Sample Problem Solutions for Exam Three

1. Find the linear approximation to $f(x) = x^{frac14}$ near the point (16,2) and use use this to estimate $18^{\frac{1}{4}}$.

$f'(x) = \frac{1}{4}x^{\frac{-3}{4}}$, so $f'(16) = \frac{1}{32}$.

Then $L(x) = 2 + \frac{1}{32}(x - 2)$.

Thus $L(18) = 2 + \frac{1}{16}$ is our estimate for $18^{\frac{1}{4}}$.

2. Find the absolute maximum and minimum of $f(x) = x + cos\ x$ on $[0, \pi]$.

$f' = 1 - sin\ x$, so the critical point is $x = \pi/2$.

$f(0) = 1$ is the minimum, $f(\pi/2) = \pi/2$ and $f(\pi) = \pi - 1$ is the maximum.

3. Given that $f(x) = \frac{2(x+1)^2}{x^2+1}$, $f'(x) = \frac{-4(x^2-1)}{(x^2+1)^2}$, and $f''(x) = \frac{8x(x^2-3)}{(x^2+1)^3}$, find the intercepts, asymptotes, critical and inflection points, and sketch the graph.

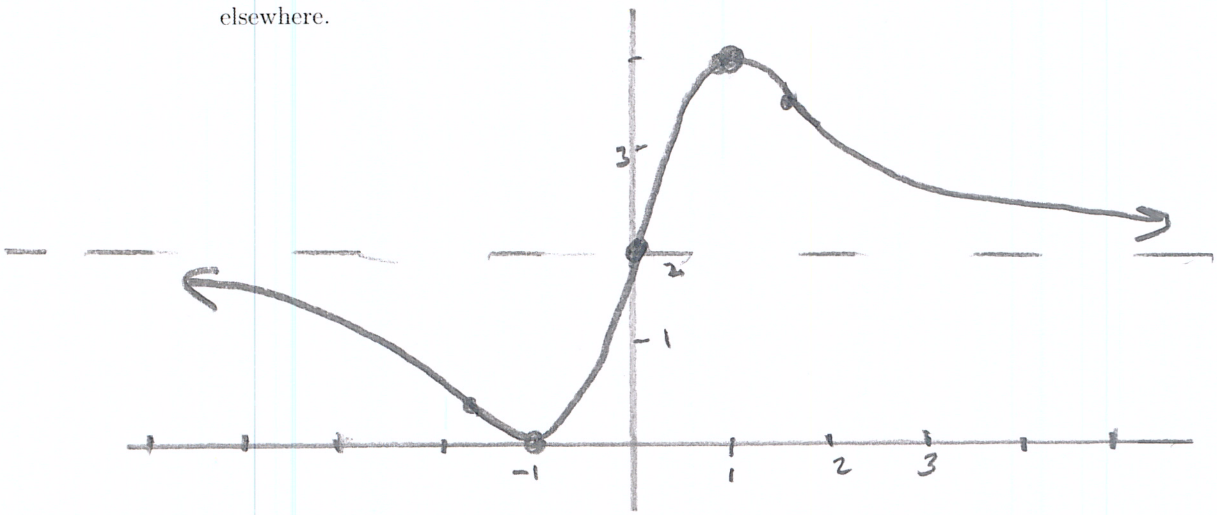
Note that $\sqrt{3} \simeq 1.73$, $f(-\sqrt{3}) \simeq 0.27$, and $f(\sqrt{3}) \simeq 3.73$.

$f(-1) = 0$, $f(x) \geq 0$; horizontal asymptote $y = 2$ ($lim_{x \to \pm\infty} f(x) = 2$)

Critical points $x = \pm 1$, f is increasing on $(-1, 1)$ and decreasing elsewhere.

$f(-1) = 0$ is a min and $f(0) = 2$ is a max.

Inflection points $x = 0$, $x = \pm\sqrt{3}$, f concave up on $(-\sqrt{3}, 0$ and $(\sqrt{3}, \infty)$, down elsewhere.



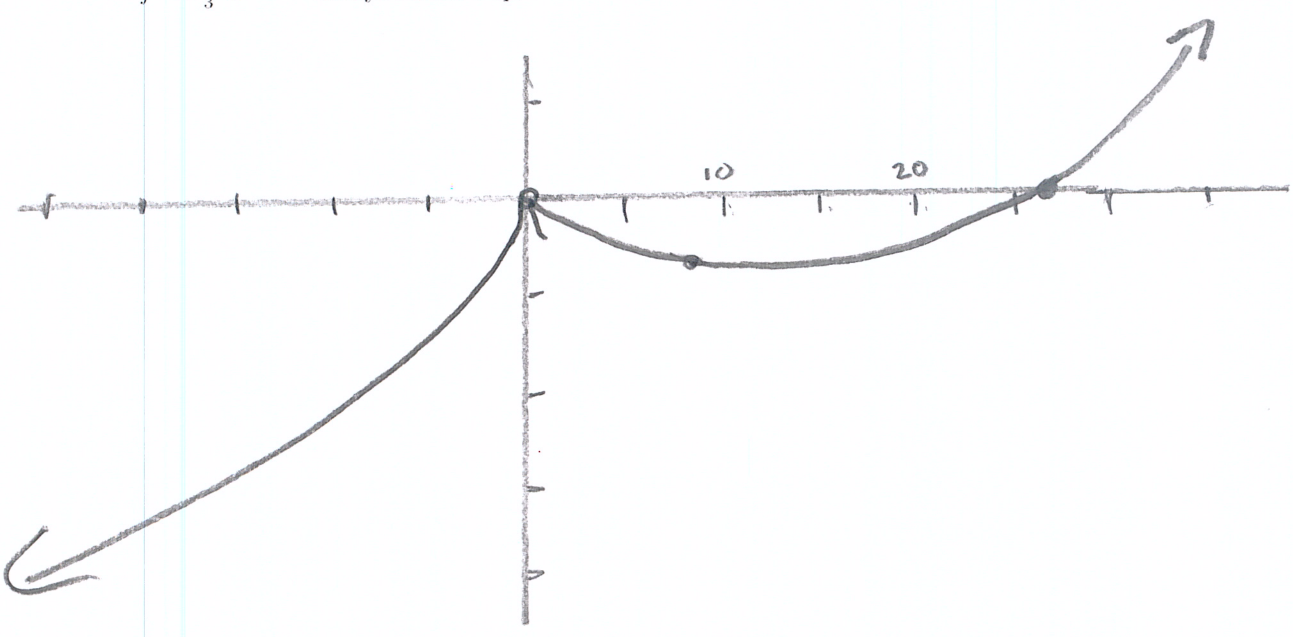
4. Let $f(x) = x - 3x^{2/3}$. Find the domain, intercepts, asymptotes, critical numbers, inflection points, intervals where increasing/decreasing and concave up/down. Sketch the graph.

Intercepts: $(0, 0)$ and $(27, 0)$; Domain: $(-\infty, \infty)$. No Asymptotes.

$f' = 1 - 2x^{-1/3}$. Increasing on $(0, 8)$ and decreasing on $(-\infty, 0)$ and $(8, \infty)$.

Critical points: $x = 0$–local max of $y = 0$ and $x = 8$–local min of $y = -4$.

$f'' = \frac{2}{3}\ x^{-4/3}$ – always concave up.



5. Evaluate the following limits:

(a) $\lim_{x\to\infty} x^3/e^x$

(a) $\lim_{x\to\infty} x^3/e^x = \frac{\infty}{\infty} = \lim_{x\to\infty} 3x^2/e^x [by L'Hopital's rule] = \lim_{x\to\infty} 6x/e^x = \lim_{x\to\infty} 6/e^x = 0$.

(b) $\lim_{x\to 0} \frac{sin\ x - x}{x^3}$.

$\lim_{x\to 0} \frac{sin\ x - x}{x^3} = 0/0$, by L'Hopital's rule, which equal $\lim_{x\to 0} \frac{cos\ x - 1}{3x^2}$, which equals $\lim_{x\to 0} \frac{-sin\ x}{6x} = 0/0 = \lim_{x\to 0} \frac{-cos\ x}{6} = -\frac{1}{6}$.

(c) $\lim_{x\to 1}(x^2-1)^{x-1}$

$\lim_{x\to 1}(x^2-1)^{x-1} = 0^0 = L$.

Then $ln\ L = \lim_{x\to 1}(x-1)ln(x^2-1) = (0)(-\infty) = \lim_{x\to 1} ln(x^2-1)/\frac{1}{x-1} =$ [by L'Hopital's rule] $\lim_{x\to 1} \frac{2x}{x^2-1}/\frac{-1}{(x-1)^2} = \lim_{x\to 1} \frac{-2x(x-1)}{x+1} = 0$.

Hence $L = e^0 = 1$.

6. A window has the shape of a rectangle with a semicircle on one end. Find the dimensions of the window with the largest area with a perimeter of 20 feet.

$P = 20 = 2r + 2s + \pi r$, so $s = 10 - r - \frac{\pi}{2} r$.

$A = \frac{\pi}{2} r^2 + 2rs = 20r - 2r^2 - \frac{\pi}{2} r^2$.

$A' = 20 - 4r - \pi 4$, so the critical number is $r = \frac{20}{4+\pi}$.

Then $s = \frac{20}{4+\pi}$ and $A = \frac{200}{4+\pi}$.

7. Find the point on the line $y = 2 - x$ which is closest to the point $(1,0)$.

The distance from $(1,0)$ to $(x, 1/x)$ is $\sqrt{(x-1)^2 + (2-x)^2}$.

So we need to minimize $f(x) = x^2 - 2x + 1 + x^2 - 4x + 4 = 2x^2 - 6x + 5$.

$f'(x) = 4x - 6$, so $x = 1.5$ and $y = 2 - x = .5$.

8. A ball is tossed into the air with initial velocity 24.5 m/sec and subject to gravity, which is $-9.8 m/sec^2$. Find equations for the velocity $v(t)$ and height $s(t)$. Give the maximum height and the impact velocity.

$v = \int(-9.8)dt = -9.8t + 24.5$

$s = \int(-9.8t + 24.5)dt = -4.9t^2 + 24.5t$.

Maximum height occurs when $v = 0$, so t $= 24.5/9.8 = 2.5$ seconds. Then $s = -4.9(2.5)^2 + 24.5(2.5) = 30.625$.

Impact occurs when $s = 0$, so $t = 5$ and $v = -9.8(5) + 24.5 = -24.5$.

9. A point moves along the x-axis with acceleration given by $a(t) = -1/t^2$. If $v(1) = 3$, find the distance it will travel between $t = 1$ and $t = 3$ seconds.

$v(t) = \int -1/t^2 = t^{-1} + c$.

Since $v(1) = 3 = 1 + c$, we have $c = 2$.

Now distance $s = \int(t^{-1} + 2 = ln\ t + 2t + b$.

Setting $s(1) = 0$, we get $b = -2$.

Then $s(3) = ln\ 3 + 6 - 2 = 4 + ln\ 3$.

10. Suppose that $f'(x) = x^{2/3} + sec\ x\ tan\ x$ and that $f(0) = 2$. Find $f(x)$.
$f(x) = \int x^{2/3} + sec\ x\ tan\ x = \frac{3}{5}x^{5/3} + sec\ x + C$.
Then $2 = f(0) = 0 + 1 + C$, so $C = -1$.
Thus $f(x) = \frac{3}{5}x^{5/3} + sec\ x - 1$.

11. Find the following antiderivatives:
(a) $\int 7x^{4/3}$

$3x^{7/3}$.

(b) $\int (\frac{6}{x^3} + \frac{4}{x})$
(c) $\int 10e^{-5x}$
$-2e^{-5x}$.
(d) $\int cos(2t + 1)$.
$\frac{1}{2}sin(2t + 1)$.