{0,1\}^\omega$. Since $\Re$ is computably homeomorphic to the open interval (0,1) via the order-preserving map $\frac{e^x}{1+e^x}$, we will frequently identify $\Re$ with $(0,1)$ and treat it as a subset of $[0,1]$.

Let $\mathcal{D}$ be the set of dyadic rationals in $[0,1]$. Then $[0,1]$ has a basis of open intervals (a,b),$[0,c)$ or $(d,1]$ where $a,b,c,d \in \mathcal{D}$. Thus an open subset of [0,1] is a countable union

$$U = \cup_n (a_n, b_n) \bigcup \cup_n [0, c_n) \bigcup \cup_n (d_n, 1]$$

of dyadic intervals. The open set U is said to be *effectively open*, or Σ^0_1, if the sequences a_n, b_n, c_n and d_n are computable. Then a closed set C is said to be *effectively closed*, or Π^0_1, if it is the complement of an effectively open set.

Any $x \in \{0,1\}^{\mathbb{N}}$ represents a real $r_x = \sum_n x(n)/2^n \in [0,1]$. In addition, for any $\sigma \in \{0,1\}^*$, $\sigma^\frown 0^\omega$ represents the dyadic rational $q_\sigma = \sum_{i<n} \sigma(i)/2^i$. Some difficulty arises from the fact that q_σ has another representation, $\sigma\lceil(n-1)^\frown 0^\frown 1^\omega$ (assuming that σ ends in a 1). Each dyadic rational is of course computable, so that we may unambiguously say that r is a *computable real* if $r = r_x$ for some computable sequence $x \in \{0,1\}^\omega$. Then a subset P of $\{0,1\}^\omega$ represents a subset of $[0,1]$ if and only if, for all x, y such that $r_x = r_y$, we have $x \in P$ if and only if $y \in P$. For any $\sigma \in \{0,1\}^{<\omega}$ of length n, the members of $I(\sigma)$ represent the members of the real closed interval $[q_\sigma, q_\sigma + 2^{-n}]$, which we denote by $U(\sigma)$. More generally, if $r < s$ are computable reals, then the interval $[r,s]$ is a Π^0_1 class, since, if $r = r_x$ and $s = r_y$, then

$$r_z \in [r,s] \iff (\forall n)[q_{x\lceil n} - 2^{-n} \leq q_{z\lceil n} \leq q_{y\lceil n} + 2^{-n}].$$

Lemma XV.0.17. *(a) The following are equivalent for any subset K of* $[0,1]$.

(1) K is a Π^0_1 class

(2) K is closed and $\{\langle p,r\rangle \in \mathcal{D}^2 : K \cap [p,r] = \emptyset\}$ is a c. e. set.

(3) K is represented by a Π^0_1 class $P \subset \{0,1\}^\omega$.

(b) K may be represented by a computable binary tree with no dead ends if and only if $\{\langle p,r\rangle \in \mathcal{D}^2 : K \cap [p,r] = \emptyset\}$ is computable.

Proof. (a) We show that both (1) and (3) are equivalent to (2). Suppose first that K is a Π^0_1 class and let

$$[0,1] \setminus K = \cup_n (a_n, b_n) \bigcup \cup_n [0, c_n) \bigcup \cup_n (d_n, 1].$$

Then

$$K \cap [p,r] = \emptyset \iff (\exists n)[p,r] \subseteq \cup_{m<n}(a_m, b_m) \bigcup \cup_{m<n}[0, c_n) \bigcup \cup_{m<n}(d_n, 1].$$

Suppose next that $A = \{\langle p,r\rangle : K \cap [p,r] = \emptyset\}$ is an c. e. set. Then K is a Π^0_1 class since

$$[0,1] \setminus K = \bigcup \{(p,r) : \langle p,r\rangle \in A\}.$$

Furthermore, K is represented by $[T]$ where the Π^0_1 tree T is defined as follows. Given σ of length n, let

$$\sigma \in T \iff [q_\sigma, q_\sigma + 2^{-n}] \not\subseteq \cup\{(p,r) : \langle p,r\rangle \in A^n\}.$$

Here we replace $q + 2^{-n}$ with 1 if $q = 1$.)

Finally suppose that $K = \{r_x : x \in P\}$ for some Π^0_1 class $P = [T] \subseteq \{0,1\}^\omega$. Then for any σ,

$$K \cap [q_\sigma, q_\sigma + 2^{-n}] = \emptyset \iff \sigma \notin Ext(T)$$

Since any dyadic interval $[p,r]$ may be decomposed into a finite union of intervals of the form $[q_\sigma, q_\sigma + 2^{-n}]$, it follows that $\{\langle p,r\rangle : K \cap [p,r] = \emptyset\}$ is an c. e. set.

(b) This follows from the observation that, if K is represented by $[T]$, then

$$\sigma \in Ext(T) \iff K \cap [q_\sigma, q_\sigma + 2^{-|\sigma|}] \neq \emptyset.$$

□

An arbitrary Π^0_1 class $P \subseteq \{0,1\}^\omega$ can be represented by a Π^0_1 subclass of [0,1] by the following lemma.

Lemma XV.0.18. *For any Π^0_1 class $P \subseteq \{0,1\}^\omega$, there is a Π^0_1 subclass $Q \subseteq \{0,1\}^\omega$ which represents a subset of $[0,1] \setminus \mathcal{D}$ which is computably homeomorphic to P.*

Proof. Let the computable homeomorphism Φ be defined by

$$\phi(x(0), x(1), \ldots) = (1, 0, x(0), 1, 0, x(1), \ldots)$$

and let $Q = \phi[P]$. Q represents a subset of [0,1] since every element of Q has both infinitely many "1"s and infinitely many "0"s. □

We can characterize those intervals which are Π^0_1 classes using the notion of the *Dedekind cut* $L(r) = \{q \in \mathcal{D} : q \leq r\}$ of a real number r. Soare showed in [196, 197] that if $x \in \{0,1\}^\omega$ is the characteristic function of a Π^0_1 set (respectively a Σ^0_1 set), then $L(r_x)$ is a Π^0_1 set (resp. a Σ^0_1 set) and that these implications are not reversible.

The set $\omega^{<\omega}$ and the space ω^ω may be linearly ordered by the lexicographic ordering $\leq_L$, where $x <_L y$ if, for some n, $x(n) < y(n)$ and $x(i) = y(i)$ for all $i < n$. This ordering is computable on $\omega^{<\omega}$ and thus is Π^0_1 on ω^ω, since

$$x \leq_L y \iff (\forall n) x\lceil n \leq y\lceil n.$$

We now define the interval $[x,y] = \{z : x \leq_L z \leq_L y\}$ and also $[x,\infty] = \{z : x \leq_L z\}$. Then we let $L(x) = \{\sigma \in \omega^{<\omega} : \sigma^\frown 0^\omega \leq x\}$. These notions may also be restricted to $\{0,1\}^\omega$ and $\{0,1\}^{<\omega}$. Observe that for non-dyadic rationals r_x and r_y, $r_x < r_y$ if and only if $x <_L y$.

Lemma XV.0.19. *(a) For any $x < y$ in either $[0,1]$, $\{0,1\}^\omega$, or ω^ω, the interval $[x,y]$ is a Π^0_1 class if and only if $L(x)$ is a Σ^0_1 set and $L(y)$ is a Π^0_1 set.*

(b) In either $[0,1]$, $\{0,1\}^\omega$, or ω^ω: $L(x)$ is a computable set (respectively computable in A) if and only if x is computable (resp. in A)

(c) For any $x \in \{0,1\}^\omega$, if x is the characteristic function of a $\Sigma^{0,A}_1$ (respectively $\Pi^{0,A}_1$) set, then $L(x)$ is a $\Sigma^{0,A}_1$ (resp. $\Pi^{0,A}_1$) set.

Proof. (a) First consider the case, where $x < y$ and $x, y \in \omega^\omega$. We claim that $[x, y]$ is a Π^0_1 class if and only if both $[x, \infty]$ and $[0, y]$ are Π^0_1 classes. The if direction follows from the fact that $[x, y] = [x, \infty] \cap [0, y]$. For the other direction, choose $\sigma \in \omega^\omega$ and n such that $x\lceil n \leq_L \sigma <_L y\lceil n$ and observe that $[x, \infty] = [x, y] \cup [\sigma^\frown 1^\omega, \infty]$ and $[0, y] = [0, \sigma^\frown 0^\omega] \cup [x, y]$.

Thus we need only show that $[x, \infty]$ is a Π^0_1 class iff $L(x)$ is a Σ^0_1 set and that $[0, y]$ is a Π^0_1 class iff $L(x)$ is a Π^0_1 set. Suppose that $[x, \infty] = [T]$ for some computable tree T. Then

$$\sigma \in L(x) \iff \sigma^\frown 0^\omega \notin [T] \iff (\exists n)\sigma^\frown 0^n \notin T$$

and hence $L(x)$ is Σ^0_1 set. Vice versa, suppose that $L(x)$ is a Σ^0_1 set. Then we have

$$z \in [x, \infty] \iff (\forall m)(z\lceil m \notin L(x))$$

so that $[x, \infty]$ is a Π^0_1 class.

Similarly, if $[0, y] = [T]$ for some computable tree, then

$$\sigma \in L(y) \iff (\forall n)(\sigma^\frown 0^n \in T)$$

so that $L(y)$ is a Π^0_1 set. Vice versa, if $L(y)$ is Π^0_1 set, then

$$z \in [0, y] \iff (\forall n)(z\lceil n \in L(y))$$

so that $[0, y]$ is a Π^0_1 class.

For $x, y \in \{0, 1\}^\omega$, the argument is similar, except that $[x, \infty]$ is replaced by $[x, 1^\omega]$.

For $r_x, r_y \in [0, 1]$, the problem reduces to the previous case of $\{0, 1\}^\omega$, as long as we take x to end in 0^ω whenever $r_x \in \mathcal{D}$ and y to end in 1^ω whenever $r_y \in \mathcal{D}$, so that $q_\sigma \in [r_x, r_y] \iff \sigma \in [x, y]$.

(b) We give the argument for ω^ω. $L(x)$ is computable in x, since $\sigma \in L(x) \iff \sigma \leq_L x\lceil|\sigma|$. Also, x is computable in $L(x)$, since for each n, $x(n+1)$ is the least a such that $x\lceil n^\frown a \in L(x)$ & $x\lceil n^\frown a + 1 \notin L(x)$.

(c) Now suppose that x is the characteristic function of a Π^0_1 set, i.e. x is the characteristic function of $\omega \setminus A$ where A is an r.e. set. Then let A^s for $s \geq 0$ be some effective enumeration of A. Thus x is the decreasing limit of a sequence $(x_0, x_1, \ldots)$ where x_s is the characteristic function of A^s. Then $\sigma \in L(x) \iff (\forall n)(\sigma \leq_L x_n\lceil|\sigma|)$. Similarly, if x is the characteristic function of a Σ^0_1 set A then x is the increasing limit of the sequence (x_n). Hence $\sigma \in L(x) \iff (\exists n)(\sigma \leq_L x_n\lceil|\sigma|)$. □

It follows from part (b) that $L(x)$ is Δ^0_2 if and only if x is Δ^0_2, and that if x is Π^0_2 (respectively, Σ^0_2), then $L(x)$ is Π^0_2 (resp. Σ^0_2.)

Theorem XV.0.20. *(a) Let $x \in \omega^\omega$. If x is the maximum element of a c. b. Π^0_1 class, $L(x)$ is a Π^0_1 set. If $L(x)$ is a Π^0_1 set and, in addition, x is not hyperimmune, i.e. there is a computable function f such that $x(e) \leq f(e)$ for all e, then x is the maximum element of some r.b. Π^0_1 class. If x is the* *minimum element of some r.b.* Π^0_1 *class, then* $L(x)$ *is a* Σ^0_1 *set. If* $L(x)$ *is* Σ^0_1 *set and, in addition,* x *is not hyperimmune, then* x *is the minimum element of some r.b.* Π^0_1 *class.*

(b) *For any* x *in* $[0,1]$*,* x *is the maximum element of some* Π^0_1 *class if and only if* $L(x)$ *is* Π^0_1 *and* x *is the minimum element of some* Π^0_1 *class if and only if* $L(x)$ *is* Σ^0_1*.*

(c) *For any* $x \in \omega^\omega$ *or* $[0,1]$*,* x *is the maximum (respectively, minimum) element of a* Π^0_1 *class represented by a tree with no dead ends if and only if* x *is computable.*

(d) *For any* $x \in \omega^\omega$*, if* x *is the maximum element of a bounded* Π^0_1 *class, then* $L(x)$ *is a* Π^0_2 *set and if* x *is the minimum element of a bounded* Π^0_1 *class, then* $L(x)$ *is a* Σ^0_2 *set.*

(e) *For any* $x \in \omega^\omega$*, if* x *is the maximum element of a* Π^0_1 *class, then* $L(x)$ *is a* Σ^1_1 *set and if* x *is the minimum element of a* Π^0_1 *class, then* $L(x)$ *is a* Π^1_1 *set.*

Proof. We just give proofs for the maximum element versions.

(a) Suppose that $L(x)$ is a Π^0_1 set and there is a computable function f such that $x(e) \leq f(e)$ for all e. Then x is the maximum element of the Π^0_1 interval $[0,x]$ by Lemma XV.0.19. Hence x is the maximal element of the *r.b.* Π^0_1 class $[0,x] \cap [T]$ where T is the computable tree such that $\sigma \in T \iff (\forall i \leq |\sigma|)(\sigma(i) \leq f(i))$.

Now let x be the maximum element of a *r.b.* Π^0_1 class $P = [T]$. Then $\sigma \in L(x)$ if and only if

$$(\exists y)[y \in P \;\&\; \sigma \leq_L y] \iff (\exists \tau \in \omega^{|\sigma|})[\tau \in Ext(T) \;\&\; \sigma \leq_L \tau].$$

Since T is *r.b.*, the search for τ is bounded and, since $Ext(T)$ is a Π^0_1 set, $L(x)$ is a Π^0_1 set.

If T has no dead ends, then $Ext(T)$ is computable, so that $L(x)$ is computable. This completes the proof of part (a) as well as part (c).

Part (b) now follows from Lemma XV.0.17.

(c) Any computable x is the maximum element of the *r.b.* class $\{x\}$. The maximum element of a *r.b.* Π^0_1 class $P = [T]$ is computed by letting $x(n)$ be the largest i such that $(x\lceil n-1)^\frown i \in T$.

Parts (d) and (e) follow from the characterization of $L(x)$ given above, since $Ext(T)$ is always Σ^1_1 and is Π^0_2 if P is bounded. □

XV.0.1 Computable continuous functions

Next we turn to the definition of computably continuous functions. For functions on ω^ω or $\{0,1\}^\omega$, a computable function $y = F(x)$ is given by an oracle Turing machine which uses input x as an oracle to compute the values $y(n)$ and is continuous since each value $y(n)$ depends on only finitely many values of x.

Lemma XV.0.21. *A function $F : \omega^\omega \to \omega^\omega$ (respectively, $F : \{0,1\}^\omega \to \{0,1\}^\omega$) is computably continuous if and only if there is computable function $f : \omega^{<\omega} \to \omega^{<\omega}$ (resp. $f : \{0,1\}^{<\omega} \to \{0,1\}^{<\omega}$) such that*

(1) for all $\sigma \prec \tau$, $f(\sigma) \preceq f(\tau)$,

(2) for all $x \in \omega^\omega$, $\lim_{n\to\infty} |f(x\lceil n)| = \infty$, and

(3) for all $x \in \omega^\omega$, $\lim_{n\to\infty} f(x\lceil n) = F(x)$.

Proof. Given such a representation f for F, clearly we can compute $y(n)$ for $y = F(x)$ from x by computing $f(x\lceil k)$ for sufficiently large k.

Given a computable function F, define the representation f as follows. On input σ of length n, compute the values of $\tau(i)$ where $\tau = f(\sigma)$ for each $i < n$ by applying the algorithm for F for n steps, using oracle σ. The length of τ will be the least $k < n$ such that $\tau(k)$ does not converge in n steps. □

In general, a function F on the Baire space is continuous if and only if it has a representation f as above. Thus F is continuous if and only if it is computable in some parameter $x \in \omega^\omega$.

The definition of computably continuous real functions is more difficult.

Definition XV.0.22. *A function $F : [0,1] \to [0,1]$ is computable (or computably continuous if there is a uniformly computable sequence of functions $f_n : \mathcal{D} \to \mathcal{D}$ such that, for any $x \in \{0,1\}^\omega$, $F(r_x) = \lim_i f_i(q_{x\lceil i})$ and a computable function $\nu : \omega \to \omega$ such that, for all natural numbers m, n, k and all dyadic rationals q, r, if $|q - r| < 2^{-\nu(k)}$ and $m, n > \nu(k)$, then $|f_m(q) - f_n(r)| < 2^{-k}$.*

This definition is easily seen to be equivalent to other standard definitions, such as those given by Lacombe [124]. See Pour-El–Richards [169] for some history.

Note for any computable real function, $F(x)$ is computable real for any computable real x.

Functions of several variables are treated similarly, thus a uniformly computable sequence of functions $\{f_n\}_{n\in\omega}$ and a computable function ν represent a continuous function $F : [0,1]^2 \to [0,1]$ if $\lim f_i(q_{x\lceil i}, q_{y\lceil i}) = F(x,y)$ for any reals x, y and if $|f_m(q_1,q_2) - f_n(r_1,r_2)| < 2^{-k}$ whenever $m, n > \nu(k)$ and both $|q_1 - r_1|, |q_2 - r_2| < 2^{-\nu(k)}$. For example, the standard distance function $|x - y|$ may be represented by taking $f_n(q,r) = |q - r|$ for all n and $\nu(k) = k + 1$.

We say a function $F : \{0,1\}^\omega \to \{0,1\}^\omega$ *represents* a real function G provided $y = F(x)$ whenever $r_y = G(r_x)$.

Lemma XV.0.23. *If F is a continuous (respectively computably continuous) map on $\{0,1\}^\omega$ such that $F(x) = F(y)$ whenever $r_x = r_y$, then F represents a continuous (respectively computably continuous) map on $[0,1]$.*

Proof. Given the representation function f for F, let $f_i(q_\sigma) = q_{f(\sigma)}$ for all i and let $\nu(k)$ be the least n such that $|f(\sigma)| > k$ for all $\sigma \in \{0,1\}^n$. □

We remark that not every computably continuous real function may be represented by a computable function on $\{0,1\}^\omega$; the distance function $|x-r|$ for any fixed rational $r \in (0,1)$ is a counterexample. For example, suppose that $r = \frac{1}{6}$ and $G(x) = |x - \frac{1}{6}|$. Now suppose that $F : \{0,1\}^\omega \to \{0,1\}^\omega$ represents G and that $f : \{0,1\}^{<\omega} \to \{0,1\}^{<\omega}$ represents F. Now $G(\frac{2}{3}) = \frac{1}{2}$ which has two representations $x_1 = 1^\frown 0^\omega$ and $x_0 = 0^\frown 1^\omega$. Let $x_2 = (10)^\omega$ so that x_3 represents $\frac{2}{3}$. Then either $F(x_2) = x_1$ or $F(x_2) = x_0$. Suppose first that $F(x_2) = x_0$. Then for some n, $0 \prec f((10)^n)$. But then $(10)^{n\frown}1^\omega$ is a number greater than $\frac{2}{3}$ so that $1 \prec f((10)^n 1^k)$ for some k which is a contradiction. Similarly if $F(x_2) = x_1$, then for some n, $1 \prec f((01)^n)$. But then $(10)^{n\frown}0^\omega$ is a number less than $\frac{2}{3}$ so that $0 \prec f((10)^n 0^k)$ for some k which is again a contradiction.

A computable metric on the Baire space is defined by $\delta(x,y) = 1/2^n = 0^n 1^\omega$, where n is the least such that $x(n) \neq y(n)$, and $\delta(x,y) = 0 = 0^\omega$ if $x = y$.

The graph of a function $F : X \to X$ is defined as usual to be $gr(F) = \{(x, F(x)) : x \in X\}$. For $X = \omega^\omega$, we can view the graph as a subset of X by associating the pair (x,y) with the element $z = x \otimes y$, where $z(2n) = x(n)$ and $z(2n+1) = y(n)$. For any class P and any $x \in X$, let $\pi_x(P) = \{y : x \otimes y \in P\}$. For a function F from $[0,1]$ to $[0,1]$, the graph may be represented by a subset of $\{0,1\}^\omega$, namely $\{x \otimes y : f(r_x) = r_y\}$.

A classical result says that a function on the interval is continuous if and only if the graph is closed. We give the effective version here.

Theorem XV.0.24. *(a) The graph of a computably continuous function on ω^ω is a Π^0_1 class.*

(b) Let X be either $\{0,1\}^\omega$ or $[0,1]$. Then a function $F : X \to X$ is computably continuous if and only if the graph of F is a Π^0_1 class. Furthermore, the graph of any computably continuous function may be represented by a tree with no dead ends.

Proof. (a) Suppose first that $F : \omega^\omega \to \omega^\omega$ is computably continuous and is represented by $f : \omega^{<\omega} \to \omega^{<\omega}$. Define the computable tree T with $[T] = gr(F)$ by putting $\sigma \otimes \tau \in T$ if and only if τ is consistent with $f(\sigma)$.

(b) Given a computable $F : \{0,1\}^\omega \to \{0,1\}^\omega$, define the computable tree T with $gr(F) = [T]$ as in (a). Then $Ext(T)$ is Σ^0_1, and therefore computable, by the following easily verified claim.

$$\text{CLAIM: } \sigma \otimes \tau \in Ext(T) \iff (\exists \sigma' \succ \sigma)\tau \prec f(\sigma').$$

Given a computable tree T so that $gr(F) = [T]$, define the computable representing function f by letting $f(\sigma)$ be the common part of $\{\tau : \sigma \otimes \tau \in T\}$.

Next suppose that F is a computably continuous function on $[0,1]$ and let the computable sequence f_i of dyadic rational functions and the computable modulus function ν be given as in Definition XV.0.22. We can assume that $\nu(k) > k$ for all k. In this case, we can define our desired computable tree T with $gr(F) = [T]$ to be the set of pairs $\sigma \otimes \tau$ of length $2n-1$ or $2n$ such that $|f_n(q_\sigma) - q_\tau| \leq 2^{1-k}$ for all k such that $|\sigma| \geq \nu(k)$. Again it is easy to see that $EXT(T)$ is computable.

Suppose now that $gr(F)$ is a Π^0_1 class and, by Lemma XV.0.17, let $T \subseteq \{0,1\}^{<\omega}$ be a computable tree so that $[T]$ represents $gr(F)$. T may not be the graph of a function, since each dyadic real has two representations. However any two representations of length n differ by 2^{-n}. Thus, for any i and any σ of length n, we let $f_i(q_\sigma) = q_\tau$ for the lexicographically least τ of length $|\sigma|$ such that $\sigma \otimes \tau \in T$, and let $\nu(k)$ be the least n such that, for all σ of length n and any τ_1, τ_2 with $\sigma \otimes \tau_1$ and $\sigma \otimes \tau_2$ both in T, $\delta_q(\tau_1, \tau_2) < 2^{-k}$. □

We next examine the complexity of the image of a Π^0_1 class under a computablly continuous function. The classical results are that the image of any compact set under a continuous function is compact and that the image of a closed set is an analytic set.

Theorem XV.0.25. *Let F be a computably continuous function on a Π^0_1 subclass $P = [T]$ of ω^ω or $[0,1]$ and let $F[P] = \{F(x) : x \in P\}$. Then*

(a) $F[P]$ is a Σ^1_1 class,

(b) if P is bounded, then $F[P]$ is a strong Π^0_2 class, and

(c) if P is computably bounded, then $F[P]$ is a computably bounded Π^0_1 class and, furthermore, if there is a computable tree T with no dead ends such that $P = [T]$, then there is a computable tree S with no dead ends such that $F[P] = [S]$.

Proof. (a) This part follows immediately from the fact that $y \in F[P] \iff (\exists x)(x \in P \;\&\; \langle x, y\rangle \in gr(F))$.

(b) Suppose that T is a finitely branching, computable tree and let S be a computable tree such that $gr(F) = [S]$. Then it follows from König's Lemma that $F[P] = [R]$, for the finitely branching Σ^0_1 tree R defined by

$$\tau \in R \iff (\exists \sigma)[\sigma \in T \textit{ and } \sigma \otimes \tau \in S].$$

(c) Now suppose that T is computably bounded and let F be represented by the computable function $f : \omega^{<\omega} \to \omega^{<\omega}$. Then it is easy to see the definition above in (b) becomes computable.

To find a bound for the possible value of $\tau(n)$ for $\tau \in R$, compute the least m such that $|f(\sigma)| > n$ for all $\sigma \in T$ of length m. Then we compute the maximum value $h(r)$ of $f(\sigma(n))$ for all $\sigma \in T$ of length n. Thus R is highly computable.

ly continuous map $F(x) = r + (s - r)x$.

Suppose that K is the image of the computably continuous map F. It follows from the Intermediate Value Theorem that $K = [r, s]$ where the reals r and s are the maximum and minimum elements of P. It follows from Theorem XV.0.25 that K may be represented by a tree with no dead ends and then from Theorem XV.0.20 that r and s are computable. □

Corollary XV.0.26. *Let F be a computably continuous function on ω^ω, $\{0,1\}^\omega$, or $[0,1]$. Then the maximum and minimum values of F on P are computable reals (if they exist).*

Theorem XV.0.27. *Each of the following sets is a Π^0_1 class for any computably continuous function $F : X \to X$, where the space X may be $\{0,1\}^\omega$, $[0,1]$, ω^ω or $\Re$. In case (3), the class always has a compuable member when $X = [0,1]$. In case (4), the class is always bounded when $X = \Re$.*

(a) *The set of points x where $F(x) = x_0$ for any fixed computable x_0.*

(b) *The set of points where F attains its maximum (minimum).*

(c) *The set of fixed points of F.*

(d) *The Julia set of F where $X = \omega^\omega$ or $\Re$.*

(e) *The complement of the basin of attraction of a computable periodic point.*

Proof. (a) This is immediate from Theorem XV.0.24.

(b) It follows from Corollary XV.0.26 that the maximum and minimum are computable if they exist. The result now follows from part (b).

(c) This is easily reduced to part (b). For $\Re$, x is a fixed point of F if and only if F is a zero of $G(x) = F(x) - x$. For $[0,1]$, take $G(x) = |F(x) - x|$. For $\{0,1\}^\omega$ or ω^ω, define $z = G(x)$ by $z(n) = |F(x)(n) - x(n)|$.

A computable fixed point r may be found for a computably continuous function on $X = [0,1]$ by the standard procedure. If F has a dyadic fixed point, then there is nothing to do. If not, then repeatedly split the interval in two and choose the subinterval with $F(x) < x$ on one end and $F(x) > x$ on the other. Then r is the unique element in the intersection of these intervals.

(d) This is immediate from the characterization of the Julia set as $\{x : (\forall n)|F^n(x)| \leq c\}$ for a fixed computable point c. Note that in ω^ω, $\{x : x \leq_L x_0\}$ is not a bounded Π^0_1 class in our sense of being the paths through a finite branching tree.

(e) Given an attracting point c for F, there is some computable interval $(a,b) \subseteq B(x)$ containing c. Then the complement of the basin of attraction may be characterized as

$$\{x : (\forall n)(F^n(x) \leq a \vee F^n(x) \geq b)\}$$

□

As usual, we give a few immediate corollaries from the results of Part One.

Theorem XV.0.28. *Let $F : X \to X$ be a computably continuous map, where X is either $\{0,1\}^\omega$, $[0,1]$ or $\Re$.*

(a) *If F attains a maximum M, then there are two points x_1 and x_2 with $F(x_1) = F(x_2) = M$ such that any function computable in both x_1 and x_2 is computable.*

(b) *If F has only countably many zeroes, then F has a computable zero.*

(c) *If F has only finitely many fixed points, then every fixed point of F is computable.*

(d) *If the Julia set of $F : \Re \to \Re$ has no computable member, then it contains a continuum of pairwise Turing incomparable elements.*

(e) *If the basin of attraction $B(x_0)$ of a computable fixed point x_0 of F is not all of X, then there is a point x of c. e. degree which is not in $B(x_0)$.*

Proof. We just note that in each case, a function defined on $\Re$ may be restricted to a finite interval and thus be treated as a map on the interval. For example, if F has a zero, take a computable interval $[a, b]$ on F has a zero and let $[c, d]$ be the image of $[a, b]$ under F. Then F may be composed with maps between $[0, 1]$ and the two intervals to obtain a map $G : [0, 1] \to [0, 1]$ so that the set of zeroes of G is homeomorphic to a subset of the set of zeroes of F. □

Theorem XV.0.29. *Let $F : \omega^\omega \to \omega^\omega$ be a computably continuous map.*

(a) *If F attains a maximum M, then $F(x) = M$ for a point x which is computable in some Σ^1_1 set.*

(b) *If F has only countably many zeroes, then F has a hyperarithmetic zero.*

(c) *If F has only finitely many fixed points, then every zero of F is hyperarithmetic.*

Next we give the collection of converses to Theorem XV.0.28. The first three parts are due to Nerode and Huang [158] and may also be found in Ko [112].

Next we give the collection of converses to Theorem XV.0.28. The first three parts are due to Nerode and Huang [158] and may also be found in Ko [112].

Theorem XV.0.30. *Let P be a Π^0_1 subclass of the space X, either $\{0, 1\}^\omega$, $[0, 1]$, ω^ω or $\Re$.*

(1) *There is a computably continuous function F such that P is the set of zeroes of F.*

(2) *There is a computably continuous function F with maximum value M such that $P = \{x : F(x) = M\}$.*

(3) (a) *If X is either $\{0, 1\}^\omega$, ω^ω or $\Re$, then there is a computably continuous function F such that P is the set of fixed points of F.*

(b) *If X is $[0, 1]$ and P has a computable member, then there is a computably continuous function F such that P is the set of fixed points of F.*

(4) *If P is bounded and has both a computable maximum and a computable minimum element, then there is a computably continuous function such that*

(a) *P is the complement of the basin of attraction of a computable periodic point, where* $X = [0,1]$ *or* $\Re$.

(b) *P is the Julia set of F where* $X = \Re$

Proof. (1) First suppose $P \subseteq \omega^\omega$ and let T be a computable tree such that $P = [T]$. Define the computable function F by

$$F(x) = \begin{cases} 0^\omega, & \text{for } x \in P \\ 0^n \frown 1 \frown 0^\omega, & \text{if } n \text{ is the least with } x\lceil n \notin T. \end{cases}$$

If P represents a subset of $[0,1]$, then the function F is modified when x represents a dyadic, so that $F(\sigma \frown 1 \frown 0^\omega) = F(\sigma \frown 0 \frown 1^\omega)$ for all σ. Thus when $r_x = r_y$ is dyadic, we let

$$F(x) = F(y) = \begin{cases} 0^\omega, & \text{for } x \in P \\ 0^n \frown 1 \frown 0^\omega, & \text{if } n \text{ is the least with } x\lceil n \notin T \text{ and } y\lceil n \notin T. \end{cases}$$

For a subset P of $\Re$, let Q be the image of P under the isomorphism G with $(0,1)$ together with the point 0, if P has no lower bound and the point 1, if P has no upper bound. Then let H be the computably continuous map with set Q of zeroes. It follows that P is the set of zeroes of $H \circ G$.

(2) Let F be the function defined in the proof of (1) and observe that 0 is the minimum value of F in each case. For the maximum argument on $[0,1]$ or $\Re$, just take $G(x) = 1 - F(x)$. For the maximum argument on ω^ω, note that the range of F is a subset of $\{0,1\}^\omega$ and take $G(x)(n) = 1 - F(x)(n)$.

(3) (a) Let F be given by (1) so that P is the set of zeroes of F. Now let $G(x) = F(x) + x$ for the real line, and, for ω^ω or $\{0,1\}^\omega$, let $G(x)(n) = x(n)$ if $F(x)(n) = 0$ and $G(x)(n) = 1 - x(n)$, if $F(x)(n) \neq 0$.

(b) Let x_0 be a computable member of the Π^0_1 class P and let F be the function given by (1) so that $x \in P$ if and only if $F(x) = 0$. Define $G(x)$ to be $x + (x_0 - x)F(x)$.

(4) (a) Let P be a Π^0_1 proper subclass of $[0,1]$. Then there is some computable element $x_0 \notin P$. Let F be the computable function given by part (1) such that $F(x) = 0$ for $x \in P$ and $F(x) > 0$ for $x \notin P$. Let $P_1 = P \cap [0, x_0]$ and $P_2 = P \cap [x_0, 1]$. Let M_1 be the maximal element of P_1 and let M_2 be the minimal element of P_2, so that both M_1 and M_2 are computable. Now define the function G by cases.

$G(x) = M_1 + F(x)(x_0 - M_1)$, for $x \leq M_1$;
$G(x) = x + (x - M_1)(x_0 - x)$, for $M_1 \leq x \leq x_0$;
$G(x) = x - (M_2 - x)(x - x_0)$, for $x_0 \leq x \leq M_2$;
$G(x) = M_2 - F(x)(M_2 - x_0)$, for $M_2 \leq x$.

Then x_0, M_1 and M_2 are all fixed points of G. We claim that P is the complement of the basin of attraction of x_0. The following inequalities are immediate from the above definition.

$M_1 \leq G(x) \leq x_0$, for $x < M_1$;
$x < G(x) < x_0$, for $M_1 < x < x_0$;
$x_0 < G(x) < x$, for $x_0 < x < M_2$;

$x_0 \leq G(x) < M_2$, for $M_2 < x$.

First we show that the basin of attraction of x_0 for G includes $[M_1, M_2]$. Given $M_1 < x < x_0$, we see that $x < G(x) < x_0$. It follows that $G^n(x)$ is an increasing sequence with limit L such that $G(L) = L$ and $M_1 < L \leq x_0$. Thus we must have $L = x_0$. A similar argument works for $x_0 < x < M_1$.

Next suppose that $x \notin P$ and either $x < M_1$ or $x > M_2$. Then either $G(x) \in [M_1, M_2]$, so that x is in the basin of attraction of G.

Now suppose that $x \in P$, so that either $x \in P_0$ or $x \in P_1$. For $x \in P_0$, we have $F(x) = 0$ and $x \leq M_1$, so that $G(x) = M_1$ and thus $G^n(x) = M_1$ for all $n > 0$. Thus x is not in the basin of attraction of G. Similarly for $x \in P_1$, $G^n(x) = M_2$ for all $n > 0$, so that x is not in the basin of attraction of G.

For $X = \Re$, just identify X with a subclass of $(0,1)$ as in (1) above.

(b) Let P be a bounded Π^0_1 class of reals with a computable minimal element m and a computable maximal element M and let the computably continuous function F be given by (1) so that $F(x) = 0$ for all $x \in K$ and $F(x) > 0$ for all $x \notin K$. Now define the function F in the following cases.

$G(x) = m + M - x$, for $x \leq m$.

$G(x) = M + F(x)$, for $m \leq x \leq M$.

$G(x) = 2x - M$ for $x \geq M$. □

Since any countable Π^0_1 subset of $[0,1]$ and any Π^0_1 subset which may be represented by a tree with no dead ends has a computable member, we have the following immediate corollary.

Corollary XV.0.31. (a) *If the nonempty Π^0_1 subclass K of [0,1] may be represented by a tree with no dead ends, then K is the set of fixed points of some computably continuous function from [0,1] into [0,1].*

(b) *Any countable, nonempty Π^0_1 subclass K is the set of fixed points of some computable function from [0,1] into [0,1].*

As usual, we have a number of immediate corollaries, of which we state only a few.

Theorem XV.0.32. *Let X be $\{0,1\}^\omega$, ω^ω, $\Re$, or $[0,1]$.*

(a) For any r.e. degree $\mathbf{c}$, there is a computably continuous function F on X such that the set of c. e. degrees which contain zeroes of F equals the set of c. e. degrees $\geq_T \mathbf{c}$.

(b) There is a computably continuous function F on X which has a fixed point and such that any two distinct fixed points are Turing incomparable if X is $\{0,1\}^\omega$, ω^ω, $\Re$. There is a computably continuous function F on $[0,1]$ which has a unique computable fixed point and uncountable many non-computable fixed points and such that any two distinct non-computable fixed points are Turing incomparable.

(c) *There is a computably continuous function F which has a maximum M on X, such that there is a unique non-comptable point x_0 where M is attained and x_0 is also the unique accumulation point of the set where M is attained.*

(d) *There is a computably continuous function on $\Re$ with attracting point at infinity such that every computable point is attracted to infinity but not every point is attracted to infinity.*

(e) *There is a computably continuous function on $[0,1]$ with a attracting point at infinity such that every computable point is attracted to infinity but not every point is attracted to infinity.*

Proof. Note that in part (b) when $X = [0,1]$, we may add a single computable point to the Π^0_1 class so that it can represent the set of fixed points. □

Theorem XV.0.33. (a) *There is a computably continuous function on ω^ω which has a zero but has no hyperarithmetic zero.*

(b) *There is a computably continuous function on ω^ω which attains a maximum M such that $F(x) \neq M$ for any hyperarithmetic point x.*

(c) *There is a computably continuous function on ω^ω which has a fixed point but has no hyperarithmetic fixed point.*

Ko [113] improved part (4) of Theorem 10.15 by showing that if the Π^0_1 class P has either a p-time maximum element or a p-time minimum element, then there is a p-time computable function f with Julia set P. Furthermore, Ko shows in [113] that there is such a set P which has a non-computable Hausdorff dimension, which implies that there is a p-time computable function f such that the Julia set of f has non-computable Hausdorff dimension.

XV.1 Symbolic Dynamics

In this section, we examine computable dynamical systems and symbolic dynamics associated with computable functions on the Cantor space $\{0,1\}^{\mathbb{N}}$ and the unit interval $[0,1]$.

Computable real dynamical systems have been studied by Cenzer [**?**], where the the Julia set of a computably continuous real function is shown to be a Π^0_1 class and Ko [113], who examined fractal dimensions and Julia sets. Computable complex dynamical systems have recently been investigated by Braverman and Cook [**?**] and Braverman and Yampolsky [**?**], who showed that there is a complex number c such that the Julia set corresponding to the function $f(z) = z^2 + c$ is not decidable.

In particular, the computability of a closed set K in a computable metric space (X,d) may be defined in terms of the distance function d_K, where $d_K(x)$ is the infimum of $\{d(x,y) : y \in K\}$. K is a Π^0_1 class if and only if d_K is upper semi-computable and K is a *decidable* (or *computable*) Π^0_1 class if d_K is computable.

xxx

put this in earlier

zzz

For any finite k, the *shift* function on $\{0, 1, \ldots, k\}$ is defined by $\sigma(x) = y$, where $y(n) = x(n+1)$. A closed set $Q \subseteq \{0, 1, \ldots, k\}$ is said to be a *subshift* if it is closed under the shift function. We will refer to a Π^0_1 class which is also a subshift as a *subsimilar* Π^0_1 class.

Fix a finite alphabet Σ, let $F : \Sigma^{\mathbb{N}} \to \Sigma^{\mathbb{N}}$ be a computable function and let a partition $\{U_0, U_1, \ldots, U_k\}$ of $\Sigma^{\mathbb{N}}$ into clopen sets be given. The *itinerary* of a point $x \in \Sigma^{\mathbb{N}}$ is the sequence $It(x) \in \{0, 1, \ldots, k\}^{\mathbb{N}}$ where

$$It(x)(n) = i \iff F^n(x) \in U_i.$$

Now let $IT[F] = \{It(x) : x \in \Sigma^{\mathbb{N}}\}$. We show that $IT[F]$ is a decidable subsimilar Π^0_1 class and that, for any decidable subsimilar Π^0_1 class $Q \subseteq \Sigma^{\mathbb{N}}$, there exists a computable F such that $Q = IT[F]$.

XV.1.1 Undecidable subshifts

In this section, we construct a subsimilar Π^0_1 class with no computable element. We will give the construction in $\{0, 1\}^{\mathbb{N}}$, but it can be generalized to $\Sigma^{\mathbb{N}}$ for any finite Σ. Now every decidable Π^0_1 class has a computable element (in fact, the leftmost path is computable). Hence we have an undecidable subsimilar Π^0_1 class.

Let us say that a string v is a *factor* of a string w if there exist w_1 and w_2 such that $w = w_1 {}^\frown v {}^\frown w_2$. For any set S of strings, we may define a closed set P_S, where $x \in P_S$ if and only if, for all n and all $w \in S$, w is not a factor of $x \lceil n$. If the set P_S is nonempty, then S is said to be *avoidable*. For this section, we restrict ourselves to $\Sigma = \{0, 1\}$

Lemma XV.1.1. *Given any sequence $x_0, x_1, \ldots$ of elements of $\{0, 1\}^{\mathbb{N}}$, there is a nonempty subshift containing no x_i.*

Proof. Define the sequence $l_0, l_1, \ldots$ by $l_0 = 3$ and, for $n > 0$,

$$l_n = 3(2^{\frac{n(n+3)}{2}}).$$

This will imply that $l_{n+1} = 2^{n+2} l_n$. Now let $w_n = x_n \lceil 2l_n$ for each n and define subshift P to consist of all x which do not contain any w_n as a factor. Clearly $x_i \notin P$ for all i. It remains to show that P is nonempty, that is, $\{w_n : n \in \mathbb{N}\}$ is avoidable.

It is important to notice that given any word w of length $2k$, it has at most $k+1$ distinct factors of length k. Since there are 2^k words of length k, for k large enough so that $2^k > k+1$, there are words of length k that do not appear as a factor of w. With this in mind, we construct recursively two sequences of words $< A_n >_{n \in \mathbb{N}}$ and $< B_n >_{n \in \mathbb{N}}$ such that, for all n:

1. $|A_n| = |B_n| = l_n$.
2. A_0 and B_0 are not factors of w_0; this is possible since $|w_0| = 6$ so w_0 has at most 4 distinct factors of length 3.
3. A_{n+1} and B_{n+1} are taken from $\{A_n, B_n\}^*$, have A_n as a prefix, and have length $m = 2^{n+2} = l_{n+1}/l_n$. This is possible since there are 2^{m-1} such words, but there are at most $l_{n+1} + 1$ factors of length l_{n+1} in w_{n+1} and $2^{m-1} \geqq l_{n+1} + 1 + 2$.

Now let $x = lim_n A_n$. This exists since each $A_n \prec A_{n+1}$. We claim that $x \in P$. Suppose by way of contradiction that some w_n is a factor of x. We can view x as an infinite concatenation of blocks length l_n, where each block is either A_n or B_n. Since w_n has length 2 l_n, it must completely contain one of the blocks, which would imply that either A_n or B_n is a factor of w_n. This contradiction shows that $x \in P$. □

We need to improve this lemma in two ways. First, we may have only a subset of words w_k of length l_{n_k}. Second, we need an effective version.

Theorem XV.1.2. *There is a recursive sequence of natural numbers $l_0, l_1, \ldots$ such that if for any subsequence $< l_{n_k} >_{k \in \mathbb{N}}$ and any set $S = \{v_k : k \in \mathbb{N}\}$ of words such that $|v_k| = l_{n_k}$, S is avoidable. Furthermore, if ϕ is a partial computable function such that $\phi(n_k) = v_k$, then there is a nonempty subsimilar Π^0_1 class P such that no element of P contains any factor v_k.*

Proof. For the first part, simply let $w_{n_k} = v_k$ and choose arbitrary words w_i of length l_i for $i \notin \{n_k : k \in \mathbb{N}\}$ and apply the lemma.

For the second part, we have

$$x \in P \iff (\forall n)(\forall k)[v_k \text{ is not a factor of } x\lceil n]$$

In more detail, notice that v_k is not a factor of $x\lceil n$ if and only if, for all v, if $\phi_s(k) = v$, then v is not a factor of $x\lceil n$. □

Theorem XV.1.3. *There is a nonempty subsimilar Π^0_1 class P with no computable element.*

Proof. Let the sequence $< l_n >$ be given as in Lemma XV.1.1. Let Let $\phi_0, \phi_1, ..., \phi_e, \ldots$ be an enumeration of partial computable functions. Now define the partial recursive function ϕ by

$$\phi(k) = \begin{cases} \phi_k\lceil l_k, & \text{if } \phi_k(i)\downarrow \text{ for all } i < 2l_k; \\ \text{undefined}, & \text{otherwise.} \end{cases}$$

By Theorem XV.1.2, there is a nonempty subsimilar Π^0_1 class P such that no element of P has any word $\phi(k)$ as a factor. Now let y be any computable element of $\{0,1\}^{\mathbb{N}}$. Then $y = \phi_k$ for some k such that ϕ_k is a total function. Thus $\phi(k) = \phi_k\lceil k$ is defined and is not a factor of any $x \in P$ and hence certainly $\phi_k \notin P$. □

XV.1.2 Symbolic Dynamics of Computable Functions

Fix a finite alphabet $\Sigma = \{0, 1, \ldots, k\}$, let $F : \Sigma^{\mathbb{N}} \to \Sigma^{\mathbb{N}}$ be a computable function and let a partition $\{U_0, U_1, \ldots, U_k\}$ of $\Sigma^{\mathbb{N}}$ into clopen sets be given. The *itinerary* of a point $x \in SN$ is the sequence $It(x) \in \{0, 1, \ldots, k\}^{\mathbb{N}}$ where

$$It(x)(n) = i \iff F^n(x) \in U_i.$$

Now let $IT[F] = \{It(x) : x \in \Sigma^{\mathbb{N}}\}$. We observe that $IT[F]$ is a subshift. That is, suppose $y = It(x) \in IT[F]$. Then $\sigma(y) = It(F(x))$, so that $\sigma(y) \in IT[F]$ as well. The function It is continuous and hence $IT[F]$ is a closed set, as seen by the proof of the following lemma.

Lemma XV.1.4. *The function from $\Sigma^{\mathbb{N}} \to \{0, 1, \ldots, k\}^{\mathbb{N}}$ mapping x to $I(x)$ is computable.*

Proof. Given clopen sets $U_0, \ldots, U_k$, there exists a finite j and a finite subset W of $\{0,1\}^j$ such that each U_i is a finite union of intervals $I[w]$ for some set of $w \in W$. Thus one can determine from $y \lceil j$ the unique i for which $y \in U_i$. Given $x \in \Sigma^{\mathbb{N}}$, let $y = I(x)$. To compute $y(n)$, it suffices to find the first j values of $F^n(x)$, which can be computed uniformly from x and n. □

Theorem XV.1.5. *Fix a computable function $F : \Sigma^{\mathbb{N}}$ to $\Sigma^{\mathbb{N}}$, let a partition $\{U_0, U_1, \ldots, U_k\}$ of $\Sigma^{\mathbb{N}}$ into clopen sets be given and let $I(x)$ denote the itinerary of x under F. Then*

(a) For any computable $x \in \Sigma^{\mathbb{N}}$, $I(x)$ is computable.

(b) The set $IT[F]$ of itineraries is a decidable, subsimilar Π^0_1 class.

Proof. Part (a) follows from the well-known result that computable functions map computable points to computable points and (b) follows from the fact that the image of a decidable Π^0_1 class under a computable function is a decidable Π^0_1 class. See [**?**, **?**]. □

Next we prove the converse. Note that $F^0(x) = x$ for all $x \in \Sigma^{\mathbb{N}}$ and therefore $IT[F]$ meets every U_i. Note that if Q is a subshift and Q does not meet $I[i]$, then $Q \subseteq \{0, 1, \ldots, i-1, i+1, \ldots, k\}$.

Theorem XV.1.6. *Let $\Sigma = \{0, 1, \ldots, k\}$ be a finite alphabet and let $Q \subseteq \Sigma^{\mathbb{N}}$ be a decidable, subsimilar Π^0_1 class which meets $I[i]$ for all i. Then there exists a partition $\{U_0, \ldots, U_k\}$ and a computable $F : \Sigma^{\mathbb{N}} \to \Sigma^{\mathbb{N}}$ such that $Q = IT[F]$.*

Proof. We will use the partition given by $U_i = I[i]$. Since Q is decidable, we can define a function $G : \Sigma^{\mathbb{N}} \to Q$ such that $G(x) = x$ for all $x \in Q$. Let $Q = [T]$ where T is a computable tree without dead ends. The approximating function g for G is defined as follows. For any $w \in \{0, 1, \ldots, k\}^n$, find the longest initial segment v such that $v \in T$ and let $g(v)$ be the lexicographically least (or leftmost) extension of v which is in $T \cap \{0, 1, \ldots, k\}^n$; this exists since T has no dead ends. Now let $F(x) = \sigma(G(x))$. We claim that $\Sigma_F = Q$.

For any $x \in Q$, we have $F(x) = \sigma(x)$ and $\sigma(x) \in Q$, since Q is a subshift. Hence $F^n(x) = \sigma^n(x)$, so that $F^{n+1}(x)(0) = x(n+1)$ and belongs to the set $U_{x(n)}$. Thus the itinerary $I(x) = x$. This shows that $Q \subseteq IT[F]$.

Next consider any $x \in \Sigma^{\mathbb{N}}$. We will show by induction that $F^n(x) = \sigma^n(G(x))$. For $n = 1$, this is the definition. Then

$$F^{n+1}(x) = \sigma(G(F^n(x))) = \sigma(G(\sigma^n(G(x)))),$$

by induction. But $G(x) \in Q$, so that $\sigma^n(G(x)) \in Q$ by subsimilarity and therefore $G(\sigma^n(G(x))) = \sigma^n(G(x))$ and finally $F^{n+1}(x) = \sigma^{n+1}(G(x))$, as desired. It follows that for $n > 0$, $It(x)(n) = G(x)(n)$. But for $n = 0$, the assumption that Q meets $I[x(0)]$ implies that $G(x)(0) = x(0)$ and hence $It(x)(0) = x(0) = G(x)(0)$ as well. Therefore $It(x) \in Q$ as desired. $\square$

Chapter XVI

Feasible versions of combinatorial problems

The main goal in this chapter is to apply the results of Chapter VIII to the mathematical problems discussed above in Chapters IX to XV

We observe that any feasible structure is computable, therefore the set of solutions to a feasible problem is also the set of solutions to a computable problem. Thus results such as Theorems **??**, XI.1.1 and XII.1.1 have feasible versions. The reverse direction is more interesting.

We consider computable representation theorems such as Theorems **??**, XI.1.3 and XII.1.3, and corollary results such as Theorems **??**, XI.1.5 and XII.1.4.

These representation theorems showed that the set of solutions to a computable problem of various sorts can represent either every c. b. Π^0_1 class or at least every Π^0_1 class of separating sets. In this section we obtain better results, in most cases, by improving "recursive" to "polynomial-time". Now an infinite computable problem may be assumed to have universe ω, since any two infinite computable sets are recursively isomorphic. (Here the universe of a graph-coloring problem, for example, is the set the vertices.) However, it is not true that any two polynomial-time sets are polynomial-time isomorphic. (For example, it is clear that there is no p-time map from $Tal(\omega)$ onto $Bin(\omega)$.) Thus a polynomial-time structure with some p-time set for its universe may not be computably isomorphic to any p-time structure with universe ω. For example, a p-time Abelian groups with all elements of finite order is constructed by Cenzer and Remmel in [35] which is not even isomorphic to any p-time group with standard universe either ($Tal(\omega)$ or $Bin(\omega)$). For many of the problems considered above, we will show that any computable problem can be reduced first to a p-time problem and then to a p-time problem with standard universe.

We illustrate the general strategy with the graph coloring problem. Recall from Section XI.XI.2 that, for $k \geq 3$, the set of k-colorings of a recursive graph can be represented by a c. b. Π^0_1 class and conversely can represent an arbitrary c. b. Π^0_1 class. Let $G = (V, E)$ be a computable graph. Then the set of

k-colorings of G can be represented as the Π_1^0 class $[T]$ of infinite paths through a computable tree T. Now Theorem VIII.1.4 constructs for us a p-time tree P such that $[T] = [P]$. Then the converse representation creates from P a graph whose k-colorings are in an effective degree-preserving finite-to-one correspondence with the infinite paths through P. Furthermore, inspection of the proof from [174] shows that this graph will actually be polynomial time, since P is polynomial time. This shows that the k-colorings of any computable graph can always be placed in an effective degree-preserving correspondence with the k-colorings of some p-time graph, and, therefore, that the k-colorings of a p-time graph can strongly represent any c. b. Π_1^0 class.

However, there is no natural correspondence between the recursive graph and the p-time graph constructed in this manner. We can do better using Theorem VIII.2.1.

Theorem XVI.0.7. *For each computable instance P of any of the following problems, there is a p-time instance Q of the problem which is computably isomorphic to P. Furthermore, except in cases (13) and (14), if P has a computable solution, then we can take Q to have a p-time solution.*

(1) Finding a k-coloring for a k-colorable highly computable graph, for any $k \geq 3$.

(2) Finding a marriage in a highly computable society.

(3) Finding a surjective marriage in a symmetrically highly computable society.

(4) Finding a surjective marriage in a symmetrically highly computable society where each person knows at most two other people.

(5) Finding a k-partition of a highly computable graph such that no set in the partition is adjacent to m other sets, for $m > 2$.

(6) Finding a one-way (or two-way) Hamiltonian (or Euler) path starting from a fixed vertex for a highly computable graph.

(7) Covering a computablee poset of width k by k chains, for any $k \geq 2$.

(8) Covering a computable poset of height k by k antichains, for any $k \geq 2$.

(9) Expressing a computable partial ordering on a set as the intersection of d linear orderings on the set.

(10) Finding a subordering of type ω (or of type ω^*) of a computable ordering.

(11) Finding an ω-successivity (or an ω^*-successivity) in a computable linear ordering.

(12) Finding a non-trivial self-embedding of a computable linear ordering.

(13) Finding a winning strategy for an effectively closed binary game.

(14) Finding a prime ideal of a recursive Boolean algebra.

Proof. For problems (1) through (12), this follows immediately from Theorem VIII.2.1, since each of these problems can be viewed as a relational structure and the given solution can be viewed as a function mapping to a fixed range. In the dimension of posets problem, we can interpret the solution as a finite set of relations. For problem (13), Theorem 4.4 of [36] shows that any computable game may viewed as a p-time game in that the set of infinite paths which are winning for Player I will be the set of infinite paths through a p-time tree. For problem (14), Theorem 2.6 of [34] shows that any computable Boolean algebra is computably isomorphic to a p-time Boolean algebra. □

We note that a computable game with a computable winning strategy is not necessarily isomorphic to a p-time game with a p-time winning strategy, since by Theorem 4.5 of [36], there is a computable game with unique winning strategy, which is computable but not p-time.

Corollary XVI.0.8. *For each recursive instance P of any of the problems listed in Theorem XVI.0.7, there is a p-time instance Q of the problem such that the Π^0_1 class of solutions to P is computably homeomorphic to the Π^0_1 class of solutions to Q.*

Proof. In each case, it is easy to see that the computable isomorphism between P and Q gives rise to a computable homeomorphism between the Π^0_1 classes of solutions.

For example, we consider the coloring problem. Recall that the Π^0_1 class of k-colorings on a computable graph $G = (V, E)$ (where V may be assumed to equal ω) is the set $[T]$ of infinite paths through the computable k-ary tree T, where a finite sequence $(\sigma(0), \ldots, \sigma(n-1)) \in \{1, 2, \ldots, k\}^n$ is in T if and only if $\sigma(i) \neq \sigma(j)$ for all $(i, j) \in E$. Now suppose that f is a computable isomorphism mapping G to the computable graph $G' = (V', E')$, so that $V' = \{f(0), f(1), \ldots\}$ and $(f(i), f(j)) \in E'$ if and only if $(i, j) \in E$. Then we can define the tree $k+1$-ary T' by having $(\tau(0), \ldots, \tau(n-1)) \in \{0, 1, \ldots, k\}$ in T' if and only if

(1) $\tau(v) = 0 \iff v \notin V'$;

(2) $\tau(u) \neq \tau(v)$ whenever $(u, v) \in E'$.

Then $[T']$ represents in a reasonable way the set of legal k-colorings on G' and we have a natural homeomorphism from $[T]$ to $[T']$ defined by $H(x)(f(i)) = x(i)$ and $H(x)(v) = 0$ if $v \notin V'$. □

We can now represent Π^0_1 classes as the set of solutions to p-time problems of the types listed above. We list only some of the results.

Corollary XVI.0.9. *For each of the problems (1) through (9), and (13) listed in Theorem XVI.0.7,*

(a) *The problem of finding a computable solution to a p-time problem can strongly represent the c. b.* Π^0_1 *class of separating sets for any pair of disjoint infinite c. e. sets.*

(b) *There is a p-time instance of the problem with no computable solution.*

(c) *If* $\mathbf{a}$ *is a Turing degree and* $\mathbf{0} <_T \mathbf{a} <_T \mathbf{0}'$, *then there is a p-time instance* P *of the problem such that* P *has a solution of degree* $\mathbf{a}$ *but has no computable solution.*

For problems (1), (3), (6) and (13), we have also:

(d) *The problem of finding a computable solution to a p-time problem can strongly represent an arbitrary c. b.* Π^0_1 *class.*

(e) *There exists a p-time instance* P *of the problem such that*

  (i) P *has a unique non-computable solution* y *which is also the unique limit solution and has degree* $\mathbf{0}'$ *and such that any other solution is computable;*

  (ii) *if* R *is any computable sub-problem of* P *and* z *is any computable solution of* R, *then either (i) there are only finitely many solutions of* P *which extend* z, *or (ii) all but finitely many solutions of* P *extend* z.

  (iii) *if* x *is any computable solution of* P, *then there is some finite sub-problem* F *of* P *such that any solution of* P *which agrees with* x *on* F *must equal* x.

We have seen that, by changing the names of the vertices, we can transform a computable graph into a p-time graph. However, we would prefer for a countably infinite graph to have the set V of vertices equal to some standard universe such that the tally or binary representation of the set of natural numbers. This would, for instance, allow us to define the homeomorphism of Corollary XVI.0.8 without worrying about the set of non-vertices. The p-time graph constructed by Theorem VIII.1.4 will have a rather sparse set of vertices and this appears to be an essential part of the theorem. We will next indicate how to fill out the p-time structure given by Theorem XVI.0.7 to a structure with universe $Bin(\omega)$ such that there is a degree-preserving correspondence, which is one-to-one (up to a finite permutation), between the Π^0_1 classes of solutions of the associated problems. For example, in the coloring problem, we add vertices whose colors will be determined, up to a permutation, by the coloring of the vertices of Q.

Theorem XVI.0.10. *For each computable instance* P *of the combinatorial problems listed below, there is a p-time instance* Q *with universe* $Bin(\omega)$ *and a degree-preserving correspondence between the solutions of* P *and the solutions of* Q.

(1) *Finding a* k*-coloring for a* k*-colorable highly computable graph, for any* $k \geq 3$.

(2) Finding a marriage in a highly computable society.

(3) Finding a surjective marriage in a symmetrically highly computable society.

(4) Finding a surjective marriage in a symmetrically highly computable society where each person knows at most two other people.

(5) Finding a k-partition of a highly computable graph such that no set in the partition is adjacent to m other sets.

(6) Finding a (one-way or two-way) Hamiltonian or Euler path for a highly computable graph.

(7) Covering a computable poset of width k by k chains, for any $k \geq 2$.

(8) Covering a computable poset of height k by k antichains, for any $k \geq 2$.

(9) Expressing a computable partial ordering on a set as the intersection of d linear orderings on the set.

(10) Finding an ω-successivity (or an ω^*-successivity) in a computable linear ordering.

Proof. In each case, we may assume by Corollary XVI.0.8 that we start with a p-time instance of the problem which is a relational structure $\mathcal{B}$ with some universe $B \subseteq Bin(\omega)$. Now it follows from Lemma 2.3 of [35] that $\mathcal{B}$ is computably isomorphic to a p-time structure $\mathcal{A}$ with universe $A \subseteq Tal(\omega)$. Then Lemma 2.6 of [35] says that the disjoint union $A \oplus Bin(\omega)$ is p-time isomorphic to $Bin(\omega)$, where $X \oplus Y = \{\langle 0, x\rangle : x \in X\} \cup \{\langle 1, y\rangle : y \in Y\}$. Then we will create a p-time structure $\mathcal{C}$ with universe $A \oplus Bin(\omega)$ which has a copy of $\mathcal{A}$ together with a copy of $Bin(\omega)$, where the relations will be defined on $Bin(\omega)$ and between A and $Bin(\omega)$ so as to determine the degree-preserving correspondence between the solutions of $\mathcal{A}$ and those of the extension $\mathcal{C}$. Since the universe C of $\mathcal{C}$ is p-time isomorphic to $Bin(\omega)$, it follows from Lemma 2.2 of [35] that $\mathcal{C}$ is computably isomorphic to a p-time structure with universe $Bin(\omega)$. Then we will let Q be the problem associated with this structure. It follows that there will be a degree preserving correspondence between the set of solutions of Q and the set of solutions of the original problem P. In each case, we will assume that our original structure is p-time and has for its universe a p-time subset A of $Tal(\omega)$ and that there is a p-time list of $Bin(\omega) \setminus A$. These assumptions are justified by the above discussion. In each case, the correspondence will be one-to-one unless otherwise indicated.

(1) Finding a k-coloring for a k-colorable highly computable graph, for any $k \geq 3$.

This is Theorem 2.1 of [37]. Here the correspondence is one-to-one, up to a finite permutation of the colors on the new vertices.

(2) Finding a marriage in a highly computable society.

Let $S = (B, G, K)$ be a p-time society. Then we will directly extend S to a highly recursive p-time society $S' = (B', G', K')$ where $B' = G' = Bin(\omega)$. Let $Bin(\omega)\setminus B = \{b_0, b_1, \ldots\}$ and let $Bin(\omega)\setminus G = \{g_0, g_1, \ldots\}$ be p-time lists of the new boys and girls in the society S'. Then K' is defined by putting $(b_i, g_i) \in K'$ for all i. It is clear that any marriage f on S has a unique extension f' to S' defined by letting $f'(b_i) = g_i$ for all i. It follows that f and f' have the same degree.

(3) Finding a surjective marriage in a symmetrically highly computable society.

The extension is the same as in (2). It is clear that f' will be onto if and only if f is onto.

(4) Finding a surjective marriage in a symmetrically highly computable society where each person knows at most two other people.

The extension is again the same as in (2). It is clear that if each person in S knows at most two other people, then each person in the extension S' also knows at most two other people.

(5) Finding a k-partition of a highly computable graph such that no set in the partition is adjacent to m other sets, with $m > 2$.

Let the p-time graph $G = (V, E)$ be given. We define a p-time graph $G_1 = (V_1, E_1)$ to be a regular $m-1$-ary tree of complete k-graphs. That is, define the regular $(m-1)$-ary tree T_{m-1} to consist of a root node $\emptyset$ together with the set $\{bin(0), bin(1), \ldots, bin(m-1)\} \times \{bin(1), bin(2), \ldots, bin(m-2)\}^*$, where $\emptyset$ has $m-1$ successors $(bin(i), \emptyset)$ for $i < m$ and $(bin(i), \sigma)$ has $m-2$ successors $(bin(i), \sigma^\frown bin(j))$ for $j < m-1$. Then we let

$$V_1 = \{bin(1), bin(2), \ldots, bin(k)\} \times T_{m-1},$$

and we let $((bin(i), \sigma), (bin(j), \tau)) \in E_1$, where $\sigma = (\sigma(0), \ldots, \sigma(s-1))$ and $\tau = (\tau(0), \ldots, \tau(t-1))$, provided that $\sigma = \tau$ or either τ is a successor of σ or σ is a successor of τ. It is clear that if the graph is computably partitioned into the complete k-graphs corresponding to the nodes of T_{m-1}, then each set in the partition is adjacent to at most $m-1$ other sets. We see also that each node of T_{m-1} has $m-1$ neighbors, so that any two distinct nodes have at least $2m-4$ other neighbors. Now let $\{A_i : i < \omega\}$ be a k-partition of G_1. Suppose that some A_i contains vertices u and v corresponding to different nodes of the T_{m-1}. Then u and v taken together have at least $2(k-1) + (2m-4)k = (2m-2)k - 2$ other adjacent vertices in G_1. Since $k-2$ of these could belong to A_i, we see that A_i has at least $(2m-3)k$ adjacent vertices. Since each set in the partition has at most k vertices, it follows that A_i is adjacent to at least $2m-3$ sets in the partition. Thus since $m > 2$, we have $2m-3 > m-1$ so that the set A_i is adjacent to too many sets.

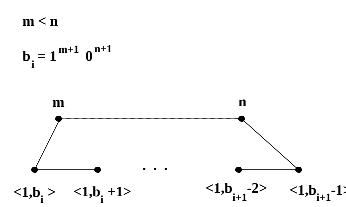
Now let $G' = G \oplus G_1$. It is clear that for any k-partition of G, there is a partition of G' of the same degree which is given by adding the recursive partition of G_1 into the nodes of the tree as defined above. We claim that these are the only possible partitions. That is, suppose $\{B_j : j < \omega\}$ is a k-partition of G'. It suffices to show that for any $u \in V_1$ and any j, if $u \in B_j$, then the entire node to which u belongs must be included in B_j. Suppose that this is false. It follows from the argument above that B_j may not contain an element of a different node of T. Thus the set B_j has all $k(m-1)$ vertices from the adjacent nodes as neighbors as well as at least one vertex from the node of u. But this clearly implies that at least m sets of the partition must be adjacent to B_j.

(6) Finding a one-way (or two-way) Hamiltonian Euler path starting from a fixed vertex for a highly recursive graph.

Let the p-time graph $G = (V, E)$ be given with $V = \{v_0 < v_1 < \cdots\}$ a subset of $Tal(\omega)$. Let each edge $(tal(m), tal(n))$ of V with $tal(m) < tal(n)$ be coded as $0^{n+1}1^{m+1}$ in $Bin(\omega)$. Let $b_0, b_1, \ldots$ enumerate the codes of edges in increasing order and let $b_i = 0^{n_i+1}1^{m_i+1}$ for each i. Now let $V' = V \oplus Bin(\omega)$ and let E' be defined by joining $\langle 1, b_i\rangle$ to $\langle 0, tal(m_i)\rangle$, joining $\langle 0, tal(n_i)\rangle$ to $\langle 1, b_{i+1}-1\rangle$, joining $\langle 1, b\rangle$ to $\langle 1, b+1\rangle$ whenever $b+1 \neq b_i$ for any i, and joining $\langle 1, b_0-1\rangle$ to $\langle 0, tal(m_0)\rangle$. Note that other than the initial sequence of edges connecting $\langle 1, 0\rangle$ to $\langle 1, b_0-1\rangle$ and then to $\langle 0, tal(m_0)\rangle$, this has the effect of replacing an edge $(m, n) \in E$ where $m < n$ and $b_i = 1^{m+1}0^{n+1}$ by a sequence of edges $(\langle 0, m\rangle, \langle 1, b_i\rangle), (\langle 1, b_i\rangle, \langle 1, b_i+1\rangle), \ldots, (\langle 1, b_{i+1}-2\rangle, \langle 1, b_{i+1}-1\rangle), (\langle 1, b_{i+1}-1\rangle, \langle 0, n\rangle)$. See Figure XVI.

Thus to test whether $\langle 1, b\rangle$ and $\langle 1, c\rangle$ are joined by an edge, where $b < c$, we simply check that $c = b+1$ and that, if $c = 0^{n+1}1^{m+1}$ with $m < n$, then $(tal(m), tal(n)) \notin E$. To determine whether $\langle 0, v\rangle$ and $\langle 1, c\rangle$ are joined by an edge, we first check that $v = tal(m) \in V$ and that either (i) $c = 0^{s+1}1^{r+1}$ or (ii) $c = 0^{s+1}1^{r+1} - 1$ for some edge $(tal(r), tal(s))$ in G with $r < s$. Finally, in case (i), we check that $r = m$ and, in case (ii), we either check that $m = m_0$ and that $c + 1 = b_0$ or else we compute the largest code $0^{q+1}1^{p+1}$ of an edge of G less than $c+1$ and check that $m = q$. If everything checks, then there is an edge and otherwise there is not. Thus G' is a p-time graph.

Now suppose that f is a one-way Euler path on G starting from $f(0) = v_0 = tal(m_0)$. Then we can define a corresponding Euler path on G' starting from $\langle 1, bin(0)\rangle$ by beginning with the sequence
$\langle 1, bin(0)\rangle, \langle 1, bin(1)\rangle, \ldots, \langle 1, b_0-1\rangle, \langle 0, v_0\rangle$ and then replacing in turn each edge $(f(i), f(i+1))$ which joins $tal(m)$ to $tal(n)$ with $m < n$, either by the sequence $\langle 0, f(i)\rangle, \langle 1, bin(b)\rangle, \langle 1, bin(b+1)\rangle, \ldots, \langle 1, bin(c-1)\rangle, \langle 0, f(i+1)\rangle$, if $f(i) = tal(m) < f(i+1)$, or by the sequence $\langle 0, f(i)\rangle, \langle 1, bin(c-1)\rangle, \langle 1, bin(c-2)\rangle, \ldots, \langle 1, bin(b+1)\rangle, \langle 1, bin(b)\rangle, \langle 0, f(i+1)\rangle$ if $f(i) = tal(n) > f(i+1)$, where $b = 0^{n+1}1^{m+1}$ and c is the least code greater than b for an edge of G. It is clear


m < n
b_i = 1^{m+1} 0^{n+1}
m
n
. . .
<1,b_i >
<1,b_i +1>
<1,b_{i+1}-2>
<1,b_{i+1}-1>

that this one-way Euler path is computable in f.

Conversely, let g be a one-way Euler path in G' starting from $\langle 1, bin(0)\rangle$. It is clear that the path must proceed through
$\langle 1, bin(1)\rangle, \langle 1, bin(2)\rangle, \ldots, \langle 1, b_0 - 1\rangle$ and then to $\langle 0, v_0\rangle$. Now let $f : \omega \to V$ be defined by letting $\langle 0, f(i)\rangle$ be the i-th vertex of the form $\langle 0, x\rangle$ in the path g. Then f is a one-way Euler path for G and it follows from the construction that g is the corresponding path as defined above, since there is only one way in G' to go from $\langle 0, f(i)\rangle$ to $\langle 0, f(i+1)\rangle$.

For two-way Euler paths, modify the construction by eliminating the finite initial sequence $\langle 1, bin(0)\rangle, \langle 1, bin(1), \ldots, \langle 1, bin(m_0) - 1\rangle$ of vertices of G' along with the edges through those vertices. Then the remaining vertex set is still p-time isomorphic to $Bin(\omega)$ and the argument goes through as above.

The Hamiltonian paths require a different construction. We assume without loss of generality that the vertices of G include 0 and that all are multiples of 4 (in binary) and let $4m_0, 4m_1, \ldots$ enumerate the vertices of G in increasing order. Define the graph G' to have vertex set $V' = Bin(\omega)$ with edges defined as follows. For each i, there will be two sequences of edges joining the set of binary numbers from $4m_i + 1$ up to $4m_{i+1}$, as follows:

(i) $4m_{i+1}, 4m_{i+1} - 4, 4m_{i+1} - 8, \ldots, 4m_i + 4, 4m_i + 2, 4m_i + 6, \ldots, 4m_{i+1} - 2,$

(ii) $4m_{i+1}, 4m_{i+1} - 3, 4m_{i+1} - 7, \ldots, 4m_i + 1, 4m_i + 3, \ldots, 4m_{i+1} - 1.$

These are the vertices associated with $4m_{i+1}$. In addition, for each edge joining $4m_i$ and $4m_j$ in G with $m_i, m_j \neq 0$, there are edges joining $4m_i - 1$ with $4m_j - 2$ and joining $4m_i - 2$ with $4m_j - 1$. For an edge in G joining $4m_i$ with 0, there is an edge joining $4m_i - 1$ with 0. The procedure for determining whether there is an edge joining a and b is the following. First look for the largest m and n such that $4m \in V$ and $4m < a$ and $4n \in V$ and $4n < b$. In the special case that $a = 0$, a and b are joined if and only if $b + 1$ is joined to 0 as vertices in G. Otherwise there are several cases. First suppose that $m = n$; then a are b are joined if and only if, either they differ by exactly 4 or $\{a, b\} = \{4m + 1, 4m + 3\}$ or $\{a, b\} = \{4m + 2, 4m + 4\}$. Next suppose that $m \neq n$. Then a and b are joined if and only if either $a + 1$ and $b + 2$ are joined as vertices in G or $a + 2$ and $b + 1$ are joined as vertices in G. Thus G' is a p-time graph.

Now let f be a one-way Hamiltonian path on G starting from $v_0 = 0$ and suppose that $f(i) = 4m_{r_i}$. Then there is a corresponding Hamiltonian path g in G' obtained by replacing the edge from v_0 to $4m_{r_1}$ with the sequence of edges joining v_0 to $4m_{r_1} - 1$ and then on to $4m_{r_1} - 3$ and $4m_{r_1}$ as described above, and for $i > 0$, replacing each edge $(f(i), f(i+1))$ with the sequence of edges first joining $4m_{r_i}$ to $4m_{r_i} - 4$ and then on to $4m_{r_i} - 2$ as described above, then joining $4m_{r_i} - 2$ to $4m_{r_{i+1}} - 1$, and closing with the sequence joining $4m_{i+1} - 1$ to $4m_{i+1} - 3$ and then $4m_{i+1}$. Thus for each $i > 0$, the even vertices associated with $f(i)$ are joined to the odd vertices associated with $f(i+1)$.

Conversely, let g be a one-way Hamiltonian path in G' starting from $v_0 = 0$ and define $f(i) = 4m_{r_i}$ so that $0, 4m_{r_1}, \ldots$ lists the members of G in order of appearance in the path g. It follows from the construction that f is a one-way

Hamiltonian path for G' starting from v_0 and that g is the corresponding path as defined above.

For the two-way Hamiltonian paths, the construction is modified by adding an edge joining v_0 with $4m_i - 2$ for each edge joining v_0 with $4m_i$ in G. Then for any two way Hamiltonian path f in G, there will be two corresponding two-way paths in G', one in which the even vertices associated with $f(i)$ are joined to the odd vertices associated with $f(i+1)$ and one in which the odd vertices associated with $f(i)$ are joined with the even vertices of $f(i+1)$. Thus the correspondence here is two-to-one.

(7) The problem of covering a computable poset of width k by k chains, for any $k \geq 2$.

Let $\mathcal{P} = (P, \leq_P)$ be a p-time poset where $P \subseteq Tal(\omega)$. Then define a p-time poset $\mathcal{R} = (R, \leq_R)$ where $R = P \oplus (\{bin(1), \ldots, bin(k)\} \times Bin(\omega))$ and $\langle 0, p\rangle \leq_R \langle 0, q\rangle$ iff $p \leq_P q$, $\langle 0, p\rangle \leq_R \langle 1, n\rangle$ for all p and n, and
$\langle 1, m\rangle \leq_R \langle 1, n\rangle$ iff $m = \langle bin(i), bin(r)\rangle$ and $n = \langle bin(i), bin(s)\rangle$ where $r \leq s$. Then it is clear that for any covering f of $\mathcal{P}$ by k chains induces a covering f' of covering of $\mathcal{R}$ by k chains where

(i) $f'(\langle 0, p\rangle) = f(p)$ for all $p \in P$ and

(ii) $f'(\langle 1, \langle bin(i), n\rangle\rangle) = f'(\langle 1, \langle bin(i), m\rangle\rangle)$ for all i,m and n.

Thus the covering is determined by the value of f' on the finitely many new points $\langle 1, \langle bin(1), 0\rangle\rangle, \ldots, \langle 1, \langle bin(1)0\rangle\rangle$. This shows that f' has the same degree as f and that f' is unique up to a permutation of the names of chains. Then $\mathcal{R}$ is p-time isomorphic to p-time linear ordering $\mathcal{S}$ whose universe is $Bin(\omega)$.

(8) The problem of covering a computable poset of height k by k antichains, for any $k \geq 2$.

This is the dual of problem (7). The partial order is now defined by making $\langle 1, \langle bin(i), n\rangle\rangle \leq \langle 1, \langle bin(j), m\rangle\rangle \iff (i < j \;\&\; m = n)$.

(9) The dimension of posets problem.

Let $\mathcal{P} = (P, \leq_P)$ be a poset and let $Bin(\omega) \setminus P = \{v_i : i < \omega\}$. The partial order $\leq'$ is defined on $Bin(\omega)$ by making $p \leq' v_i$ for all $p \in P$ and all i and making $v_i \leq' v_j$ if and only if $v_i \leq v_j$ (where $<$ is the usual ordering on $Bin(\omega)$.

(10) Finding an ω-successivity (or an ω^*-successivity) in a computable linear ordering.

Given a p-time linear ordering $L_1 = (A, <_1)$ on a p-time set $A = \{a_0 < a_1 < \cdots\}$, we may assume $a_0 = 0$ and that each $a_i = bin(4m_i)$. Now define the p-time ordering $L_2 = (Bin(\omega), <_2)$ by replacing each point $a = bin(4m_i)$ with a block $B(a)$:

$$\begin{aligned}&bin(4m_i+1) < bin(4m_i+5) < \cdots < bin(4m_{i+1}-3)\\ &< bin(4m_{i+1}-1) < bin(4m_{i+1}-5) < \cdots < bin(4m_i+3)\\ &< bin(4m_i) < bin(4m_i+4) < bin(4m_i+8) < \cdots < bin(4m_{i+1}-4)\\ &< bin(4m_{i+1}-2) < bin(4m_{i+1}-6) < \cdots < bin(4m_i+2)\end{aligned}$$

That is, we use the elements between $4m_i$ and $4m_{i+1}$ which are equivalent to 1 mod 4 to form a chain between $4m_i+1$ and $4m_{i+1}-1$, then we use the elements between $4m_i$ and $4m_{i+1}$ which are equivalent to 3 mod 4 in reverse order to form a chain between $4m_{i+1}-1$ and $4m_i$, etc.

Now suppose that f is an ω-successivity in L_1. Then we can recursively obtain an ω-successivity g in L_2 by replacing each point $f(i)$ with the block $B(f(i))$. Conversely, given an ω-successivity g in L_2, the ω-successivity f of L_1 may be defined by making $f(i)$ the i-th binary number in the successivity g which is divisible by 4 and it then follows that g is the successivity obtained from f as above. The argument for ω^*-successivities is similar. □

We remark that, for the three-coloring problem, it is possible to improve this result by having the 3-colorings of the the original computable graph be restrictions of the 3-colorings of the p-time graph to the original recursive vertex set.

Theorem XVI.0.10 can now be applied to obtain improved versions of Corollary XVI.0.9. We list only a few here.

Corollary XVI.0.11. *(a) There exists a p-time graph G with universe $Bin(\omega)$ which has a unique non-computable Hamiltonian path π, where π has degree $\mathbf{0}'$ and such that any other Hamiltonian path is the unique extension to G of a Hamiltonian path on some finite subgraph F of G.*

(b) There is a p-time partial ordering with universe $Bin(\omega)$ of width k which has no computable covering by k chains.

(c) For any $x \leq_T \mathbf{0}'$, there is a p-time linear ordering $\mathcal{A}$ with universe $Bin(\omega)$ such that there is ω-successivity (respectively ω^*-successivity) of $\mathcal{A}$ of degree x and every ω-successivity (respectively ω^*-successivity) of $\mathcal{A}$ is either computable or has the same Turing degree as x.

Theorem XVI.0.10 and Corollary XVI.0.11 demonstrate that the problem of finding solutions to feasible problems is just as difficult as the problem of finding solutions to recursive problems. Therefore more conditions will have to be put on a problem than just feasibility if our goal is to guarantee the existence of a feasible solution, or even the existence of a recursive solution. There are many possible approaches to this goal, some of which were explored in [37] for the graph-coloring problem.

Finally, we consider the problem of finding a prime ideal of a recursive Boolean algebra, or more generally, of a recursive ring.

Theorem XVI.0.12. *For any recursive Boolean algebra $\mathcal{B}$, there is a p-time commutative ring $\mathcal{R}$ with unity, having universe $Bin(\omega)$, and a one-to-one degree-preserving map between the class of prime ideals of $\mathcal{R}$ and the class of prime ideals of $\mathcal{B}$.*

Proof. By Theorem XVI.0.7, we may assume that $\mathcal{B}$ is a p-time Boolean algebra, and thus a Boolean ring. Now define the ring $\mathcal{R} = \mathcal{B} \oplus \mathbf{Q}$. $\mathbf{Q}$ is chosen here because it has no (proper) prime ideals. The ring $\mathbf{Q}$ of rationals may be represented as a p-time ring with universe $Bin(\omega)$ and it follows from Lemmas 2.2 and 2.6 of [35] that $\mathcal{R}$ is p-time isomorphic to a ring with universe $Bin(\omega)$. For any prime ideal I of $\mathcal{B}$, it is easy to check that $I \oplus \mathbf{Q}$ is a prime ideal of $\mathcal{R}$ and that these are the only prime ideals of $\mathcal{R}$. □

Corollary XVI.0.13. *(i) For any degree $\mathbf{a} <_T \mathbf{0}'$, there exists a recursive commutative ring $\mathcal{R}$ with a prime ideal I of degree $\mathbf{a}$ such that I is the unique non-recursive prime ideal of $\mathcal{B}$ and such that any other prime ideal of $\mathcal{B}$ is finitely generated.*

(ii) There is a computable commutative ring with unity, $\mathcal{R}$, which has a unique non-computable prime ideal I, such that any other prime ideal of $\mathcal{R}$ is finitely generated, and such that for any c. e. ideal J of $\mathcal{R}$, either there are only finitely many prime ideals of $\mathcal{R}$ extending J or else all but finitely many of the prime ideals of $\mathcal{R}$ extend J.

Part C

Advanced Topics and Current Research Areas

Chapter XVII

The Lattice of Π^0_1 classes

The inclusion lattice $\mathcal{E}_\Pi$ of Π^0_1 classes has an interesting algebraic structure, in some ways analogous to the dual of the lattice $\mathcal{E}$ of c.e. sets. Recent work has focused on comparing and contrasting the two lattices. Important issues include the definability and complexity of various properties, automorphisms of the lattice and orbits under automorphisms, and the analysis of certain substructures of the lattice.

Here is an example. Given two Π^0_1classes $P \subset Q$, the interval $[P, Q] = \{R : P \subseteq R \subseteq Q\}$ of P and in particular $[\emptyset, Q]$ is an *initial segment* of $\mathcal{E}_\Pi$. A Π^0_1class P is said to be *thin* if $[\emptyset, Q]$ is a Boolean algebra. P is *perfect* if every element of P is a limit point. Cholak et al. [50] have shown that the family of all perfect thin classes is in certain ways analogous to the hyper-hypersimple c.e. sets. That is, any two perfect thin classes are automorphic in $\mathcal{E}_\Pi$, the family of perfect thin classes is definable in $\mathcal{E}_\Pi$ and the degrees of perfect thin classes are exactly the c.e. array noncomputable degrees. (Here the degree of $P = [T]$ is the degree of the set of nodes of T which have an extension in P.)

An infinite Π^0_1 class P is *minimal* if every Π^0_1 subclass of P is either finite or cofinite in P. This is of course dual to the notion of a maximal c.e. set. For any lattice $\mathcal{L}$, let $\mathcal{L}^*$ be the quotient lattice of $\mathcal{L}$ modulo finite difference. Then P is minimal if and only if $[0, P]^*$ is the trivial Boolean algebra. Cenzer, Downey, Jockusch and Shore [25] first constructed a minimal thin class. Cenzer and Nies [32] characterized the order types of the finite intervals of $\mathcal{E}^*_\Pi$ as finite distributive lattices with the dual reduction property. Furthermore, for each such lattice L, the theory of L is decidable. In particular, this means that there are intervals (in fact, initial segments) of order type n for any finite ordinal n. This contrasts with the classic result that finite intervals of $\mathcal{E}^*$ are all Boolean algebras. However, it is shown in [32] that for any *decidable* Π^0_1 class P, if $[0, P]^*$ is finite, then it must be a Boolean algebra. Finally, if P is decidable and $[0, P]$ is not a Boolean algebra, then the theory of $[0, P]$ interprets the theory of arithmetic and is therefore undecidable.

End intervals $[P, TN]$ were studied in [33] where it was shown that there are exactly two possible isomorphism types of *end* intervals (where P is either clopen or not).

In his thesis (and continued in joint work with Cenzer), Riazati [177, 45] studied minimal extensions of Π^0_1 classes (an analogue of maximal subsets). He proved an analogue of the Owings Splitting Theorem and used it to prove that decidable minimal extensions are not possible.

Lawton [126] introduced the notion of *minor* superclasses of Π^0_1classes, as an analogue of major subsets of c.e. sets and gave a characterization of the Π^0_1 classes which have strong minor superclasses.

It is easy to see that the family of finite classes is invariant under automorphism. Cenzer and Nies [33] showed that the property of being finite is definable in the lattice and in general, the family of countable Π^0_1 classes of rank α is definable if and only if $\alpha < \omega$.

XVII.1 Countable thin classes

Theorem XVII.1.1. *(C-D-J-S) For any computable ordinal α, there is a thin Π^0_1 class P_α with Cantor-Bendixson rank α. Furthermore, we may take P_α as the set of paths through a computable tree with no dead ends.*

Proof. We first sketch the proof for $\alpha = 1$. We construct a sequence $\tau_e \in \{0,1\}^{<\omega}$ such that $\tau_e^\frown 1 \prec \tau_{e+1}$ for all e, a set $A = \cup_e \tau_e$ and a Π^0_1 class $P = [T]$ such that

(1) $D(P) = \{A\}$, and

(2) for any e, if $A \in [T_e]$ then $P \cap I(\tau_e) \subset [T_e]$.

These conditions imply that A is non-computable, since if A were computable, then $\{A\} = [T_e]$ for some e, so that by (2), $P \cap I(\tau_e) = \{A\}$, contradicting (1).

These conditions imply that A is non-computable, since if A were computable, then $\{A\} = [T_e]$ for some e, so that by (2), $P \cap I(\tau_e) = \{A\}$, contradicting (1).

These conditions also imply that P is minimal (and therefore thin by Theorem 5.2). To see this, suppose that $[T_e] \subset P$. If $A \notin [T_e]$, then $[T_e]$ has no limit point and is therefore finite. If $A \in [T_e]$, then $P \setminus [T_e] \subset P \setminus I(\tau_e)$ by (2) has no limit point and is finite.

The construction is in stages, so that at stage s we have a tree T^s and strings τ^s_e. At stage $s+1$, we simply look for $e \leq s$ such that some $\tau \succ \tau^s_e$ is in $T^s \setminus T_e$ and let $\tau^{s+1}_e = \tau$ for the least such e and τ. For $i < e$, let $\tau^{s+1}_i = \tau^s_i$ and for $\tau^{s+1}_{e+i} = 0\tau^\frown 1^i$. We leave the details to the reader.

The general construction for a computable ordinal α is accomplished using a computable system of notations for α and a uniformly computable family of trees of rank up to α as in the proof of Theorem V.4.8. The details are omitted. □

Next we consider the possible degrees of members of thin Π^0_1 classes.

Theorem XVII.1.2. *(C-D-J-S) There is a Π^0_1 set A of degree $\mathbf{0}'$ and a minimal, thin Π^0_1 class P such that $D(P) = \{A\}$.*

Proof. Let $B = \mathbf{0}'$ be the union of uniformly computable sets B^s. Let $T_0, T_1, \ldots$ be an effective enumeration of the primitive recursive trees on $2^{<\omega}$. We will define a Π^0_1 retraceable set $A = \{a_0 < a_1 < \cdots\}$ and a corresponding Π^0_1 class $P = I(A)$ of initial subsets of A, by Theorem III.6.2, such that

(1) For any e, $e \in B \iff e \in B^{a_e}$.

(2) For any Π^0_1 class $P_e = [T_e]$, if $A \in P_e$, then $A_n \in P_e$ for all $n \geq e$.

By property (1), $\mathbf{0}'$ is recursive in A, so that, since A is Π^0_1, A has degree $\mathbf{0}'$. It then follows from (2) as in the proof of Theorem XVII.1.1 that P is minimal and thin.

The sequence $a_0 < a_1 < \cdots$ is defined by Π^0_1 recursion in the style of Theorem III.6.5 by making a_n the least a which satisfies the following:

(i) For all $m < n$, $a_m < a$.

(ii) $n \in B \rightarrow n \in B^a$.

(iii) For all $m < n$, either $< a_0, \ldots, a_{n-1}, a > \notin T_m$ or $(\forall x)(< a_0, \ldots, a_{n-1}, x > \in T_m)$.

(iv) For all $x < a$, either

(a) $x \leq x_{n-1}$ or

(b) $n \in B^a \setminus B^x$ or

(c) for some $m < n$, $< x_0, x_1, \ldots, x_{n-1}, x > \in T_m \& < x_0, x_1, \ldots, x_i, a > \notin T_m$.

The details are left to the reader. □

The following result is Theorem 2.13 of [25] (p. 102).

Theorem XVII.1.3. *(C-D-J-S) Let T be a recursive tree and P a Π^0_1 class such that $P = [T]$. Then for any set $A \in P$,*

(a) If $P \subset \mathcal{P}(A)$, then $A \leq_T Ext(T)$.

(b) If A is a Π^0_1 set and P is thin, then $A \leq_T Ext(T)$

(c) If T has no dead ends and A is either r. e. or co-r. e., then A is recursive.

Proof. (a) To test whether $n \in A$, simply see if there is a $\sigma \in Ext(T)$ of length $n+1$ such that $\sigma(n) = 1$.

(b) Note that $\mathcal{P}(A)$ is a Π^0_1 class, so that $Q = P \cap \mathcal{P}(A)$ is a Π^0_1 subclass of P and is nonempty since $A \in Q$. Since P is thin, we must have $Q = P \cap U$ for some clopen $U = I(\sigma_0) \cup \cdots \cup I(\sigma_k)$. If we now define

$T_Q = \{\sigma \in T : \sigma \mathrm{iscompatiblewith} \sigma_i, \mathrm{forsome} i \leq k\}$,

then it is clear that $Q = [T_Q]$ and that

$Ext(T_q) = \{\sigma \in Ext(T) : \sigma \mathrm{iscompatiblewith} \sigma_i, \mathrm{forsome} i \leq k\}$,

so that $Ext(T_Q)$ is recursive in $Ext(T)$. Now $Q \subset \mathcal{P}(A)$, so that by (a) we have $A \leq_T Ext(T_Q) \leq_T Ext(T)$.

(c) It is immediate from (b) that if A is Π^0_1 set, then A is recursive. If A is an r.e. set, then $\omega \setminus A$ is Π^0_1 and belongs to the thin Π^0_1 class $\{\omega \setminus X : X \in P\}$. □

Theorem XVII.1.4. *(C-D-J-S) Let T be a recursive tree such that $P = [T]$ is a thin Π^0_1 class and let $A \in P$. Then*

(a) $A' \leq_T A \oplus \mathbf{0}''$ (so that it is not possible that $A \geq_T \mathbf{0}''$.)

(b) If T has no dead ends, then $A' \leq_T A \oplus \mathbf{0}'$ (so that it is not possible that $A \geq_T \mathbf{0}'$.)

Proof. (a) Let $P = [T]$ be thin and suppose $A \in P$. For each e, let $Q_e = \{C : \phi_e^C(e) \uparrow\}$. Then Q_e is a Π^0_1 class, so there is a clopen set $U(e)$ such that $P \cap Q_e = P \cap U(e)$. Thus if $\phi_e^A(e) \uparrow$, then there is some $\sigma = A\lceil n$ such that σ forces $\phi_e^A(e) \uparrow$, that is, such that, for any $B \in P$, if $\sigma \prec B$ then $\phi_e^B(e) \uparrow$. Now define the Π^0_2 relation $R(e, \sigma)$ which says that σ forces $\phi_e^B(e) \uparrow$, by
$R(e,\sigma) \iff (\forall \tau \succ \sigma)[(\tau \in T \& \phi_e^\tau(e) \downarrow) \rightarrow \tau \notin Ext(T)]$.

Then we can compute from A together with $\mathbf{0}''$, whether $e \in A'$ by searching for the least n such that, for $\sigma = x\lceil n$, either $\phi_e^\sigma(e) \downarrow$, in which case $e \in A'$, or $R(e, \sigma)$, in which case $e \notin A'$.

(b) 0bserve that if $Ext(T)$ is recursive, then the relation R defined above will be recursive in $\mathbf{0}'$. □

It follows from (b) and Theorem 4.2 (b) above that if A has rank one in a thin Π^0_1 class $P = [T]$, where T has no dead ends, then A has *low* degree $\mathbf{a}$, that is, $\mathbf{a}' = \mathbf{0}'$.

Part (a) of this theorem is best possible in the sense that, as shown in Theorem 2.18 of [25], there is a minimal thin Π^0_1 class P and a set A such that $D(P) = \{A\}$ and $A \oplus \mathbf{0}' \equiv_T \mathbf{0}''$.

We conclude this section by stating without proof several further results from [25].

Theorem XVII.1.5. *(C-D-J-S) Between any two distinct r. e. degrees $\mathbf{b} < \mathbf{c}$, there is a degree $\mathbf{a}$, a set A of degree $\mathbf{a}$ and a minimal, thin Π^0_1 class P with $D(P) = \{A\}$.*

There is a family of c. e. degrees which contain members of thin Π^0_1 classes. In particular, it follows from Theorem 4.9 of Downey-Jockusch-Stob [65] that all array non-computable (*a.n.c.*) degrees and hence all non $-$ low$_2$ degrees contain members of thin Π^0_1 classes.

Theorem 5.8 tells us that no set of degree $\mathbf{0}''$ can even belong to a thin Π^0_1 class. Two further results give lower degrees which also contain no members of thin classes.

Theorem XVII.1.6. *(C-D-J-S) (a) There is an r.e. degree $\mathbf{a}$ such that no set B of degree $\mathbf{a}$ belongs to any thin Π^0_1 class.*

(b) There is a minimal degree $\mathbf{a} < \mathbf{0}'$ such that no set A of degree $\mathbf{a}$ is a member of any thin Π^0_1 class.

In contrast, we have the following improvement of Theorem V.4.14.

Theorem XVII.1.7. *(C-D-J-S) There is a non-recursive set $A \leq_T \mathbf{0}''$ such that every non-recursive set $B \leq_T A$ is a rank 1 member of a minimal, thin Π^0_1 class.*

Finally, there is another connection with maximal c. e. sets.

Theorem XVII.1.8. *(C-D-J-S) There is a maximal set A which is not a member of any thin Π^0_1 class.*

XVII.2 Initial Segments of the Lattice

XVII.3 Global Properties of the Lattice

XVII.4 Almost complemented classes

XVII.5 Perfect thin classes

The proof of the Low Basis Theorem IV.1.4 shows that every nonempty c.b. Π^0_1 class contains a member of c. e. degree $\mathbf{a}$ such that $\mathbf{a}$ is low, that is, $\mathbf{a} \oplus \mathbf{0}' = \mathbf{a}' = \mathbf{0}''$. The method of Theorem III.8.1 can be used to construct a nonempty Π^0_1 class with no computable members and no members of *high* degree, where the c. e. degree $\mathbf{a}$ is high if $\mathbf{a}' = 0''$.

Theorem XVII.5.1. *There exists a perfect, thin, c.b. Π^0_1 class P with no computable members such that if $\mathbf{a}$ is the degree of a member of P, then $\mathbf{a}' \leq a \oplus 0'$.*

Proof. Let P be the Π^0_1 class constructed in Theorem III.8.1 and let f be the function defined therein. Then f is the limit of a uniformly computable sequence of functions and is therefore computable in $\mathbf{0}'$ by the Limit Lemma. Now let $U_e = \{\sigma : \phi^\sigma_e(e)\uparrow\}$. It follows from the s-m-n theorem that there is a computable function ϕ such that $U_e = T_{\phi(e)}$ for each e. Then for any element A of P and any e, we have

$e \in A' \iff (\exists \sigma \prec A)\sigma \in U_e \iff x\lceil f(2e+2) \in T_{\phi(e)}$.

This shows that A' is computable in $A \oplus \mathbf{0}'$. □

xxx

There will be results here from the paper [50]

Chapter XVIII

Degrees of Difficulty

The Medvedev lattice was introduced in [149] to classify problems according to their degree of difficulty. A *mass problem* is a subset of $\mathbb{N}^{\mathbb{N}}$ and is thought of as representing the set of solutions to some problem. For example, the *problem of separability* of sets A and B is $S(A,B) = \{f : i \in A \to f(i) = 0 \ \&\ i \in B \to f(i) = 1\}$. The *coloring problem* for a given countably infinite graph G may be given as a set of functions each mapping ω into $\{1,2,3,4\}$. A mass problem is said to be *solvable* if it contains a computable function. See the survey paper by Sorbi [200] for more background.

In this chapter we study the Medvedev and Muchnik degrees of nonempty Π^0_1 classes. Each of these partial orderings is in fact a distributive lattice with top element, which can be viewed as the degree of the set of completions of Peano arithemtic, and bottom element, which is the degree of any set containing a computable member.

XVIII.1 Reducibility

P is *Medvedev reducible* to Q ($P \leq_M Q$) if there is a partial computable functional Φ which is defined for all $X \in Q$ and maps Q into P. Thus any solution of Q may be used to compute a solution of P, so we say that P has a lower (Medvedev) degree of difficulty than Q. There is also a nonuniform notion, Muchnik reducibility, given by $P \leq_w Q$ if every member X of Q computes a member of P, that is, $Y \leq_T X$ for some $Y \in P$. As usual, $P \equiv_M Q$ means that both $P \leq_M Q$ and $Q \leq_M P$, $P <_M Q$ means $P \leq_M Q$ but not $Q \leq_M P$, and the Medvedev degree $\mathbf{dg}_M(P)$ of P is the class of all sets Q such that $P \equiv_M Q$. Similar notations applies to Muchnik reducibility. Observe that $P \leq_M Q$ implies $P \leq_w Q$, so that the Medvedev degree of P is a subset of the Muchnik degree of P. Let $\mathcal{P}_M$ denote the lattice of Medvedev degrees of Π^0_1 classes and let $\mathcal{P}_w$ denote the lattice of Muchnik (or *weak*) degrees.

We will focus primarily on the Medvedev degrees.

First we show that only total functionals are needed for Medvedev reducibility of Π^0_1 classes.

Lemma XVIII.1.1. *For any Π^0_1 subclasses P and Q of ω^ω, if $P \leq_M Q$, then there exists a total computable functional $F : \mathbb{N}^\mathbb{N} \to \mathbb{N}^\mathbb{N}$ such that $F[Q] \subseteq P$.*

Proof. Given that $P \leq_M Q$, there is a partial computable functional Φ which maps Q into P. This means that there is a partial computable function ϕ mapping finite sequences to finite sequences such that $\Phi(X) = \cup_n \phi(X\lceil n)$ and with the property that $\sigma \prec \tau$ implies $\phi(\sigma) \prec \phi(\tau)$. Now Q may be expressed as the set of infinite paths through some computable tree T. Then we can extend the mapping Φ from Q to a total mapping F representing function f defined recursively as follows. Let $f(\emptyset) = \emptyset$. Then for any finite sequence σ and any n, define $f(\sigma^\frown n)$ in two cases. If $\sigma^\frown n \in T$, let $f(\sigma^\frown n) = \phi(\sigma^\frown n)$, which must be defined. If $\sigma^\frown n \notin T$, let $f(\sigma^\frown n) = f(\sigma)^\frown 0$. □

Simpson [194] has observed that Medvedev reducibility can be viewed as the analog for degrees of difficulty of the truth-table degrees for the Turing degrees, whereas Muchnik reducibility is the analog of Turing reducibility.

Proposition XVIII.1.2. *If $P \leq_M Q$, then for any $x \in Q$, there is a $y \in P$ such that $y \leq_{tt} x$.*

Proof. Suppose $P \leq_M Q$ for Π^0_1 classes P and Q. Then by Lemma XVIII.1.1, there is a total computable Φ which maps Q into P. Then for any element x of Q, it follows from Lemma II.5.10 that $\Phi(x) \leq_{tt} x$. □

The meet and join operations of the Medvedev lattice turn out to be the standard sum $P \oplus Q$ and product $P \otimes Q$ defined earlier. We summarize here some basic facts about these meet and join operations.

Proposition XVIII.1.3. *For any Π^0_1 classes P, Q and R,*

(i) *$P \oplus Q \equiv_M Q \oplus P$ and $P \otimes Q \equiv_M Q \otimes P$ (so that also $P \oplus Q \equiv_w Q \oplus P$ and $P \otimes Q \equiv_w Q \otimes P$*

(ii) *The Medvedev (Muchnik) degree of $P \oplus Q$ is the meet, or greatest lower bound, of the Medvedev (resp. Muchnik) degrees of P and Q;*

(iii) *The Medvedev degree of $P \otimes Q$ is the join, or least upper bound, of the Medvedev degrees of P and Q*

(iv) *$P \otimes (Q \oplus R) \equiv_M (P \otimes Q) \oplus (P \otimes R)$ and $P \otimes (Q \oplus R) \equiv_M (P \otimes Q) \oplus (P \otimes R)$.*

(v) *If $P \leq_M Q$, then, for any R, $(P \otimes R) \oplus Q \equiv_M P \otimes (Q \oplus R)$; if $P \leq_w Q$, then, for any R, $(P \otimes R) \oplus Q \equiv_w P \otimes (Q \oplus R)$*

Proof. (i) is obvious since these sets are in fact computably homeomorphic.

(ii) $P \oplus Q \leq_M P$ via the map $\Phi(x) = 0^\frown x$ and similarly $P \oplus Q \leq_M Q$. Suppose now that $R \leq P$ via Φ and $R \leq Q$ via Ψ. Then $R \leq P \oplus Q$ via the map taking $0^\frown x$ to $\Phi(x)$ and $1^\frown x$ to $\Psi(x)$.

(iii) $P \leq_M P \otimes Q$ via the map $\Phi(x) = (x(0), x(2), \dots)$ and similarly $Q \leq_M P \otimes Q$. Suppose now that $P \leq_M R$ via Φ and $Q \leq_M R$ via Ψ. Then $P \otimes Q \leq_M R$ via the map taking x to $\langle \Phi(x), \Psi(x) \rangle$.

(iv) To see that $P \oplus (Q \otimes R) \equiv_M (P \oplus Q) \otimes (P \oplus R)$, we define computable functionals in each direction. First define $\Phi : P \oplus (Q \otimes R) \to (P \oplus Q) \otimes (P \oplus R)$ by

$$\Phi(0^\frown X) = \langle 0^\frown X, 0^\frown X \rangle$$

and

$$\Phi(1^\frown \langle Y, Z \rangle) = \langle 1^\frown Y, 1^\frown Z \rangle.$$

Then define $\Psi : (P \oplus Q) \otimes (P \oplus R) \to P \oplus (Q \otimes R)$ as follows. Given $Z = \langle V, W \rangle \in (P \oplus Q) \otimes (P \oplus R)$, there are three cases.

$$\text{If } V = 0^\frown X, \text{ let } \Psi(Z) = V;$$
$$\text{if } V = 1^\frown Y \text{ and } W = 0^\frown X, \text{ let } \Psi(Z) = W;$$
$$\text{if } V = 1^\frown Y \text{ and } W = 1^\frown Z, \text{ let } \Psi(Z) = 1^\frown \langle Y, Z \rangle.$$

The other equivalence of (iv) is left as an exercise.

(v) Since $P \leq_M Q$, we have $P \oplus Q \equiv_M P$ and $P \otimes Q \equiv_M Q$. Then

$$(P \otimes R) \oplus Q \equiv_M (P \oplus Q) \otimes (R \oplus Q) \equiv_M P \otimes (Q \oplus R).$$

The same argument works for the Muchnik degrees. □

Corollary XVIII.1.4. *Both $\mathcal{P}_M$ and $\mathcal{P}_w$ are distributive lattices.*

Next we observe that $\mathcal{P}_M$ has both a least and a greatest element. The least element $\mathbf{0}$ consists of all classes P which contain a computable element. To see this, just let X_0 be a computable element of P and define $F(X) = X_0$ for any X. Then F maps any class Q into P, so that $P \leq_M Q$. In particular, the classes $\{0,1\}^\omega$ and $\{0^\omega\}$ are both in $\mathbf{0}$. Sorbi points out in [200] that this means that the solvable Medvedev degree is *definable* in the lattice (as the least element).

Proposition XVIII.1.5. *$\mathcal{P}_M$ has a greatest element.*

Proof. Since there is an enumeration $\{P_e\}_{e \in \omega}$ of the Π^0_1 classes, it seems natural to take the product of these classes as the univeral set and hence the top Medvedev degree. There is one hitch in that the empty set is not included in $\mathcal{P}_M$. To enumerate the *nonempty* Π^0_1 classes, first recall the usual enumeration $\{T_e\}_{e \in \mathbb{N}}$ of the primitive recursive trees in $\{0,1\}^*$ and let

$$\sigma \in T_e^+ \iff [\sigma \in T_e \ \vee \ (\forall m \leq |\sigma|)(\sigma \lceil m \notin T_e \to (\forall \tau \in \{0,1\}^m)\tau \notin T_e)].$$

Now if $P_e = \emptyset$ and m is the least such that $T_e \cap \{0,1\}^{m+1} = \emptyset$, then it is clear that $P_e^+ = [T_e^+] = \bigcup \{I(\sigma) : \sigma \in \{0,1\}^m \cap T_e\}$ and is still a Π^0_1 class. If $P_e \neq \emptyset$, $P_e^+ = P_e$. We claim that $\prod_e P_e^+$ is the greatest element of $\mathcal{P}_M$ and hence also of $\mathcal{P}_w$. That is, for any nonempty Π^0_1 class P_e, the projection map π_e takes $\prod_e P_e^+$ to P_e^+. □

XVIII.2 Completeness

Let $\mathcal{B}$ denote the computable Boolean algebra of clopen sets in $\{0,1\}^{\mathbb{N}}$; recall that these are finite unions of intervals $I(\sigma)$. Note that $\mathcal{B}$ is computably isomorphic to the Boolean algebra of propositional logic over an infinite set of variables–see Section IX.1 of Chapter IX. In particular, let $b_n \in \mathcal{B}$ denote $\{x : x(n) = 1\}$.

Definition XVIII.2.1. *Let P be a nonempty subset of $\{0,1\}^{\mathbb{N}}$. A* splitting function *for P is a computable function $g : \mathbb{N} \to \mathcal{B}$ such that, for all e, if $P_e \subseteq P$ and $P_e \neq \emptyset$, then $P_e \cap g(e)$ and $P_e - g(e)$ are both nonempty. P is said to be* productive *if it has a splitting function.*

Clearly a productive Π^0_1 class can have no subset that is a singleton and hence can have no computable member. It follows in particular that P is nowhere dense.

We observe that the class DNC_2 of diagonally non-computable functions in $\{0,1\}^{\mathbb{N}}$ is productive. This will be shown in the next section.

Theorem XVIII.2.2. *(Simpson) For any productive Π^0_1 class $P \subseteq \{0,1\}^{\mathbb{N}}$ and any nonempty Π^0_1 class $Q \subseteq \{0,1\}^{\mathbb{N}}$, there is a computable functional Φ from P onto Q. Thus any productive class is Medvedev complete.*

Proof. Let P and Q be Π^0_1 subsets of $\{0,1\}^{\mathbb{N}}$ and suppose that P is productive. We will define a computable monomorphism $f : \mathcal{B} \to \mathcal{B}$ such that, for all $b \in \mathcal{B}$, $Q \cap b = emptyset \iff P \cap f(b) = \emptyset$. Then the computable map $\Phi : P \to Q$ is defined by letting $Y = \Phi(X)$ be the unique element of Q such that $X \in f(Y\lceil n) \neq \emptyset$ for all n. The approximating function for Φ is something like the inverse of f. For an arbitrary element $Y \in Q$, $Q \cap I(Y\lceil n) \neq \emptyset$ for all n and hence $P \cap I(f(Y\lceil n)) \neq \emptyset$ for all n, so that $\cap_n[P \cap I(f(Y\lceil n))] \neq \emptyset$ and any element of this set will map to Y. This shows that Φ will map X onto Y.

It clearly suffices to define f for intervals $I(\sigma)$ and for ease of notation we will just write $f(\sigma)$ for $f(I(\sigma))$. Then $f(\sigma) = \Phi^{-1}(I(\sigma))$ under the function Φ defined above. That is, if $Y = \Phi(X)$, then $X \in f(\sigma) \iff \sigma \preceq Y$.

The function f is defined recursively beginning with $f(\emptyset) = \emptyset$.

For the recursive step, suppose that $f(\sigma) = a$ is given so that $Q \cap I(\sigma) = \emptyset \iff P \cap a = \emptyset$ and that $a \subseteq I(\tau)$ for some τ with $|\tau| \geq |\sigma|$. We will show how to compute $f(\sigma^\frown 0) = a_0$ and $f(\sigma^\frown 1) = a_1$ such that each a_i is included in some $I(\tau_i)$ with $|\tau_i| > |\tau|$ and such that

$$\sigma^\frown i \in T_Q \iff P \cap a_i \neq \emptyset.$$

Since P is productive, it is nowhere dense so we can partition a non-trivially into $b_0 \cup b_1 \cup b_2$ so that $P \cap b_0 = P \cap b_1 = \emptyset$ and hence $P \cap a = P \cap b_2$.

By the Recursion Theorem, we can compute $e \in \mathbb{N}$ such that

$$P_e = \begin{cases} P \cap b_2 & \text{if } \sigma^\frown 0 \in T_Q \text{ and } \sigma^\frown 1 \in T_Q; \\ P \cap b_2 \cap g(e) & \text{if } \sigma^\frown 0 \notin T_Q \text{ and } \sigma^\frown 1 \in T_Q; \\ P \cap b_2 - g(e) & \text{if } \sigma^\frown 0 \in T_Q \text{ and } \sigma^\frown 1 \notin T_Q; \\ \emptyset & \text{if } \sigma^\frown 0 \notin T_Q \text{ and } \sigma^\frown 1 \notin T_Q. \end{cases}$$

Now let $a_0 = b_0 \cup (b_2 \cap g(e))$ and $a_1 = \{0,1\}^{\mathbb{N}} - a_0 = b_1 \cup (b_2 - g(e))$. We claim the following:

1. $Q \cap I(\sigma^\frown 0) = \emptyset \iff P \cap a_0 = \emptyset$;
2. $Q \cap I(\sigma^\frown 1) = \emptyset \iff P - a_1 = \emptyset$.

There are several cases to check. Suppose first that $\sigma \notin T_Q$. Then by assumption $P \cap a = \emptyset$, so that we have $P \cap a_0 = P \cap a_1 = \emptyset = Q \cap I(\sigma^\frown 0) = Q \cap I(\sigma^\frown 1)$. Now suppose that $\sigma \in T_Q$, so that $Q \cap I(\sigma)$ and $P \cap a$ are both nonempty. There are three cases. First suppose that both $\sigma^\frown 0$ and $\sigma^\frown 1$ are in T_Q. Then $Q \cap I(\sigma^\frown 0)$ and $Q \cap I(\sigma^\frown 1)$ are both nonempty, so that $P_e = P \cap a \neq \emptyset$. It follows that $P \cap a_0 = P \cap b_2 \cap g(e) = P_e \cap g(e) \neq \emptyset$ (since g is a splitting function for P) and similarly $P \cap a_1 \neq \emptyset$. Next suppose that $\sigma^\frown 0 \notin T_Q$ but $\sigma^\frown 1 \in T_Q$, so that $P_e = P \cap b_2 \cap g(e) = P \cap a_0$. Then $P_e - g(e) = \emptyset$ and, since g is a splitting function for P, it follows that $P_e = \emptyset$, and hence

$$P \cap a_1 = (P \cap b_1) \cup (P \cap b_2 \cap g(e)) = \emptyset.$$

The remaining case where $\sigma^\frown 0 \in T_Q$ and $\sigma^\frown 1 \notin T_Q$ is similar. □

This result can be improved for two productive classes.

Theorem XVIII.2.3. *(Simpson) Any two productive Π^0_1 classes $P, Q \subseteq \{0,1\}^{\mathbb{N}}$ are computably homeomorphic.*

Proof. We simply add a back-and-forth argument to the proof of Theorem XVIII.2.3 to make the monomorphism onto. That is, at stage n, we will have a finite isomrphism $f_n : \mathcal{B}_n \simeq \mathcal{B}'_n$ where each of $\mathcal{B}_n$ and $\mathcal{B}'_n$ are finite subalgebras including $b_0, \ldots, b_n$ such that $P \cap a = \emptyset$ if and only if $Q \cap f_n(a) = \emptyset$. We start as above with $f((0)) = a_0$ and $f((1)) = a_1$ as above so that $P \cap a_i = \emptyset \iff Q \cap I((i)) = \emptyset$. Now use the splitting function for P to obtain $b_{ij} \subset I(i)$ for $i, j \in \{0,1\}$ so that $a_i \cap (j) \cap Q = \emptyset \iff b_{ij} \cap P = \emptyset$ and let $\mathcal{B}_0$ be generated by $\{b_{ij} : i, j \in \{0,1\}\}$ and $\mathcal{B}_1$ be generated by $\{I((0)), I((1)), a_0, a_1\}$. Note that $f^{-1}(I(e)) = b_{e0} \cup b_{e1}$. We leave the details to the reader. □

Question XVIII.2.4. *Suppose in general that P and Q are Π^0_1 subsets of $\{0,1\}^{\mathbb{N}}$ and that there there exist mappings Φ from P onto Q and Ψ from Q onto P. Does it follows that P and Q are computably homeomorphic?*

Lemma XVIII.2.5. *(Simpson) Let P and Q be nonempty subsets of $\{0,1\}^{\mathbb{N}}$. If $P \leq_M Q$ and P is productive, then Q is productive.*

Proof. Since $P \leq_M Q$, there is a computable function $\Phi : Q \to P$. Define $f : \mathcal{B} \to \mathcal{B}$ by $f(b) = \Phi^{-1}(b)$. Let $g : \mathbb{N} \to \mathcal{B}$ be a splitting function for P. By the s-m-n Theorem, let $h : \mathbb{N} \to \mathbb{N}$ be primitive recursive such that $P_{h(e)} = \Phi(P_e \cap Q)$ for all e. We claim that the composition $f \circ g \circ h$ is a splitting function for Q. To see this, suppose that $P_e \subseteq Q$ and $P_e \neq \emptyset$. Then $P_{h(e)} = \Phi(P_e) \subset P$ and $P_{h(e)} \neq \emptyset$, so that $P_{h(e)} \cap g \circ h(e) \neq \emptyset$ and $P_{h(e)} - g \circ h(e) \neq \emptyset$ and therefore $P_e \cap f \circ g \circ h(e) \neq \emptyset$. □

Corollary XVIII.2.6. *Let $P \subset \{0,1\}^{\mathbb{N}}$ be a nonempty Π^0_1 class. Then P is productive if and only if P is Medvedev complete.*

Proof. By Theorem XVIII.2.2, if P is productive, then P is Medvedev complete. Lemma XVIII.2.5 implies that any complete Π^0_1 class is productive. □

Simpson and Slaman (unpublished) have shown the following.

Theorem XVIII.2.7. *Every nonzero degree in $\mathcal{P}_w$ contains infinitely many Medvedev degrees from $\mathcal{P}_M$.*

Proof. xxxx

□

Exercises

XVIII.2.1. Prove parts (ii) and (iii) of Proposition XVIII.1.3 for the Muchnik degrees.

XVIII.2.2. Show that $P \otimes (Q \oplus R) \equiv_M (P \otimes Q) \oplus (P \otimes R)$.

XVIII.2.3. Show that the map taking the Medvedev degree of P to the Muchnik degree of P is a lattice homomorphism of $\mathcal{P}_M$ onto $\mathcal{P}_w$.

XVIII.3 Separating Classes

Here is a general result which will provide a large class of Medvedev complete Π^0_1 classes. A pair of disjoint c. e. sets A and B are said to be *effectively inseparable* if there is a computable function ϕ such that, for any x and y, if $A \subset W_x$ and $B \subset W_y$ and $W_x \cap W_y = \emptyset$, then $\phi(x,y) \notin W_x \cup W_y$. For example, it is well known that the set $PA = A_1$ of theorems of Peano Arithmetic and the set B_1 of negations of theorems of Peano Arithmetic are *effectively inseparable*–see Odifreddi [163], p 356. The following lemma will then imply that $S(A_1, B_1)$ is Medvedev complete.

Proposition XVIII.3.1. *If A are B are effectively inseparable c.e. sets, then $S(A,B)$ is a productive Π^0_1 class.*

Proof. Let $P = S(A,B)$ where A and B are effectively inseparable c. e. sets and and let ϕ be given as above. Define $W_{f(e)} = \{n : (\forall X \in P_e) n \in X\}$ and $W_{h(e)} = \{n : (\forall X \in P_e) n \notin X\}$. To see that these are indeed c. e. sets, note that $W_{f(e)}$ has an alternate definition, that is,

$$n \in W_{f(e)} \iff (\forall \sigma \in \{0,1\}^{n+1})(\sigma \in T_e \implies \sigma(n) = 1),$$

Clearly $W_{f(e)} \cap W_{h(e)} = \emptyset$, and if $P_e \subset P$, then $A \subset W_{f(e)}$ and $B \subset W_{h(e)}$. Thus $\phi(f(e), h(e)) = n \notin W_{f(e)} \cup W_{h(e)}$. Hence there exist X and Y in P_e such that $n \in X$ and $n \notin Y$. The splitting function for P can thus be defined by $g(e) = \{X : \phi(f(e), h(e)) \in X\}$. □

Proposition XVIII.3.2. *The Π^0_1 class DNC_2 is the separating class of a pair of effectively inseparable c. e. sets.*

Proof. Let $A = \{e : \phi_e(e) = 0\}$ and $B = \{e : \phi_e(e) = 0\}$. Then $S(A, B) = DNC_2$. Now suppose that $A \subseteq W_x$ and $B \subseteq W_y$ and suppose that $W_x \cap W_y = \emptyset$. We will show how to compute $\phi(x, y) = e$ such that $e \notin W_x \cup W_y$. Let $\psi(e, x, y) = 1$, if $e \in W_x$ and $= 0$, if $e \in W_y$. That is, $\psi(e, x, y)$ searches for the least s such that $e \in W_{x,s} \cup W_{y,s}$ and then outputs 1 if $e \in W_{x,s}$ and outputs 0 if $e \in W_{y,s} - W_{x,s}$. Let $\phi(x, y) = e$ so that $\phi_e(i) = \psi(e, x, y)$. We claim that $e \notin W_x \cup W_y$. To see this, suppose that $e \in W_x$, so that by definition of ψ, ϕ and e, $\phi_e(e) = 1$. Then $e \in B$, which implies that $e \in W_y$ and contradicts $W_x \cap W_y = \emptyset$. The argument when $e \in W_y$ is similar. □

Corollary XVIII.3.3. *DNC_2 is Medvedev complete.*

Proof. By Propositions XVIII.3.2 and XVIII.3.1, DNC_2 is productive and hence by Theorem XVIII.2.2, it is Medvedev complete. □

The family of c.e. separating classes are closed under join, since

$$S(A, B) \otimes S(C, D) = S(\langle A, C\rangle, \langle B, D\rangle).$$

However, there is no non-trivial meet for c.e. separating classes, as shown by the following.

Lemma XVIII.3.4. *For any Π^0_1 class P and any clopen sets G, and H, if $P \cap G \leq_M P \cap H$, then $P \cap G \equiv_M P \cap (G \cup H)$.*

Proof. First, $P \cap (G \cup H) \leq_M P \cap G$ via the identity map. Fix a computable functional $\Phi : P \cap H \to P \cap G$ and define $\Psi : P \cap (G \cup H) \to P \cap G$ by

$$\Psi(X) := \begin{cases} X, & \text{if } X \in G; \\ \Phi(X), & \text{otherwise.} \end{cases}$$

Note that Ψ is computable since clopen sets are simply finite unions of intervals. □

Lemma XVIII.3.5. *For any c.e. separating class P and any clopen set G, if $P \cap G \neq \emptyset$, then $P \cap G \equiv_M P$.*

Proof. By Lemma XVIII.3.4, it suffices to prove this for intervals, and we proceed by induction on the length n of σ. If $n = 0$, then $I(\sigma) = 2^\omega$, so $P \cap I(\sigma) = P$. Assume as induction hypothesis that $P \cap I(\sigma) \equiv_M P$ for some σ of length n, and suppose that $P \cap I(\sigma^\frown(e)) \neq \emptyset$. If $P \cap I(\sigma^\frown(1-e)) = \emptyset$, then $P \cap I(\sigma^\frown(e)) = P$. Otherwise, $P \cap I(\sigma^\frown(e)) \equiv_M P \cap I(\sigma^\frown(1-e))$ via the computable maps $X \mapsto X \cup \{0\}$ and $X \mapsto X/\{0\}$. Then by Lemma XVIII.3.4 again,

$$P \cap I(\sigma^\frown(e)) \equiv_M P \cap \big(I(\sigma^\frown(e)) \cup I(\sigma^\frown(1-e))\big) = P. \qquad \square$$

Proposition XVIII.3.6. *For any Π^0_1 classes P and Q and any c.e. separating class R, if $P \oplus Q \leq_M R$, then either $P \leq_M R$ or $Q \leq_M R$.*

Proof. Fix a computable functional $\Phi : R \to P \oplus Q$ and set $G := \{X : \Phi(X) \in I((0))\}$. G is clopen as the continuous inverse image of an interval. $P \leq_M R \cap G$ via the map $X \mapsto \big(k \mapsto \Phi(X)(k+1)\big)$. If $R \cap G \neq \emptyset$, then by Lemma XVIII.3.5 $R \cap G \equiv_M R$, so $P \leq_M R$. Otherwise $R \setminus G \neq \emptyset$ and we have similarly $Q \leq_M R$. $\square$

This suggests that we should consider the sublattice of $\mathcal{P}_M$ generated by the family of c.e. separating degrees. This turns out to have a simple direct characterization.

Definition XVIII.3.7. *For any tree $T \subseteq \{0,1\}^{<\omega}$ and any Π^0_1 class $P \subseteq \{0,1\}^\omega$,*

(i) T is homogeneous *iff* $(\forall \sigma, \tau \in T)(\forall i < 2)$,

$$|\sigma| = |\tau| \implies (\sigma^\frown i \in T \iff \tau^\frown i \in T);$$

(ii) T is almost homogeneous *iff* $\exists n (\forall \sigma, \tau \in T)(\forall i < 2)$,

$$n \leq |\sigma| = |\tau| \wedge \sigma \restriction n = \tau \restriction n \implies (\sigma^\frown i \in T \iff \tau^\frown i \in T);$$

The least such n is called the modulus *of T;*

(iii) P is (almost) homogeneous *iff T_P is (almost) homogeneous; a Medvedev degree is* (almost) homogeneous *iff it contains an (almost) homogeneous class;* **AH** *denotes the family of almost homogeneous degrees.*

Proposition XVIII.3.8. *For any Π^0_1 class P,*

$$P \text{ is homogeneous} \iff P \text{ is a c.e. separating class.}$$

Proof. If $P = S(A,B)$ for c.e. sets A and B, then

$$T_P = \{\sigma : (\forall i < |\sigma|)\big[\sigma(i) = 0 \wedge i \notin A) \vee (\sigma(i) = 1 \wedge i \notin B)\big]\}.$$

This is clearly a homogeneous tree. Conversely, if T_P is homogeneous, then $P = S(A,B)$ for

$$A = \{n : 0^{n\frown}0 \notin T_P\} \qquad \text{and} \qquad B = \{n : 0^{n\frown}1 \notin T_P\}.$$

$\square$

Corollary XVIII.3.9. *For any Π^0_1 class P, if P is almost homogeneous with modulus n, then P is the disjoint union of 2^n c.e. separating classes.*

Proof. Given $P \in \mathbf{AH}$ with modulus n, for each sequence σ of length n, let $P[\sigma] := \{X \in P : \sigma \prec X\}$. Each $P[\sigma]$ is homogeneous, so is a c.e. separating class, and clearly P is the disjoint union of the $P[\sigma]$. □

Proposition XVIII.3.10. *For any Π^0_1 classes P and Q, if P and Q are almost homogeneous, then also $P \oplus Q$ and $P \otimes Q$ are almost homogeneous.*

Proof. If P and Q are almost homogeneous with moduli m and n, respectively, then easily $P \oplus Q$ is almost homogeneous with modulus $\max\{m, n\} + 1$ and $P \otimes Q$ is almost homogeneous with modulus $2\max\{m, n\}$. □

Theorem XVIII.3.11. **AH** *is the smallest sublattice of $\mathcal{P}_M$ which includes the family of c.e. separating degrees.*

Proof. By the preceding two propositions, **AH** is a sublattice of $\mathcal{P}_M$ which includes the family of c.e. separating degrees. Let L be any other such lattice; we prove by induction that for all n,

$$P \text{ is almost homogeneous with modulus } n \quad \Longrightarrow \quad \mathbf{dg}_M(P) \in L.$$

For $n = 0$ this is true by Proposition XVIII.3.8, so assume as induction hypothesis that it holds for n and that P is almost homogeneous with modulus $n + 1$. Then if for $i < 2$ we set $P_i := \{X : (i)X \in P\}$, P_i is almost homogeneous with modulus n, so $\mathbf{dg}_M(P_i) \in L$ and clearly $P = P_0 \oplus P_1$ so also $\mathbf{dg}_M(P) \in L$. □

Of particular interest are the generalizations

$$DNC_k = \{X \in k^{\mathbb{N}} : (\forall n) X(n) \neq \phi_n(n)\}.$$

The next theorem is due to Jockusch [92]. We will use the following lemma.

Lemma XVIII.3.12. *(Cenzer-Hinman) For any $l < k$, any $s > 0$ and any function $F : k^s \to l$, there exists $j < l$ and a tree $T \subseteq k^{\leq s}$ such that*

(i) for all $\sigma \in T$, there exist $i_0 \neq i_1 < k$ such that $\sigma^\frown i_t \in T$ for $i = 0, 1$;

(ii) for all $\tau \in T \cap k^s$, $F(\tau) = j$.

Proof. The proof is by induction on s. For $s = 1$, this is just the pigeonhole principle. Now given $F : k^{s+1} \to l$, define $G : k^s \to l$ by

$$G(\tau) = (\text{least } j < l)[(\exists i_0 < i_1) F(\tau^\frown i_0) = F(\tau^\frown i_1) = j];$$

such a j must exist for each τ by considering the map $F_\tau : k \to l$ defined by $F_\tau(i) = F(\tau^\frown i)$.

Now by induction there exists $j < l$ and a tree $T_G \subseteq k^{\leq s}$ satisfying (i) and (ii) above with respect to G. Let

$$T = T_G \cup \{\tau^\frown i : \tau \in T_G \cap k^s \ \& \ F(\tau^\frown i) = j\}.$$

It is easy to check that T satisfies conditions (i) and (ii) with respect to F. □

Theorem XVIII.3.13. *For all $n > 1$, $DNC_{k+1} <_M DNC_k$.*

Proof. Note here that in general $\phi_e : \mathbb{N} \to \mathbb{N}$ and we let $\phi_e^k(n) = max\{k - 1, \phi_e(n)\}$ to get the eth function in $k^{\mathbb{N}}$. $DNC_{k+1} \leq_M DNC_k$ by the the map $\Phi(X) = X$. Now suppose by way of contradiction that $DNC_k \leq_M DNC_{k+1}$ and let $\Phi : DNC_{k+1} \to DNC_k$. We will show how to use Φ to compute an element Y of DNC_k, which is the contradiction.

Given n, we can compute $Y(n)$ as follows. First compute a level s such that $\Phi(\sigma, n) \downarrow$ for all $\sigma \in (k+1)^s$ and consider the map $F : (k+1)^s \to k$ defined by $F(\sigma) = \Phi(\sigma, n)$. By Lemma XVIII.3.12, there exits $j_n < k$ and a tree $T \subseteq (k+1)^{\leq s}$ such that $F(\tau) = j_n$ for all $\tau \in T \cap (k+1)^s$ and such that any $\sigma \in T \cap (k+1)^{<s}$ has at least two extensions in T. We now show that there is in fact some $\tau \in T \cap (k+1)^s$ such that $I(\tau) \cap DNC_{k+1} \neq \emptyset$ and hence some $X \in DNC_{k+1}$ such that $\Phi(X)(n) = \Phi(\sigma, n) = j_n$. Since $\Phi(X) \in DNC_k$, it will follow that $\phi_n(n) \neq j_n$ and hence we can compute $Y \in DNC_k$ by taking $Y(n) = j_n$ for each n.

The path $\tau = (e_0, e_1, \ldots, e_{s-1} \in T \cap (k+1)^s$ exists by the following (although we cannot directly compute it). For $t = 0$, we have $(i_0) \neq (i_1)$ in T and at least one of these does not equal $\phi_0(0)$; let e_0 be the least such. Given $\sigma = (e_0, \ldots, e_{t-1}) \in T$, again there exist $i_0 < i_1$ such that both $\sigma^\frown i_0$ and $\sigma^\frown i_0$ are in T and again at least on of i_0, i_1 is not equal to $\phi_t(t)$ so we can choose e_0 to be the least such. □

Exercises

XVIII.3.1. Say that c. e. sets A and B are *weakly effectively inseparable* if there is a computable function F, mapping ω^2 into the family of finite sets of natural numbers, such that, for any x and y, if $A \subset W_x$ and $B \subset W_y$ and $W_x \cap W_y = \emptyset$, then $F(x, y)$ contains at least one element which is not in $W_x \cup W_y$. Show that if $S(A, B)$ is productive, then A and B are weakly effectively inseparable.

XVIII.3.2. Recall from Section III.III.9 the class $CC(\mathcal{T})$ of complete consistent extensions of a c. e. propositional theory $\mathcal{T}$. Show that for any c. e. theory $\mathcal{U}$, $CC(\mathcal{U})$ is Medvedev complete if and only if, for every c. e. theory $\mathcal{T}$, there exists a computable function $\Phi : CC(\mathcal{U}) \to CC(\mathcal{T})$.

XVIII.3.3. Let the eth c. e. theory $\mathcal{T}_e = \{\gamma_i : i \in W_e\}$, where $\gamma_i : i \in \mathbb{N}\}$ enumerates the sentences of propositional logic. Let us say that a theory $\mathcal{T}$ is *effectively incompletable* if there exists a computable mapping $\theta : \mathbb{N} \to Sent$ such that for all a, if $\mathcal{T} \subseteq T_a$ and T_a is consistent, then both $W_a \cup \{\theta(a)\}$ and $W_a \cup \{\neg\theta(a)\}$are consistent. Show that, for any c. e. theory $\mathcal{U}$, $\mathcal{U}$ is effectively incompletable if and only if $CC(\mathcal{U})$ is productive and hence $CC(\mathcal{U})$ is Medvedev complete if and only if $CC(\mathcal{U})$ is effectively incompletable.

XVIII.3.4. For a pair A, B of disjoint c. e. sets, let $S(A, B)$ represent the logical theory $\mathcal{U}(A, B)$ with axioms $\{A_i : i \in A\} \cup \{\neg A_i : i \in B\}$. Show that for any pair A, B of effectively inseparable c. e. sets, $\mathcal{U}(A, B)$ is effectively incompletable. Hint: There exist computable functions f and g such that $W_{f(e)} = \{i : A_i \in \mathcal{T}_e\}$and $W_{g(e)} = \{i : \neg A_i \in \mathcal{T}_e\}$.

XVIII.3.5. Let $\mathcal{U}$ be an effectively incompletable c. e. theory. Then for any c. e. theory $\mathcal{T}$, there exists a computable mapping $\Theta : Sent^3 \to Sent$ such that if both $\mathcal{T} \cup \{\phi\}$ and $\mathcal{U} \cup \{\psi\}$ are consistent, then

(a) $\mathcal{T} \cup \{\phi, \chi\}$ is consistent $\longleftarrow \mathcal{U} \cup \{\psi, \theta(\phi, \psi, \chi)\}$ is consistent;

(b) $\mathcal{T} \cup \{\phi, \neg\chi\}$ is consistent $\longleftarrow \mathcal{U} \cup \{\psi, \neg\theta(\phi, \psi, \chi)\}$ is consistent.

Hint: Use the Recursion Theorem to compute an index a from ϕ, ψ, χ such that $\mathcal{T}_a$ equals $Con(\mathcal{U} \cup \{\psi, \theta_a\})$ if $\mathcal{T} \cup \{\phi, \chi\}$ is consistent, equals $Con(\mathcal{U} \cup \{\psi, \neg\theta_a\})$ if $\mathcal{T} \cup \{\phi, \neg\chi\}$ is consistent, and equals $Con(\mathcal{U} \cup \{\psi\})$ otherwise.

XVIII.3.6. Show directly that any effectively incompletable c. e. theory $\mathcal{U}$ is Medvedev complete. Hint: given any c. e. theory $\mathcal{T}$, recursively define a mapping $\psi : \{0,1\}^* \to Sent$ taking σ to ψ_σ by $\psi(\emptyset) = p_0 \vee \neg p_0$ and for any σ of length n, $\psi(\sigma^\frown 0) = \psi_\sigma \wedge \neg\Theta(q_\sigma, \psi_\sigma, A_n)$ and $\psi(\sigma^\frown 1) = \psi_\sigma \wedge \Theta(q_\sigma, \psi_\sigma, A_n)$, where q_σ denotes the conjunction over $i < n$ of $\{A_i : \sigma(i) = 1\} \cup \{\neg A_i : \sigma(i) = 0\}$. Then $\mathcal{T} \cup \{q_\sigma$ is consistent if and only if $\mathcal{U} \cup \{\psi(\sigma)\}$ is consistent and for $X \in CC(\mathcal{U})$, we may define $\Phi(X) \in CC(\mathcal{T})$ to be the unique Y such that $X \in CC(\mathcal{U} \cup \{\psi(Y\lceil n)\}$ for all n.

XVIII.4 Measure

In this section we give the result of Cenzer and Hinman [26] that there is no Medvedev complete Π^0_1 class of positive measure and indeed that no class of positive measure can be $\leq_M$ any separating class. The following lemma is due to Simpson [194].

Lemma XVIII.4.1. *(Simpson) Let $\{F_n\}_{n\in\omega}$ be a sequence of nonempty finite subsets of $\mathbb{N}$ of bounded cardinality and let $S = \prod_n F_n$. Let $P \subseteq \{0,1\}^{\mathbb{N}}$ have positive measure and let $Q \subseteq \mathbb{N}^{\mathbb{N}}$ be nonempty.*

1. *If $S \leq_M P \otimes Q$, then $S \leq_M Q$.*
2. *If $S \leq_w P \otimes Q$, then $S \leq_w Q$.*

Proof. (1) The proof is similar to that of Theorem IV.3.11. Suppose that $card(F_n) < k$ for all n. Let U and V be clopen so that $\mu(V - P) < \mu(P)$ and $\mu(V - U)$ are both $< \mu(P)/4k$ and therefore $\mu(U - P) < \mu(U)/K$. It is important here that $\mu(U)$ is rational. Let Φ be a computable function such that $\Phi(x \oplus y) \in S$ for all $x \in P$ and $y \in Q$. Given $y \in Q$ and $n \in \mathbb{N}$, we can compute $m = \Psi(y)(n)$ such that $\mu(\{x \in U : \Phi(x \oplus y)(n) = m\}) > \mu(U)/k$ and therefore $m \in F_n$. Thus Ψ maps Q onto S and hence $S \leq_M Q$.

(2) Fix $y \in Q$ and note that $S \leq_w P \otimes \{y\}$. Now for each $x \in P$, there is a function $\Phi_{e(x)}$ such that $\Phi_{e(x)}(x, y) \in S$. By countable additivity of μ, there is a single function Φ and a subset P_g of P of positive measure such that $S \leq_M P_g \otimes \{y\}$ Now by lemma XVIII.4.1, $S \leq_M \{g\}$. It follows that $S \leq_s Q$. $\Box$

Note that in particular these lemmas apply to any separating class S.

Theorem XVIII.4.2. *(Simpson) Let P, Q be nonempty Π^0_1 subsets of $\{0,1\}^{\mathbb{N}}$ and suppose that P has positive measure. If $Q <_M \mathbf{1}$, then $P \otimes Q <_M 1$ and similarly if $Q <_w \mathbf{1}$, then $P \otimes Q <_w \mathbf{1}$.*

Proof. This follows from Lemma XVIII.4.1, since there is a Medvedev complete separating class by Corollary XVIII.3.3. $\Box$

Corollary XVIII.4.3. *If $P \subset \{0,1\}^{\mathbb{N}}$ is a Π^0_1 class with positive measure, then $P <_w \mathbf{1}$.*

Proof. This follows from Theorem XVIII.4.2 by letting $Q = \{0,1\}^{\mathbb{N}}$. $\Box$

Exercises

XVIII.4.1. Show that for any Π^0_1 classes P and Q, if P has positive measure and $DNC_k \leq_M P \otimes Q$, then $DNC_k \leq_M Q$. Conclude that $P \otimes DNC_k >_M P \otimes DNC_{k+1}$ for all k.

XVIII.5 Randomness

Recall the notion of 1-randomness from Section 4.3. Let R be the class of all 1-random reals in $\{0,1\}^{\mathbb{N}}$. It follows from Theorem 4.IV.3.5 that there is a Π^0_1 class P with positive measure such that $P \subset R$.

Theorem XVIII.5.1. *(Simpson) For any Π^0_1 class $P \subset R$ with positive measure, $P \equiv_w R$.*

Proof. Since $P \subset R$, it follows that $R \leq_w P$. On the other hand, Theorem 4.IV.3.6 tells us that P has elements of every 1-random degree, so for any $x \in R$, there exists $y \in P$ with $y \equiv_T x$ and hence $P \leq_w R$. $\Box$

Corollary XVIII.5.2. *If a Π^0_1 class P contains a random real, then $P \equiv_w R$.* $\Box$

Corollary XVIII.5.3. *(Simpson) The Muchnik degree of R can be characterized as the unique largest Muchnik degree of any Π^0_1 class $P \subseteq \{0,1\}^{\mathbb{N}}$ such that $\mu(P) > 0$.*

Proof. By Theorem XVIII.5.1, there is a Π^0_1 class $P \subseteq \{0,1\}^{\mathbb{N}}$ with positive measure and with $P \equiv_w R$. Now let P be any Π^0_1 class P of positive measure. It follows from Theorem 4.IV.3.6 that $P \leq_w R$. $\Box$

Exercises

XVIII.5.1.

XVIII.6 Thin Classes

In this section, we prove the following theorem of Simpson.

Theorem XVIII.6.1. *If $Q \subseteq \{0,1\}^{\mathbb{N}}$ is a nonempty perfect thin Π^0_1 class and $R \subset \{0,1\}^{\mathbb{N}}$ is the set of all Martin-Löf random reals, then Q and R and Muchnik incomparable.*

The theorem is proved using a sequence of lemmas.

Lemma XVIII.6.2. *Let $Q \subset \{0,1\}^{\mathbb{N}}$ be nonempty thin Π^0_1 class, let x be Martin-Löf random, and let $y \in Q$ be almost computable. Then x is not Turing reducible to y.*

Proof. Suppose by way of contradiction that $x \leq_T y$. Then by Theorem II.5.13 $x \leq_{tt} y$. Now by Theorem II.5.10, there is a total computable functional $\Phi : \{0,1\}^{\mathbb{N}} \to \{0,1\}^{\mathbb{N}}$ mapping y to x. Now by Lemma III.8.4, the image $\Phi[Q]$ is a thin Π^0_1 class and hence has measure zero by Theorem **??** But $x \in \Phi[Q]$ is Martin-Löf random and hence $\mu(\Phi[Q]) > 0$ by Exercise 5. □

Since every nonempty Π^0_1 class contains an almost computable member by Theorem **??**, it follows that R is not Muchnik reducible to Q. The following lemma is due to Demuth [57].

Lemma XVIII.6.3. *Let $x \in \{0,1\}^{\mathbb{N}}$ be Martin-Löf random and let $y \leq_{tt} x$ be noncomputable. Then there exists $z \equiv_T y$ such that z is Martin-Löf random.*

Lemma XVIII.6.4. *If $Q \subseteq \{0,1\}^{\mathbb{N}}$ is a nonempty perfect thin Π^0_1 class, let $y \in Q$ and let x be Martin-Löf random and almost computable. Then y is not Turing reducible to x.*

Proof. Suppose by way of contradiction that $y \in Q$ and $y \leq_T x$. Then $y \leq_{tt} x$ since x is almost computable. It follows from Lemma XVIII.6.3 that there is a random $z \equiv_T y$. But then $z \leq_T x$, contradicting Lemma XVIII.6.2. □

To complete the proof of Theorem **??**, let x be random and almost computable and let Q be a nonempty perfect thin Π^0_1 class. It follows from Lemma XVIII.6.4 that no member of Q is computable from x. □

Corollary XVIII.6.5. *There is a Π^0_1 separating class Q and a Π^0_1 class Q' such that $Q < wQ'$, and furthermore, for any separating class P, if P is Muchnik reducible to Q', then P is Muchnik reducible to Q.*

Proof. Let Q be a perfect thin Π^0_1 class which is separating III.8.1, let R be a Π^0_1 class of randoms and let $Q' = Q \otimes R$. It follows from Theorem **??** that Q' is not Muchnik reducible to Q. The furthermore remark follows from Lemma **??**. □

Chapter XIX

Random Closed Sets

The study of algorithmic randomness has been of great interest in recent years. The basic problem is to quantify the randomness of a single real number; here we will extend this problem to the randomness of the set of paths through a finitely-branching tree. Early in the last century, von Mises [214] suggested that a random real should obey reasonable statistical tests, such as having a roughly equal number of zeroes and ones of the first n bits, in the limit. Thus a random real would be *stochastic* in modern parlance. If one considers only *computable* tests, then there are countably many and one can construct a real satisfying all tests.

An early approach to randomness was through betting. Effective betting on a random sequence should not allow one's capital to grow unboundedly. The betting strategies used are constructive martingales, introduced by Ville [213] and implicit in the work of Levy [130], which represent fair double-or-nothing gambling.

Martin-Löf [146] observed that stochastic properties could be viewed as special kinds of meaure zero sets and defined a random real as one which avoids certain effectively presented measure 0 sets. That is, a real $x \in \{0,1\}^{\mathbb{N}}$ is Martin-Löf random if for any effective sequence $S_1, S_2, \ldots$ of c.e. open sets with $\mu(S_n) \leq 2^{-n}$, $x \notin \cap_n S_n$.

At the same time Kolmogorov [114] defined a notion of randomness for finite strings based on the concept of *incompressibility.* For infinite words, the stronger notion of prefix-free complexity developed by Levin [129], Gács [74] and Chaitin [47] is needed. Schnorr later proved [185] that the notions of constructive martingale randomness, Martin-Löf randomness, and prefix-free randomness are equivalent. In this chapter, we will consider algorithmic randomness on the space $\mathcal{C}$ of nonempty closed subsets P of $\{0,1\}^{\mathbb{N}}$.

The betting approach to randomness is formalized as follows:

Definition XIX.0.6 (Ville [213]). *1. A* martingale *is a function* $d : n^{<\mathbb{N}} \to$

$[0,\infty)$ *such that for all* $\sigma \in n^{<\mathbb{N}}$,

$$d(\sigma) = \frac{1}{n}\sum_{i=0}^{n-1} d(\sigma^\frown i).$$

2. *A martingale* d succeeds *on* $X \in n^{\mathbb{N}}$ *if*

$$\limsup_{m\to\infty} d(X\lceil m) = \infty.$$

That is, the betting strategy results in an unbounded amount of money made on the binary string X.

3. *The* success set *of* d *is the set* $S^\infty[d]$ *of all sequences on which* d *succeeds.*

That is, a martingale on $2^{<\mathbb{N}}$ is the representation of a fair double-or-nothing betting strategy. When working on $3^{<\mathbb{N}}$ the strategy is triple-or-nothing.

Definition XIX.0.7. *A martingale* d *is* constructive (effective, c.e.) *if it is* lower semi-computable*; that is, if there is a computable function* $\hat{d} : n^{<\mathbb{N}} \times \mathbb{N} \to \mathbb{Q}$ *such that*

1. *for all* σ *and* t, $\hat{d}(\sigma, t) \leq \hat{d}(\sigma, t+1) < d(\sigma)$, *and*

2. *for all* σ, $\lim_{t\to\infty} \hat{d}(\sigma, t) = d(\sigma)$.

In other words, $d(w)$ is approximated from below by rationals uniformly in w. A sequence in $2^{\mathbb{N}}$ is considered random in this setting if no constructive martingale succeeds on it.

Martin-Löf randomness for reals, as defined above, is extended to closed sets by giving an effective homeomorphism with the space $\{0,1,2\}^{\mathbb{N}}$ and simply carrying over the notion of randomness from that space.

Prefix-free randomness for reals is defined as follows. Let M be a prefix-free function with domain $\subset \{0,1\}^*$; that is, if $\sigma \sqsubseteq \tau$ are strings in the domain of M, then σ must equal τ. For any finite string τ, let $K_M(\tau) = \min\{|\sigma|, \infty : M(\sigma) = \tau\}$. There is a *universal* prefix-free function U such that, for any prefix-free M, there is a constant c such that for all τ

$$K_U(\tau) \leq K_M(\tau) + c.$$

We let $K(\sigma) = K_U(\sigma)$. Then x is called *prefix-free random* if there is a constant c such that $K(x\lceil n) \geq n - c$ for all n. This means that the initial segments of x are not *compressible*.

For a tree T, we want to consider the compressibility of $T_n = T \cap \{0,1\}^n$. This has a natural representation of length 2^n since there are 2^n possible nodes of length n. We will show that any tree T_P can be compressed, that is, $K(T_n) \geq 2^n - c$ is impossible for a tree with no dead ends.

XIX.1 Martin-Löf Randomness of Closed Sets

In this section, we define a measure on the space $\mathcal{C}$ of nonempty closed subsets of $\{0,1\}^{\mathbb{N}}$ and use this to define the notion of randomness for closed sets. We then obtain several properties of random closed sets.

An effective one-to-one correspondence between the space $\mathcal{C}$ and the space $3^{\mathbb{N}}$ is defined as follows. Let a closed set Q be given and let $T = T_Q$ be the tree without dead ends such that $Q = [T]$.

Then define the code $x = x_Q \in \{0,1,2\}^{\mathbb{N}}$ for Q as follows. Let $\emptyset = \sigma_0, \sigma_1, \sigma_2, \ldots$ enumerate the elements of T in order, first by length and then lexicographically. We now define $x = x_Q = x_T$ by recursion as follows. For each n, $x(n) = 2$ if $\sigma_n^\frown 0$ and $\sigma_n^\frown 1$ are both in T, $x(n) = 1$ if $\sigma_n^\frown 0 \notin T$ and $\sigma_n^\frown 1 \in T$ and $x(n) = 0$ if $\sigma_n^\frown 0 \in T$ and $\sigma_n^\frown 1 \notin T$.

Now define the measure μ^* on $\mathcal{C}$ by

$$\mu^*(\mathcal{X}) = \mu(\{x_Q : Q \in \mathcal{X}\}).$$

Informally this means that given $\sigma \in T_Q$, there is probability $\frac{1}{3}$ that both $\sigma^\frown 0 \in T_Q$ and $\sigma^\frown 1 \in T_Q$ and, for $i = 0,1$, there is probability $\frac{1}{3}$ that only $\sigma^\frown i \in T_Q$. In particular, this means that $Q \cap I(\sigma) \neq \emptyset$ implies that for $i = 0,1$, $Q \cap I(\sigma^\frown i) \neq \emptyset$ with probability $\frac{2}{3}$.

Let us comment briefly on why some other natural representations were rejected. Suppose first that we simply enumerate all strings in $\{0,1\}^*$ as $\sigma_0, \sigma_1, \ldots$ and then represent T by its characteristic function so that $x_T(n) = 1 \iff \sigma_n \in T$. Then in general a code x might not represent a tree. That is, once we have $(01) \notin T$ we cannot later decide that $(011) \in T$. Suppose then that we allow the empty closed set by using codes $x \in \{0,1,2,3\}^*$ and modify our original definition as follows. Let $x(n) = i$ have the same definition as above for $i \leq 2$ but let $x(n) = 3$ mean that neither $\sigma_n^\frown 0$ nor $\sigma^\frown 1$ is in T. Informally, this would mean that for $i = 0,1$, $\sigma \in T$ implies that $\sigma^\frown i \in T$ with probability $\frac{1}{2}$. The advantage here is that we can now represent all trees. But this is also a disadvantage, since for a given closed set P, there are many different trees T with $P = [T]$. The second problem with this approach is that we would have $[T] = \emptyset$ with positive probability. We briefly return to this subject in Section 6.

Now we will say that a closed set Q is (Martin-Löf) random if the code x_Q is Martin-Löf random. Let Q_x denote the unique closed set Q such that $x_Q = x$. Since random reals exist, it follows that random closed sets exists. Furthermore, there are Δ^0_2 random reals, so we have the following.

Theorem XIX.1.1. *There exists a random closed set Q such that T_Q is Δ^0_2.* □

Note that if T_Q is Δ^0_2, then Q must contain Δ^0_2 elements (Theorem XV.1.6). Since there exist strong Π^0_2 classes with no Δ^0_2 elements, there are strong Π^0_2 classes Q such that T_Q is not Δ^0_2.

The following lemma will be needed throughout.

Lemma XIX.1.2. *For any $Q \subseteq 2^{\mathbb{N}}$ which is either closed or open, $\mu^*(\{P : P \subseteq Q\}) \leq \mu(Q)$.*

Proof. Let $\mathcal{P}_C(Q)$ denote $\{P : P \subseteq Q\}$. We first prove the result for (nonempty) clopen sets U by the following induction. Suppose $U = \cup_{\sigma \in S} I(\sigma)$, where $S \subseteq \{0,1\}^n$. For $n = 1$, either $\mu(U) = 1 = \mu^*(\mathcal{P}_C(U))$ or $\mu(U) = \frac{1}{2}$ and $\mu^*(\mathcal{P}_C(Q)) = \frac{1}{3}$. For the induction step, let $S_i = \{\sigma : i^\frown\sigma \in S\}$, let $U_i = \cup_{\sigma \in S_i} I(\sigma)$, let $m_i = \mu(U_i)$ and let $v_i = \mu^*(\mathcal{P}_C(U_i))$, for $i = 0, 1$. Then considering the three cases in which S includes both initial branches or just one, we calculate that

$$\mu^*(\mathcal{P}_C(U)) = \frac{1}{3}(v_0 + v_1 + v_0 v_1).$$

Thus by induction we have

$$\mu^*(\mathcal{P}_C(U)) \leq \frac{1}{3}(m_0 + m_1 + m_0 m_1).$$

Now

$$2m_0 m_1 \leq m_0^2 + m_1^2 \leq m_0 + m_1,$$

and therefore

$$\mu^*(\mathcal{P}_C(U)) \leq \frac{1}{3}(m_0 + m_1 + m_0 m_1) \leq \frac{1}{2}(m_0 + m_1) = \mu(U).$$

For a closed set Q, let $Q = \cap_n U_n$, with $U_{n+1} \subseteq U_n$ for all n. Then $P \subset Q$ if and only if $P \subseteq U_n$ for all n. Thus

$$\mathcal{P}_C(Q) = \cap_n \mathcal{P}_C(U_n),$$

so that

$$\mu^*(\mathcal{P}_C(Q)) = \lim_{n \to \infty} \mu^*(\mathcal{P}_C(U_n)) \leq \lim_{n \to \infty} \mu(U_n) = \mu(Q).$$

Finally, for an open set Q, let $Q = \bigcup_n U_n$ be the union of an increasing sequence of clopen sets. Then, by compactness,

$$\mathcal{P}_C(Q) = \cup_n \mathcal{P}_C(U_n),$$

so that

$$\mu^*(\mathcal{P}_C(Q)) = \lim_{n \to \infty} \mu^*(\mathcal{P}_C(U_n)) \leq \lim_{n \to \infty} \mu(U_n) = \mu(Q).$$

This completes the proof of the lemma. □

Next we consider the intersection of a random closed set with an interval $I(\sigma)$ and the disjoint union of random closed sets.

Let us call the coding of a closed set Q by the nodes of its representative tree with no dead ends the *canonical code* of Q. We wish now to introduce a second method of coding, the *ghost code*. A ghost code of Q is an infinite ternary string whose bits correspond to all nodes of $2^{<\mathbb{N}}$ in lexicographical order. The bits corresponding to the nodes of Q's tree (the "canonical nodes") hold the same values as the corresponding bits in the canonical code; the remaining "ghost nodes" may hold any values. Ghost codes are non-unique, and every closed set has a non-random ghost code (if the closed set itself is random take the code with ghost nodes all equal to zero).

We define randomness for closed sets in the world of ghost codes as possession of a random code. This method of coding is more convenient for some purposes; for example, we will use it to show that if Q_0, Q_1 are closed sets and $Q = \{0^\frown x : x \in Q_0\} \cup \{1^\frown x : x \in Q_1\}$, Q is random if and only if the Q_i are random relative to each other. With canonical coding it is straightforward to show relative randomness of the half trees is sufficient for randomness of the full tree, but not its necessity.

However, the utility of the ghost codes rests entirely on the following correspondence.

Theorem XIX.1.3. *The canonical code of a closed set $Q \subseteq 2^{\mathbb{N}}$ is random if and only if Q has some random ghost code.*

Proof. ($\Leftarrow$) Suppose the canonical code of Q is nonrandom. Then there is a c.e. martingale m that succeeds on it. From any initial segment σ of a ghost code g for Q, the subsequence $\hat{\sigma}$ of exactly the canonical nodes of σ is computable. Therefore it is computable whether the bit of g after σ is canonical or ghost. From m, define the martingale m' which bets as follows:

$$m'(\sigma^\frown i) = \begin{cases} \frac{m(\hat{\sigma}^\frown i)}{m(\hat{\sigma})} m'(\sigma) & \text{next bit is a canonical node} \\ m'(\sigma) & \text{next bit is a ghost node.} \end{cases}$$

That is, m' holds its money on ghost nodes and bets proportionally to m (in fact, identically) on canonical nodes. It is clear that m' succeeds on the ghost code g and thus g is nonrandom.

($\Rightarrow$) Now suppose the canonical code r for Q is random, and let q be an infinite ternary string that is random relative to r (so therefore r is also random relative to q). We claim the ghost code g obtained by using the bits of r as the canonical nodes and the bits of q in their original order as the ghost nodes is random. It is clear that g is a ghost code for Q.

Suppose m is a c.e. martingale that bets on g. We define two martingales from m, m_r and m_q, that bet on the bits of r and q respectively, with oracle q and r respectively, according to m's actions on the corresponding bits of g, and show that if they do not succeed, m does not succeed. As q and r are relatively random, m_r and m_q cannot succeed, and so g will be random.

We define m_r only; for m_q swap the roles of r and q, and of canonical and ghost nodes. From a string σ compute the unique initial segment τ of a ghost code such that

1. σ is the substring of canonical nodes of τ,

2. the node to follow τ is a canonical node, and

3. the ghost nodes of τ are an initial segment of the oracle q.

Then $m_r(\sigma^\frown i) = \frac{m(\tau^\frown i)}{m(\tau)} m_r(\sigma)$.

Since each of r, q is random relative to the other, neither m_r nor m_q can succeed. That is, there is some b such that $m_r(\sigma) \leq b$ for all $\sigma \subset r$ and likewise $m_q(\tau) \leq b$ for all $\tau \subset q$. We claim that the values of m on initial segments of g are bounded by b^2.

At each node of the code r the martingale m_r will multiply the capital held before the node by some constant factor. Let $\{c^r_n\}_{n\in\mathbb{N}}$ be the sequence of level-n multiplicative factors for m_r. Assuming an initial capital of 1, the values m_r achieves on initial segments of r are

$$\prod_{k=0}^{\ell} c^r_k$$

for $0 \leq \ell < \infty$. By assumption, every product of this form is less than or equal to b. Substituting the corresponding definition for m_q, $\{c^q_n\}_{n\in\mathbb{N}}$, gives the same result.

The original martingale m behaves on the subequences r and q exactly as the martingales m_r and m_q do, by construction of m_r and m_q. Therefore it has the same collection of multiplicative factors. Again assuming initial capital 1, the values it achieves are thus products of the form

$$\left(\prod_{k=0}^{\ell} c^r_k\right) \cdot \left(\prod_{k=0}^{\ell'} c^q_k\right).$$

As each of the subproducts is bounded by b, the entire product is bounded by b^2 and m does not succeed on g. Since m was arbitrary, no c.e. martingale succeeds on g and thus Q has a random ghost code. □

Note that this theorem relativizes so we can assert that the canonical code is, say, A-random for some A if and only if it has an A-random ghost code.

The primary purpose of the ghost codes is to remove the dependence on the particular closed set under discussion when interpreting bits of the code as nodes of the tree. This is especially useful when subdividing the tree, as in the following definition.

Definition XIX.1.4. *The* tree join *of closed sets P_0 and P_1 is the closed set $Q = \{0^\frown x : x \in P_0\} \cup \{1^\frown x : x \in P_1\}$. Given ghost codes r_0, r_1 for the P_i, their tree join $r_0 \boxplus r_1$ is the code for Q with the corresponding ghost node values.*

This is distinguished from what we will call the *recursion-theoretic join* $r_0 \oplus r_1$, where elements of r_0 and r_1 alternate.

We wish to relate the recursion-theoretic join and the tree join. First recall van Lambalgen's theorem.

Theorem XIX.1.5 (van Lambalgen [212])**.** *The following are equivalent.*

1. *$A \oplus B$ is n-random.*

2. *A is n-random and B is n-A-random (or vice-versa).*

3. *A is n-B-random and B is n-A-random.*

Lemma XIX.1.6. *Given two ghost codes r_0, r_1, the tree join $r_0 \boxplus r_1$ is random if and only if the recursion theoretic join $r_0 \oplus r_1$ is random.*

Proof. We show a Martin-Löf test that contains one version of the join may be transformed into a test that contains the other version, and therefore if one is nonrandom, both are nonrandom. To simplify the proof we ignore the initial bit of the tree join, as it has no matching bit in the recursion-theoretic join.

It is clear that initial segments of the two versions of join may be transformed into each other via a computable algorithm that is independent of the value of the bits, provided the strings are of length $\ell_n := 2^{n+1} - 1$ (i.e., the final node is the end of a level of the tree). For strings that terminate in the middle of levels, a single initial segment σ becomes a finite collection of strings τ_j, one for each extension of σ to a string of length ℓ_n for the least n such that $|\sigma| \leq \ell_n$. There will be $2^{\ell_n - |\sigma|}$ such strings and hence the measure of the intervals around them will total $2^{-\ell_n} \cdot 2^{\ell_n - |\sigma|} = 2^{-|\sigma|}$, equal to the measure of the interval around σ.

Therefore, let $\{U_i : i \in \mathbb{N}\}$ be a Martin-Löf test failed by $r_0 \oplus r_1$. When σ enters U_i for some i, enumerate the finite collection of τ_j as described above into a new set $\hat{U}_i$. The collection $\{\hat{U}_i : i \in \mathbb{N}\}$ is clearly a Martin-Löf test, as it is enumerated simultaneously with the $\{U_i\}$ and $\mu(\hat{U}_i) = \mu(U_i)$ for all $i \in \mathbb{N}$. If $r_0 \oplus r_1$ extends $\sigma \in U_i$, $r_0 \boxplus r_1$ will extend one of the corresponding τ_j in $\hat{U}_i$. Therefore since $r_0 \oplus r_1$ fails $\{U_i\}$, $r_0 \boxplus r_1$ will fail $\{\hat{U}_i\}$. Symmetrically, if $r_0 \boxplus r_1$ is nonrandom we may construct a Martin-Löf test that $r_0 \oplus r_1$ also fails. □

We now obtain the following corollary of Theorems XIX.1.3 and XIX.1.5, and Lemma XIX.1.6.

Corollary XIX.1.7. *Suppose P_i, $i = 0, 1$, are closed sets with canonical codes r_i and let P be the tree join of P_0, P_1. Then P is random if and only if $r_0 \oplus r_1$ is random.*

Proof. ($\Leftarrow$) Suppose that $r_0 \oplus r_1$ is random. Then by Theorem XIX.1.5, the r_i are mutually relatively random. By Theorem XIX.1.3, there are ghost codes g_i for the P_i that are also mutually relatively random. Again by XIX.1.5, the recursion-theoretic join $g_0 \oplus g_1$ is random; then by Theorem XIX.1.6 the tree join $g_0 \boxplus g_1$ is also random, and hence P possesses a random ghost code and is random.
($\Rightarrow$) Suppose now that P is random, and therefore possesses a random ghost code g. The code g may be thought of as a tree join $g_0 \boxplus g_1$, which is therefore random, and so by Theorem XIX.1.6, $g_0 \oplus g_1$ is random. By Theorem XIX.1.5, the individual codes g_0, g_1 are therefore mutually relatively random, and so by Theorem XIX.1.3 the canonical codes r_0, r_1 for the half trees are as well. Thus again by XIX.1.5, $r_0 \oplus r_1$ is random. □

XIX.2 Members of Random Closed Sets

Bibliography

[1]

[2] T. . A. Lewis.

[3] K. Appel, W. Haken, and J. Koch, *Every planar map is 4-colorable*, Ill. J. Math., 41 (1976), pp. 469–480.

[4] K. Apt, *Logic programming*, in Handbook of Theoretical Computer Science (J. van Leeuven,ed.), MIT Press, 1990, pp. 493–574.

[5] E. Artin and O. Schreier, *Algebraische konstruktion reeller körper*, Abh. Math. Sem. Univ. Hamburg, 5 (1927), pp. 85–89.

[6] G. Barmpalias, P. Brodhead, D. Cenzer, A. S. Dashti, and R. Weber, *Algorithmic randomness of closed sets*, Journal of Logic and Computation, 17 (2007), pp. 1041–1062.

[7] G. Barmpalias, P. Brodhead, J. Remmel, and R. Weber, *Algorithmic randomness of continuous functions*, Archive for Mathematical Logic, 45 (2008), pp. 533–546.

[8] G. Barmpalias, D. Cenzer, J. Remmel, and R. Weber, *k-trivial closed sets and continuous functions*, in Computation and Logic in the Real World: CIE 2007–Third Conference on Computability in Europe, S. Cooper, B. Loewe, and A. Sorbi, eds., vol. 4497 of Lecture Notes in Computer Science, Springer-Verlag, 2007, pp. 135–145.

[9] G. Barmpalias, D. Cenzer, J. Remmel, and R. Weber, *k-triviality of closed sets and continuous functions*, Journal of Logic and Computation, 19 (2009), pp. 3–16.

[10] G. Barmpalias, A. Lewis, and K. Ng, *The importance of π^0_1 classes in effective randomness*, Journal of Symbolic Logic, 75 (2010), p. 387400.

[11] G. Barmpalias, A. Lewis, and F. Stephan, *π^0_1 classes, lr degrees and turing degrees*, Annals of Pure and Applied Logic, 156 (2008), p. 2138.

[12] D. Bean, *Effective coloration*, Journal of Symbolic Logic, 41 (1976), pp. 469–480.

[13] ——, *Recursive euler and hamiltonian paths*, Proceedings of the American Mathematical Society, 55 (1976), pp. 885–894.

[14] S. Binns and S. G. Simpson, *Embeddings into the medvedev and muchnik lattices of π^0_1 classes*, Archive for Mathematical Logic, (2005). to appear.

[15] L. Blum, M. Shub, and S. Smale, *On a theory of computation and complexity over the real numbers*, Bulletin of the American Mathematical Society, 21 (1989), pp. 1–46.

[16] V. Brattka and K. Weihrauch, *Computability on subsets of euclidean space i: closed and compact subsets*, Theoretical Computer Science, 219 (1999), pp. 65–93.

[17] P. Brodhead, D. Cenzer, and A. Dashti, *Random closed sets*, in Logical Approaches to Computational Barriers, Proc. CIE 2006, A. Beckmann, U. Berger, B. Loewe, and J.Tucker, eds., vol. 3988 of Lecture Notes in Computer Science, Springer-Verlag, 2006, pp. 55–64.

[18] P. Brodhead, D. Cenzer, and J. Remmel, *Random continuous functions*, in Third International Conference on Computability and Complexity in Analysis, D. Cenzer, R. Dillhage, T. Grubb, and K. Weihrauch, eds., vol. 167 of Electronic Notes in Computer Science, Springer-Verlag, 2007, pp. 275–287.

[19] R. Brooks, *On colouring the nodes of a network*, Proc. Cambridge Philos. Soc., 37 (1941), pp. 194–197.

[20] H.-G. Carstens, *A 5-color extension theorem*, in Computation Theory and Logic, vol. 270 of Lecture Notes in Computer Science, Springer-Verlag, 1987, pp. 67–77.

[21] Cenzer, *Π^0_1 classes in recursion theory*, in Handbook of Computability Theory, E. Griffor, ed., Studies in Logic Found. Math #140, Elsevier, 1999, pp. 37–85.

[22] D. Cenzer, *Effective dynamics*, in Logical Methods in honor of Anil Nerode's Sixtieth Birthday, J. Crossley, J. B. Remmel, R. Shore, and M. Sweedler, eds., Birkhäuser, 1993, pp. 162–177.

[23] D. Cenzer, P. Clote, R. Smith, R. Soare, and S. Wainer, *Members of countable π^0_1 classes*, Annals of Pure and Applied Logic, 31 (1986), pp. 145–163.

[24] D. Cenzer, A. Dashti, and J. L. F. King, *Countable symbolic dynamics*, Mathematical Logic Quarterly, 54 (2008), pp. 524–533.

[25] D. Cenzer, R. Downey, C. G. Jockusch, and R. Shore, *Countable thin π^0_1 classes*, Annals of Pure and Applied Logic, 59 (1993), pp. 79–139.

[26] D. Cenzer and P. G. Hinman, *Density of the medvedev lattice of π_1^0 classes*, Archive for Mathematical Logic, 42 (2003), pp. 583–600.

[27] D. Cenzer and E. Howorka, *On vertex partitions of certain infinite graphs*, Discrete Mathematics, 23 (1978), pp. 105–113.

[28] D. Cenzer and C. J. Jockusch, *π_1^0 classes – structure and applications*, in Computability Theory and Its Applications, P. Cholak, S. Lempp, M. Lerman, and R. Shore, eds., Contemporary Mathematics #257, American Mathematical Society, 2000, pp. 39–59.

[29] D. Cenzer, W. Marek, and J. Remmel, *Index sets for finite predicate logic programs*, in FLOC '99 Workshop on Complexity-theoretic and Recursion-theoretic methods in Databases, Artificial Intelligence and Finite Model Theory, T. Eiter and G. Gottlob, eds., Technische Universitat Wien, 1999, pp. 72–80.

[30] D. Cenzer and R. D. Mauldin, *On the borel class of the derived set operator*, Bulletin Societe Math. France, 110 (1982), pp. 357–380.

[31] ——, *On the borel class of the derived set operator ii*, Bulletin Societe Math. France, 111 (1983), pp. 367–372.

[32] D. Cenzer and A. Nies, *Initial segments of the lattice of Π_1^0 classes*, Journal of Symbolic Logic, 66 (2001), pp. 1749–1765.

[33] ——, *Global properties of the lattice of Π_1^0 classes*, Proceedings of the American Mathematical Society, 132 (2004), pp. 239–249.

[34] D. Cenzer and J. B. Remmel, *Polynomial-time versus recursive models*, Annals of Pure and Applied Logic, 54 (1991), pp. 17–58.

[35] ——, *Polynomial time abelian groups*, Annals of Pure and Applied Logic, 56 (1992), pp. 313–363.

[36] ——, *Recursively presented games and strategies*, Math. Social Sciences, 24 (1992), pp. 117–139.

[37] ——, *Feasible graphs and colorings*, Methods of Logic in Computer Science, 41 (1995), pp. 327–352.

[38] ——, *Complexity theoretic model theory and algebra*, in Handbook of Recursive Mathematics, Y. Ersov, S. Goncharov, A. Nerode, and J. B. Remmel, eds., Studies in Logic Found. Math #138, Elsevier, 1998, pp. 381–513.

[39] ——, *Index sets for π_1^0 classes*, Annals of Pure and Applied Logic, 93 (1998), pp. 3–61.

[40] ——, *π_1^0 classes in mathematics*, in Handbook of Recursive Mathematics, Y. Ersov, S. Goncharov, A. Nerode, and J. B. Remmel, eds., Studies in Logic Found. Math #139, Elsevier, 1998, pp. 623–821.

[41] ——, *Index sets in computable analysis*, Theoretical Computer Science, 219 (1999), pp. 111–150.

[42] ——, *Effectively closed sets and graphs of computable real functions*, Theoretical Computer Science, 284 (2002), pp. 279–318.

[43] ——, *Index sets in ω-languages*, Mathematical Logic Quarterly, 49 (2002), pp. 22–33.

[44] ——, *Complexity, completeness and decidability*, Journal of Symbolic Logic, 71 (2006), pp. 399–424.

[45] D. Cenzer and F. Riazati, *Minimal extensions of π_1^0 classes*, Mathematical Logic Quarterly, 51 (2005), pp. 206–216.

[46] D. Cenzer and R. Smith, *The ranked points of a π_1^0 set*, Journal of Symbolic Logic, 54 (1989), pp. 975–991.

[47] G. Chaitin, *Information-theoretical characterizations of recursive infinite strings*, Theoretical Computer Science, (1976), pp. 45–48.

[48] K. Chen, *Recursive well-founded orderings*, Annals of Pure and Applied Logic, 13 (1978), pp. 117–147.

[49] H. Chernoff, *A measure of asymptotic efficiency for tests of a hypothesis based on the sums of observations*, Annals Math Statistics, 23 (1952), pp. 493–509.

[50] P. Cholak, R. Coles, R. Downey, and E. Herrmann, *Automorphisms of the lattice of π_1^0 classes*, Transactions of the American Mathematical Society, 353 (2001), pp. 4899–4924.

[51] P. Cholak and R. Downey, *On the cantor-bendixson rank of recursively enumerable sets*, Journal of Symbolic Logic, 58 (1993), pp. 629–640.

[52] A. Church and S. Kleene, *Formal definitions in the theory of ordinal numbers*, Fund. Math., 28 (1937), pp. 11–21.

[53] P. Clote, *On recursive trees with a unique infinite branch*, Proceedings of the American Mathematical Society, 93 (1985), pp. 335–342.

[54] P. J. Cohen, *Set Theory and the Continuum Hypothesis*, Benjamin, 1966.

[55] T. Craven, *The boolean space of orderings of a field*, Transactions of the American Mathematical Society, 209 (1975), pp. 225–235.

[56] J. Dekker and J. Myhill, *Retraceable sets*, Canad. J. Math., 10 (1985), pp. 357–373.

[57] O. Demuth, *A notion of semigenericity*, Commenationes Mathematicae Universitatis Carolinae, 28 (1987), pp. 71–84.

[58] R. P. DILWORTH, *A decomposition theorem for partially ordered sets*, Annals of Math., 51 (1950), pp. 161–166.

[59] R. DOWNEY, *Abstract dependence, recursion theory and the lattice of recursively enumerable filters*. Ph.D. Thesis (Monash University), 1982.

[60] ——, *On π_1^0 classes and their ranked points*, Notre Dame J. of Formal Logic, 32 (1991), pp. 499–512.

[61] R. DOWNEY, *On presentations of algebraic structures*, in Complexity, Logic and Recursion Theory, A. Sorbi, ed., Lecture Notes in Pure and Appl. Math. #197, Marcel Dekker, 1997, pp. 157–206.

[62] R. DOWNEY, *Recursion theory and linear orderings*, in Handbook of Recursive Mathematics, Y. Ersov, S. Goncharov, A. Nerode, and J. B. Remmel, eds., Studies in Logic Found. Math #139, Elsevier, 1998, pp. 823–975.

[63] R. DOWNEY AND D. HIRSCHFELDT, *Algorithmic Randomness and Complexity*, Springer-Verlag, 2010.

[64] R. DOWNEY, D. HIRSCHFELDT, A. NIES, AND F. STEPHAN, *Trivial reals*, in Proceedings of the 7th and 8th Asian Logic Conferences, R. Downey, D. Decheng, T. Ping, Q. Hui, and M. Yasugi, eds., Singapore University Press and World Scientific, 2003, pp. 103–131.

[65] R. DOWNEY, C. JOCKUSCH, AND M. STOB, *Array non-recursive sets and multiple permitting arguments*, in Recursion Theory Week (Proceedings Oberwolfach 1989), K. Ambos-Spies, G. Muller, and G. Sacks, eds., Lecture Notes in Mathematics #1432, Springer-Verlag, 1990, pp. 141–173.

[66] R. DOWNEY AND S. KURTZ, *Recursion theory and ordered groups*, Annals of Pure and Applied Logic, 32 (1986), pp. 137–151.

[67] R. DOWNEY AND S. LEMPP, *Proof-theoretic strength of the dushnik-miller theorem on countable linear orderings*, in Recursion Theory and Complexity (Proc. 1997 Kazan conference), M. Arslanov and S. Lempp, eds., W. de Gruyter, 1999, pp. 55–57.

[68] B. DUSHKNIK AND E. W. MILLER, *Partially ordered sets*, Amer. J. Math, 1941 (63), pp. 600–610.

[69] A. EHRENFEUCHT, *Separable theories*, Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys., 9 (1961), pp. 17–19.

[70] R. M. FRIEDBERG, *Three theorems on recursive enumeration*, Journal of Symbolic Logic, 23 (1958), pp. 309–316.

[71] H. FRIEDMAN, *Systems of second order arithmetic with restricted induction i, ii (abstracts)*, Journal of Symbolic Logic, 41 (1976), pp. 557–559.

[72] H. Friedman, S. Simpson, and R. Smith, *Countable algebra and set existence theorems*, Annals of Pure and Applied Logic, 25 (1983), pp. 141–181.

[73] H. Friedman, S. Simpson, and R. Smith, *Countable algebra and set existence theorems*, Annals of Pure and Applied Logic, 25 (1983), pp. 141–181.

[74] P. Gacs, *On the symmetry of algorithmic information*, Soviet Mathematics–Doklady, 15 (1974), pp. 1477–1480.

[75] D. Gale and F. M. Stewart, *Infinite games with perfect information*, Ann. Math. Studies, 28 (1953), pp. 245–266.

[76] R. O. Gandy, *Proof of mostowski's conjecture*, Bull. Acad. Polon. Sci., 8 (1960), pp. 571–575.

[77] W. Gasarch and G. Martin, *Index sets in recursive combinatorics*, in Logical Methods in honor of Anil Nerode's Sixtieth Birthday, J. Crossley, J. B. Remmel, R. Shore, and M. Sweedler, eds., Birkhäuser, 1993, pp. 352–385.

[78] M. Gelfond and V. Lifschitz, *The stable semantics for logic programs*, in Proceedings 5th Int'l. Symp. Logic Programming, MIT Press, 1988, pp. 1070–1080.

[79] ——, *Logic programs with classical negation*, in Proceedings 7th Int'l. Conference on Logic Programming, MIT Press, 1990, pp. 579–597.

[80] S. Grigorieff, *Every recursive linear ordering has a copy in dtime(n)*, Journal of Symbolic Logic, 55 (1990), pp. 260–276.

[81] M. Groszek and T. Slaman, *A recursive tree in which every path computes a minimal degree*, Annals of Pure and Applied Logic, 87 (1997), pp. 117–144. Proceedings of Logic Colloquium 95.

[82] M. Hall, *Distinct representatives of subsets*, Bulletin of the American Mathematical Society, 54 (1948), pp. 922–926.

[83] W. Hanf, *The boolean algebra of logic*, Bulletin of the American Mathematical Society, 81 (1975), pp. 587–589.

[84] D. Harel, *Hamiltonian paths in infinite graphs*, Israel J. Math., 76 (1991), pp. 317–336.

[85] C. Haught, *The degrees below a 1-generic degree below* $\mathbf{0}'$, Journal of Symbolic Logic, 51 (1996), pp. 770–777.

[86] L. Hay, A. Manaster, and J. Rosenstein, *Concerning partial recursive similarity transformations of linearly ordered sets*, Pacific J. Math., 71 (1977), pp. 57–70.

[87] P. G. Hinman, *Recursion-Theoretic Hierarchies*, Perspectives in Mathematical Logic, Springer-Verlag, 1978.

[88] J. Hirst, *Marriage theorems and reverse mathematics*, in Logic and Computation (Pittsburgh 1987), Contemporary Mathematics #106, American Mathematical Society, 1990, pp. 181–196.

[89] J. E. Hopcroft and J. D. Ullman, *Formal Languages and Their Relations to Automata*, Addison-Wesley, 1969.

[90] T. Jech, *Set Theory*, Academic Press, 1978.

[91] C. G. Jockusch, *π_1^0 classes and boolean combinations of recursively enumerable sets*, Journal of Symbolic Logic, 39 (1974), pp. 95–96.

[92] ——, *Degrees of functions with no fixed points*, in Logic, Methodology and Philosophy of Science VIII, Studies in Logic and Foundations of Mathematics, Elsevier, 1989, pp. 191–201.

[93] C. G. Jockusch, A. Lewis, and J. B. Remmel, *π_1^0 classes and rado's selection principle*, Journal of Symbolic Logic, 56 (1991), pp. 684–693.

[94] C. G. Jockusch and T. G. McLaughlin, *Countable retracing functions and π_1^0 predicates*, Pacific J. Math., 30 (1969), pp. 67–93.

[95] C. G. Jockusch and R. Shore, *Pseudo-jump operators ii*, Journal of Symbolic Logic, 49 (1984), pp. 1205–1236.

[96] C. G. Jockusch and R. Soare, *A minimal pair of π_1^0 classes*, Journal of Symbolic Logic, 36 (1971), pp. 66–78.

[97] ——, *A minimal pair of π_1^0 classes*, Journal of Symbolic Logic, 36 (1971), pp. 66–78.

[98] ——, *Degrees of members of π_1^0 classes*, Pacific J. Math., 40 (1972), pp. 605–616.

[99] ——, *π_1^0 classes and degrees of theories*, Transactions of the American Mathematical Society, 173 (1972), pp. 33–56.

[100] K. Gödel, *Über die lange der beweise*, Ergeb. Math. Koll., 7 (1936), pp. 23–24.

[101] H. Keirstead, *Recursive and on-line graph coloring*, in Handbook of Recursive Mathematics, Y. Ersov, S. Goncharov, A. Nerode, and J. B. Remmel, eds., Studies in Logic Found. Math #139, Elsevier, 1998, pp. 1233–1269.

[102] H. Kierstead, *An effective version of dilworth's theorem*, Transactions of the American Mathematical Society, 268 (1981), pp. 63–77.

[103] H. KIERSTEAD, *An effective version of hall's theorem*, Proceedings of the American Mathematical Society, 88 (1983), pp. 124–128.

[104] ——, *Recursive ordered sets*, in Combinatorics and Ordered Sets, vol. 57 of Contemporary Mathematics, Amer. Math. Soc., 1986, pp. 75–102.

[105] H. KIERSTEAD, G. MCNULTY, AND W. TROTTER, *Recursive dimension for partially ordered sets*, Order, 1 (1984), pp. 67–82.

[106] H. KIERSTEAD AND W. TROTTER, *An extremal problem in recursive combinatorics*, Congressium Numerantium, 33 (1981), pp. 143–153.

[107] N. KLARLUND, *Progress measures and finite arguments for infinite computation.* Ph.D. Thesis (Cornell University), 1990.

[108] S. C. KLEENE, *Recursive predicates and quantifiers*, Transactions of the American Mathematical Society, 53 (1943), pp. 41–73.

[109] ——, *A symmetric form of Gödel's theorem*, Ind. Math., 12 (1950), pp. 244–246.

[110] ——, *Recursive functionals and quantifiers of finite types i*, Transactions of the American Mathematical Society, 91 (1959), pp. 193–218.

[111] J. KNIGHT, A. LACHLAN, AND R. SOARE, *Two theorems on degrees of models of true arithmetic*, Journal of Symbolic Logic, 49 (1984), pp. 425–436.

[112] K. KO, *Complexity Theory of Real Functions*, Birkhauser, 1991.

[113] ——, *On the computability of fractal dimensions and julia sets*, Annals of Pure and Applied Logic, 93 (1998), pp. 195–216.

[114] A. N. KOLMOGOROV, *Three approaches to the quantitative defintion of information*, Problems of Information Transmission, 1 (1965), pp. 1–7.

[115] G. KREISEL, *A variant to hilbert's theory of the foundations of arithmetic*, British J. Philosophy of Science, 4 (1953), pp. 107–129.

[116] ——, *Analysis of the cantor-bendixson theorem by means of the analytic hierarchy*, Bull. Acad. Polon. des Sciences Ser. Math. Astronom. et Phys, 7 (1959), pp. 621–626.

[117] A. KUCERA, *Measure,* π_1^0 *classes and complete extensions of pa*, in Recursion Theory Week (Proceedings Oberwolfach 1984), H. D. Ebbinghaus, G. Muller, and G. Sacks, eds., Lecture Notes in Mathematics #1141, Springer-Verlag, 1985, pp. 245–259.

[118] ——, *An alternative, priority free, solution to post's problem*, in Mathematical Foundations of Computer Science (Bratislava 1986), vol. 233 of Lecture Notes in Computer Science, Springer-Verlag, 1986, pp. 493–500.

[119] ——, *Randomness and generalizations of fixed point free functions*, in Recursion Theory Week (Proceedings Oberwolfach 1989), K. Ambos-Spies, G. Muller, and G. Sacks, eds., Lecture Notes in Mathematics #1432, Springer-Verlag, 1990, pp. 245–254.

[120] K. Kuratowski, *Some problems concerning semi-continuous set-valued mappings*, in Set-Valued Mappings, Selections and Topological Properties of 2^X, Lecture Notes in Mathematics #171, Springer-Verlag, 1970, pp. 45–48.

[121] K. Kuratowski and A. Mostowski, *Set Theory*, Studies in Logic #86, Elsevier North-Holland, 1968.

[122] A. H. Lachlan, *The elementary theory of recursively enumerable sets*, Duke Mathematical Journal, 35 (1968), pp. 123–146.

[123] A. H. Lachlan, *On the lattice of recursively enumerable sets*, Transactions of the American Mathematical Society, 130 (1968), pp. 1–37.

[124] D. Lacombe, *Les ensembles recursivement ouverts ou fermes, et leurs applications a l'analyse recursive i*, Comptes Rendus Hebdomaires des Sciences, 245 (1957), pp. 1040–1043.

[125] ——, *Les ensembles recursivement ouverts ou fermes, et leurs applications a l'analyse recursive ii*, Comptes Rendus Hebdomaires des Sciences, 246 (1958), pp. 405–418.

[126] L. Lawton, *Almost complemented π^0_1 classes*, Archive for Mathematical Logic, 45 (2006), pp. 555–568.

[127] S. Lempp, *Hyperarithmetical index sets in recursion theory*, Transactions of the American Mathematical Society, 303 (1987), pp. 559–583.

[128] F. W. Levi, *Arithmetische gesetze im gebiete diskreter gruppen*, Rend. Circ. Mat. Palermo, 35 (1913), pp. 225–236.

[129] L. A. Levin, *The concept of a random sequence*, Dokl. Akad. Nauk SSSR, 212 (1973), pp. 548–550.

[130] P. . Lévy, *Théorie de l'Addition des Variables Aleatoires*, Gauthier-Villars, 1937 (second edition 1954).

[131] M. Li and P. Vitanyi, *Introduction to Kolmogorov Complexity and its Applications, Third Edition*, Springer-Verlag, 2008.

[132] J. Lloyd, *Foundations of Logic Programming*, Springer-Verlag, 1989.

[133] A. Manaster and J. Rosenstein, *Effective matchmaking*, Proceedings London Mathematical Society, 25 (1972), pp. 615–654.

[134] ——, *Effective matchmaking and k-chromatic graphs*, Proceedings of the American Mathematical Society, 39 (1973), pp. 371–378.

[135] V. W. Marek, *Nonmonotonic Logic*, Springer-Verlag, 1993.

[136] W. Marek, A. Nerode, and J. B. Remmel, *Nonomonotonic rule systems i*, Annals of Mathematics and Artificial Intelligence, 1 (1990), pp. 241–273.

[137] ——, *How complicated is the set of stable models of a recursive logic program?*, Annals of Pure and Applied Logic, 56 (1992), pp. 119–135.

[138] ——, *Nonomonotonic rule systems ii*, Annals of Mathematics and Artificial Intelligence, 5 (1992), pp. 229–264.

[139] ——, *A context for belief revision: Forward-chaining normal nonmonotonic rule systems*, Annals of Pure and Applied Logic, 67 (1994), pp. 269–324.

[140] ——, *The stable models of a predicate logic program*, Journal of Logic Programming, 21 (1994), pp. 129–154.

[141] ——, *Nonmonotonic rule systems with recursive sets of restraints*, Archive for Mathematical Logic, 36 (1997), pp. 339–384.

[142] D. Marker, *Degrees of models of arithmetic*, in Logic Colloquium '81 (Proceedings of the Herbrand Symposium), North Holland, 1982, pp. 233–243.

[143] D. Martin, *Classes of recursively enumerable sets and degrees of unsolvability*, Zeitschrift für Mathematische Logik und Grundlagen der Mathematik, 12 (1966), pp. 295–310.

[144] D. Martin and M. Pour-El, *Axiomatizable theories with few axiomatizable extensions*, Journal of Symbolic Logic, 35 (1970), pp. 205–209.

[145] G. Martin, *Cantor singletons, rank-faithful trees, and other topics in recursion theory.* Ph.D. Thesis, University of Maryland, 1993.

[146] P. Martin-Lof, *The definition of random sequences*, Information and Control, 9 (1966), pp. 602–619.

[147] D. McDermott, *Nonmonotonic logic ii: nonmonotonic modal theories*, Journal of the ACM, 29 (1982), pp. 33–57.

[148] D. McDermott and J. Doyle, *Nonmonotonic logic i*, Artificial Intelligence, 13 (1980), pp. 41–72.

[149] Y. Medvedev, *Degrees of difficulty of the mass problem*, Doklady Akad. Nauk SSSR, 104 (1955), pp. 501–504.

[150] G. Metakides and A. Nerode, *Recursion theory and algebra*, in Algebra and Logic, J. N. Crossley, ed., Lecture Notes in Math. #450, Springer-Verlag, 1975, pp. 209–219.

[151] G. Metakides and A. Nerode, *Effective content of field theory*, Annals of Pure and Applied Logic, 17 (1979), pp. 289–320.

[152] J. Miller, π_1^0 *classes in analysis and topology*. Ph.D. thesis, Cornell University, 2002.

[153] M. Minsky, *A framework for representing knowledge*, in The Psychology of Computer Vision, McGraw-Hill, 1975, pp. 211–272.

[154] Y. Moschovakis, *Elementary Induction on Abstract Structures*, Studies in Logic #100, North-Holland, 1974.

[155] ——, *Descriptive Set Theory*, Studies in Logic #100, North-Holland, 1980.

[156] A. A. Muchnik, A. L. Semenov, and V. A. Uspensky, *Mathematical metaphysics of randomness*, Theoretical Computer Science, 207 (1998), pp. 263–271.

[157] A. Nerode, *General topology and partial recursive functionals*, in Summaries of talks presented at the Summer Institute for Symbolic Logic, Cornell University, 1957, pp. 247–251.

[158] A. Nerode and W. Huang, *Applications of pure recursion theory to recursive analysis*, Acta Sinica, 28 (1985).

[159] A. Nerode and J. B. Remmel, *Complexity theoretic algebra ii, the free boolean algebra*, Annals of Pure and Applied Logic.

[160] ——, *Complexity theoretic algebra I, vector spaces over finite fields*, Computer Science Press, 1987, pp. 218–239.

[161] A. Nerode and J. B. Remmel, *Complexity theoretic algebra: vector space bases*, in Feasible Mathematics, S. Buss and P. Scott, eds., Progress in Computer Science and Applied Logic #9, Birkhäuser, 1990, pp. 293–319.

[162] A. Nies, *Computability and randomness*, vol. 51 of Oxford Logic Guides, Oxford University Press, 2009.

[163] P. Odifreddi, *Classical Recursion Theory*, Elsevier, 1989.

[164] V. P. Orevkov, *A constructive mapping of the square onto itself displacing every constructive point*, Sov. Math. Dokl., 4 (1963), pp. 1253–1256.

[165] J. Owings.

[166] ——, *Rank, join and cantor singletons*, Archive for Mathematical Logic, 36 (1997), pp. 313–320.

[167] C. Papadimitriou, *Computational Complexity*, Addison-Wesley, 1994.

[168] E. L. Post, *Recursively enumerable sets of positive integeres and their decision problems*, Bulletin of the American Mathematical Society, 50 (1944), pp. 284–316.

[169] M. B. Pour-El and J. I. Richards, *Computability in Analysis and Physics*, Perspectives in Mathematical Logic, Springer-Verlag, 1989.

[170] R. Rado, *Axiomatic treatment of rank in infinite sets*, Canadian J. Math., 1 (1949), pp. 337–343.

[171] R. Reiter, *A logic for default reasoning*, Artificial Intelligence, 13 (1980), pp. 81–132.

[172] J. B. Remmel, *Recursively enumerable boolean algebras*, Annals of Pure and Applied Logic, 14 (1978), pp. 75–107.

[173] ——, *On the effectiveness of the schroder-bernstein theorem*, Proceedings of the American Mathematical Society, 83 (1981), pp. 379–386.

[174] ——, *Graph colorings and recursively bounded π_1^0 classes*, Annals of Pure and Applied Logic, 32 (1986), pp. 185–194.

[175] ——, *When is every recursive linear ordering of type μ recursively isomorphic to a p-time linear order over the binary representation of the natural numbers?*, in Feasible Mathematics, S. Buss and P. Scott, eds., Prog. in Comp. Science and Appl. Logic #9, Birkhäuser, 1990, pp. 321–341.

[176] ——, *Polynomial-time categoricity and linear orderings*, in Logical Methods, J. Crossley, J. B. Remmel, R. Shore, and M. Sweedler, eds., Prog. in Comp. Science and Appl. Logic #12, Birkhäuser, 1993, pp. 321–341.

[177] F. Riazati, *The lattice of π_1^0 classes.* Ph.D. thesis, University of Florida, 2001.

[178] H. Rogers, *Theory of Recursive Functions and Effective Computability*, McGraw-Hill, 1967.

[179] J. Rosenstein, *Linear Orderings*, Acad. Press, 1982.

[180] B. Rosser, *Extensions of some theorems of Gödel and church*, Journal of Symbolic Logic, 1 (1936), pp. 87–91.

[181] G. Sacks, *Degrees of Unsolvability*, Annals of Mathematics Studies #55, Princeton University Press, 1963.

[182] J. Schmerl, *Recursive colorings of graphs*, Canadian J. Math., 32 (1980), pp. 821–830.

[183] ——, *The effective version of brooks' theorem*, Canadian J. Math, 34 (1982), pp. 1036–1046.

[184] J. Schmerl, *Jump number and width*, Order, 3 (1986), pp. 227–234.

[185] C. P. Schnorr, *A unified approach to the definition of random sequences*, Mathematical Systems Theory, 5 (1971), pp. 246–258.

[186] D. Scott, *Algebras of sets binumerable in complete extensions of arithmetic*, in Recursive Function Theory, J. Dekker, ed., Proceedings of Symposia in Pure Mathematics #5, American Mathematical Society, 1962, pp. 117–121.

[187] D. Scott and S. Tennenbaum, *On the degrees of complete extensions of arithmetic*, Notices Amer. Math. Soc., 7 (1960), pp. 242–243.

[188] J. Shoenfield, *Degrees of formal systems*, Journal of Symbolic Logic, 23 (1958), pp. 389–392.

[189] ——, *Degrees of models*, Journal of Symbolic Logic, 25 (1960), pp. 233–237.

[190] ——, *Mathematical Logic*, Addison-Wesley, 1967.

[191] S. G. Simpson, *Degrees of unsolvability*, in Handbook of Mathematical Logic, J. Barwise, ed., Studies in Logic #90, Elsevier, 1977, pp. 631–652.

[192] ——, *Subsystems of Second Order Arithmetic*, Springer-Verlag, 1999.

[193] ——, π_1^0 *sets and models of* wkl_0, in Reverse Mathematics, Lecture Notes in Logic, Association for Symbolic Logic, 2004. to appear.

[194] ——, *Mass problems and randomness*, Bulletin of Symbolic Logic, (2005). to appear.

[195] R. Smith, *Effective aspects of profinite groups*, Journal of Symbolic Logic, 46 (1981), pp. 851–863.

[196] R. Soare, *Cohesive sets and recursively enumerable dedekind cuts*, Pacific J. Math., 31 (1969), pp. 215–231.

[197] ——, *Recursion theory and dedekind cuts*, Transactions of the American Mathematical Society, 139 (1969), pp. 271–294.

[198] ——, *Recursively Enumerable Sets and Degrees*, Springer-Verlag, 1987.

[199] D. R. Solomon, π_1^0 *classes and orderable groups*, Annals of Pure and Applied Logic, 115 (2002), pp. 279–302.

[200] A. Sorbi, *The medvedev lattice of degrees of difficulty*, in Computability, Enumerability, Unsolvability, London Math. Soc. Lecture Notes #224, Cambridge University Press, 1996, pp. 289–312.

[201] C. Spector, *Recursive well-orderings*, Journal of Symbolic Logic, 20 (1955), pp. 151–163.

[202] ——, *Hyperarithmetic quantifiers*, Fundamenta Mathematica, 48 (1959), pp. 313–320.

[203] ——, *Inductively defined sets of natural numbers*, in Infinitistic Methods (Proceedings of Warsaw Symposium), Pergamon Press, 1961, pp. 301–384.

[204] L. Staiger, *Hierarchies of recursive ω-languages*, J. Inform. Process. Cybernetics EIK, 22 (1986), pp. 219–241.

[205] ——, *Recursive automata on infinite words*, in Proc. STACS '93, vol. 665 of Lecture Notes in Computer Science, Springer-Verlag, 1993, pp. 629–639.

[206] ——, *ω-languages*, in Handbook of Formal Languages, Volume 3, Springer-Verlag, 1997, pp. 339–387.

[207] L. Staiger and K. Wagner, *Recursive ω-languages*, in Fundamentals of Computation Theory, vol. 56 of Lecture Notes in Computer Science, Springer-Verlag, 1977, pp. 532–537.

[208] A. Tarski, *Logic, semantics, metamathematics*, Oxford University Press, 1956.

[209] A. Tarski, A. Mostowski, and R. M. Robinson, *Undecidable Theories*, North-Holland, 1953.

[210] S. Tennenbaum, *Non-archimedean models for arithmetic*, Notices Amer. Math. Soc., 6 (1959), p. 270.

[211] B. A. Trakhtenbrot, *Tabular representation of recursive operators*, Doklady Acad. Nauk, 101 (1955), pp. 417–420.

[212] M. van Lambalgen, *Random sequences*. Ph.D. thesis, University of Amsterdam, The Netherlands, 1987.

[213] J. Ville, *Étude Critique de la Notion de Collectif*, Gauthier-Villars, Paris, 1939.

[214] R. von Mises, *Grundlagen der wahrscheinlichkeitsrechnung*, Math Zeitschrift, 5 (1919), pp. 52–99.

[215] R. Weber, *A definable relation between c.e. sets and ideals*. Ph.D. thesis, Notre Dame University, 2004.

[216] ——, *Invariance in $\rceil^*$ and $\rceil^*_\Pi$*, Transactions of the American Mathematical Society, 358 (2006), pp. 3023–3059.

[217] K. Weihrauch, *An Introduction to Computable Analysis*, Springer-Verlag, 2000.