Sample Problem Solutions for Exam Two

1. Use the limit definition of derivatives to find $\frac{d}{dx} cos\ x$.

$lim_{h \to 0} \frac{cos(x+h) - cos(x)}{h} = lim_{h \to 0} \frac{cos\ x \cos\ h - sin\ x\ sin\ h - cos(x)}{h} =$
$cos\ x\ lim_{h \to 0} \frac{cos\ h - 1}{h} - sin\ x\ lim_{h \to 0} \frac{sin\ h}{h} = (cos\ x)(0) - (sin\ x)(1) = -sin\ x.$

2. Find the equation of the normal line to the curve $y = f(x) = (x^2 - x)^3$ at the point where $x = 2$.

$f'(x) = 3(x^2 - x)^2(2x - 1) = 3(4 - 2)^2(4 - 1) = 36.$
The slope of the normal line is $-1/36$, and
$y_0 = f(2) = 8$ when $x + 0 = 2$,
so the line has equation
$y = 8 - (x - 2)/36 = \frac{290 - x}{36}$.

3. A particle moves according the law of motion $s(t) = 2t^3 - 21t^2 + 60t$. Find the velocity function, determine when the particle is at rest and when it is moving forward. Find the average velocity from time $t = 5$ to $t = 10$ and find the total distance traveled from $t = 0$ to $t = 10$.

$v = ds/dt = 6t^2 - 42t + 60 = 6(t - 2)(t - 5).$
The particle is at rest when $v = 0$, which is $t = 2$ and $t = 5$.
$s(0) = 0$, $s(2) = 52$, $s(5) = 25$ and $s(10) = 500$.
Average velocity $= \frac{s(10) - s(5)}{10 - 5} = \frac{500 - 25}{5} = 95.$
Total distance $= [s(2) - s(0)] + [s(2) - s(5)] + [s(10) - s(5)] = 52 + 27 + 475 = 554.$

4. Find $f'(\frac{\pi}{4}$ if $f(x) = \frac{x}{sin\ x}$.

$f'(x) = \frac{sin\ x - xcos\ x}{sin^2 x} = \frac{\sqrt{2}/2 - \frac{\pi}{4}\sqrt{2}/2}{1/2} = (1 - \frac{\pi}{4})\sqrt{2}.$

5. Doctors estimate a person's body surface area S (in meters squared) by the formula $S = \sqrt{hm}/60$, where h is height in cm and m is mass in kg. Calculate the rate of change of S with respect to mass if $h = 180$ is constant. Find the rate at $m = 60$ and at $m = 80$.

$S = \sqrt{180m}/60 = m^{1/2}/2\sqrt{5}$, so $dS/dm = m^{-\frac{1}{2}}/4\sqrt{5}$.
When $m = 60$, this is $1/4\sqrt{300} = 1/40\sqrt{3}$.
When $m = 80$, this is $1/4\sqrt{400} = 1/80$.

6. Find $F'(3)$ where $F(x) = (f \circ g)(x)$, $g(3) = 5$, $g'(3) = 3$, $f'(3) = 1$ and $f'(5) = 4$.

$F'(3) = f'(g(3)g'(3) = f'(5)g'(3) = 4 \cdot 3 = 12.$

7. Find $f'(x)$ and $f''(x)$ where $f(x) = cos^2(3x)$.

$f(x) = u^2$ where $u = cos\ 3x$. Then $u' = -3sin\ 3x$ and
$f'(x) = 2uu' = 2cos\ 3x(-3sin\ 3x) = -6cos\ 3x\ sin\ 3x.$
$f'(x) = UV$, where $U = -6cos\ 3x$ and $V = sin\ 3x$.
Then $U' = 18sin\ 3x$ and $V' = 3cos\ 3x$.
So $f''(x) = U'V + UV' = 18sin^2 3x - 18cos^2 3x.$

8. Find $f'(3)$ and $f''(3)$ where $y = f(x)$, $f(3) = 1$ and $xy^3 + xy = 6$.

$y^3 + 3xy^2y' + y + xy' = 0$, so $y' = \frac{-y^3-y}{3xy^2+x} = \frac{-2}{12} = \frac{-1}{6}$.
Then
$y'' = \frac{(-3y^2-1)y'(3xy^2+x)+(y^3+y)(6xyy'+3y^2+1)}{(3xy^2+x)^2} = \frac{(-4)(-1/6)(12)+(2)(1)}{(12)^2} = \frac{10}{144}$.

9. Find $f'(x)$ and $f''(x)$ where $f(x) = \frac{e^x}{e^x+1}$.

$f'(x) = \frac{e^x(e^x+1)-e^x(e^x)}{(e^x+1)^2} = \frac{e^x}{(e^x+1)^2}$.
$f''(x) = \frac{e^x(e^x+1)^2-e^x2(e^x+1)(e^x)}{(e^x+1)^4} = \frac{e^x(e^x+1)-e^x2e^x}{(e^x+1)^3} = \frac{e^x-e^{2x}}{(e^x+1)^3}$.

10. Find the second derivative of $f(x) = x^2 ln\ x$.

$f'(x) = 2xln\ x + x$, and $f''(x) = 2ln\ x + 2 + 1 = 3 + 2ln\ x$.

11. Let $f(x) = x + \sqrt{x^3+1}$ (so $f(2) = 5$)and that $g(x)$ is the inverse function of $f(x)$. Find $g'(5)$.

$f' = 1 + 3x^2/\sqrt{x^3+1}$, so $f'(2) = 3$ and then $g'(5) = g'(f(2)) = 1/f'(2) = 1/3$.

12. Use Logarithmic Differentiation to find $f'(1)$ for
$f(x) = e^{x^2}(x+3)^2/(2x-1)$.

$f(1) = 16e$; $\quad ln\ f(x) = x^2 + 2ln(x+3) - ln(2x-1)$;
$f'/f = 2x + \frac{2}{x+3} - \frac{2}{2x-1} = 1/2$; $\quad f'(1) = 8e$.

13. Find the tangent line to $y = arcsin\ x$ at $x = \frac{1}{2}$ and sketch with the curve.
$m = dy/dx = \frac{1}{1-x^2} = \frac{2}{\sqrt{3}}$; $\quad y = \frac{\pi}{6} + \frac{2}{\sqrt{3}}(x - \frac{1}{2})$.

14. A spherical balloon is inflated at a rate of 20 cc per second. How fast is the radius of the sphere increasing when the radius equals 5 cm.

$V = \frac{4}{3}\pi R^3$, so $dV/dR = 4\pi R^2$.
$20 = dV/dt = dV/dRdR/dt = 4\pi R^2 dR/dt$.
When $R = 5$, $20 = 4\pi(25)dR/dt$, so $dR/dt = \frac{1}{5\pi}$ cm per second.