Solutions to Sample Problems for Exam Four

1. Evaluate $\sum_{i=1}^{n} i^3 - i^2$ in general and for $n = 20$.
$n^2(n+1)^2/4 - n(n+1)(2n+1)/6$; 41,230.

2. Compute the area under $y = x^2$ from $x = 0$ to $x = 2$ as a limit of (right) Riemann sums.
$R_n = \sum_{i=1^n} \frac{2}{n}(\frac{2i}{n})^2 = \frac{8}{n^3}\sum_{i=1}^{n} i^2 = 8n(n+1)(2n+1)/6.$
$lim_{n\to\infty} R_n = 8/3.$

3. Find the (left) Riemann sum which approximates $\int_1^3 2^x dx$ with n subintervals of equal length, in general and for $n = 4$.
$\sum_{i=0}^{n-1} \frac{2}{n} 2^{1+\frac{2i}{n}}.$

4. If $\int_0^3 f(x)dx = 12$ and $\int_0^6 f(x)dx = 52$, find $\int_3^6 f(x)dx$.
$\int_3^6 f(x)dx = \int_0^6 f(x)dx - \int_0^3 f(x)dx = 52 - 12 = 40.$

5. Let $f(x) = \int_1^{x^{1/3}} \frac{e^t}{t} dt$. Find $f'(1)$.
Let $g(x) = \int_1^x \frac{e^t}{t} dt$, so $g'(x) = \frac{e^x}{x}$.
$f(x) = g(x^{1/3})$, so
$f'(x) = g'(x^{1/3}) \cdot \frac{1}{3} x^{-2/3} = \frac{e^{x^{1/3}}}{3x} = e/3.$

6. Let $f(x) = \int_0^x \frac{4-t^2}{1+cos^2\ t} dt$. Where does f have a local maximum?
$f'(x) = \frac{4-x^2}{1+cos^2\ x}$ is > 0 on $(-2, 2)$ and < 0 on $(-\infty, -2)$ and $(2, \infty)$.
So f is decreasing on $(-\infty, -2)$, then increasing on $(-2, 2)$ and again decreasing on $(2, \infty)$.
This means that f has a local maximum at $x = 2$.

7. Evaluate $\int_0^1 (1 + \sqrt{x})^2$.
$\int_0^1 (1 + 2x^{\frac{1}{2}} + x)dx = [x + \frac{4}{3}x^{\frac{3}{2}} + \frac{1}{2}x^2]_0^1 = 1 + \frac{4}{3} + \frac{1}{2} = \frac{17}{6}.$

8.Evaluate $\int_1^3 12x^3 - 4x + 2$.
$3x^4 - 2x^2 + 2x]_1^3 = (243 - 18 + 6) - (3 - 2 + 2) = 228.$

9. Evaluate $\int_0^{\frac{1}{2}} \frac{dx}{\sqrt{1-x^2}}$.
$Arcsin\ x]_0^{1/2} = Arcsin(1/2) - Arcsin(0) = \pi/6 - 0 = \pi/6.$

10. Evaluate $\int_{\sqrt{e}}^{e^5} \frac{dx}{x\ ln\ x}$.
Let $u = ln\ x$, so $du = \frac{dx}{x}$. When $x = \sqrt{e}$, $u = .5$ and when $x = e^5$, $u = 5$.
Then we have $\int_{.5}^{5} u^{-1} du = [ln\ u]_{.5}^{5} = ln\ 5 - ln\ .5 = ln\ 10 \approx 2.3.$

11. Evaluate $\int \frac{e^x}{(e^x+2)^2} dx$.
Let $u = e^x + 2$, so that $du = e^x dx$. Then we have
$\int u^{-2} du = -1/u = -\frac{1}{e^x+2}$.

12. Evaluate $\int_2^3 \frac{x}{x-1} dx$.
Let $u = x - 1$ so $du = dx$, $x = u + 1$ and $\frac{x}{x-1} = \frac{u+1}{u} = 1 + u^{-1}$.
$x = 2$ becomes $u = 1$ and $x = 3$ becomes $u = 2$.
Then we have $\int_1^2 (1 + u^{-1}) du = [u + ln\ u]_1^2 = (2 + ln\ 2) - (1 + ln\ 1) = 1 + ln\ 2$.

13. Evaluate $\int_{-\pi}^{\frac{\pi}{3}} f(x)dx$, where $f(x) = -x$, if $x \le 0$ and $= sin\ 3x$, if $x \ge 0$ – sketch the area.
$\int_{-\pi}^{0} -x\ dx\ + \int_0^{\pi/3} sin\ 3x\ dx$
$= [-x^2/2]_{-\pi}^0 + [-\frac{1}{3} cos\ 3x]_0^{\pi/3}$
$= (0 + \pi^2/2) + (1/3 + 1/3) = \pi^2/2 + 2/3$.

14. A ball is thrown upwards from the top of a 160 foot tall building with initial velocity $v_0 = 80$ feet per second and subject to gravity at -32 feet per second. Find the velocity $v(t)$ and position $s(t)$ after time t. Find the maximum height of the ball, the duration of the flight and the velocity at impact.
$a = -32$, so $v = -32t + 80$ and $s = -16t^2 + 80t + 160$.
The max height is when $v = 0$, so $t = 2.5$ and $s = 260$.
Impact is when $s = 0$, so $t \approx 6.5$ and $v \approx -128$.

15. A point moves along the x-axis with acceleration given by $a(t) = 1 - 1/t^2$. If $v(1) = 1$ and $s(1) = 0$, find $v(t)$, $s(t)$, and the net and total distance traveled between $t = 1$ and $t = 3$ seconds.
$v = t + t^{-1} + c$; $1 = v(1) = 1 + 1 + c$, so $c = -1$ and $v = t + t^{-1} - 1$.
$s = \frac{1}{2}t^2 + ln\ t - t + b$; $0 = s(1) = \frac{1}{2} + 0 - 1 + b$, so $b = \frac{1}{2}$ and $s = \frac{1}{2}t^2 + ln\ t - t + \frac{1}{2}$.
Then $s(3) = \frac{9}{2} + ln\ 3 - 3 + \frac{1}{2} = 2 + ln\ 3$. .

16. A population $P(t)$ of bacteria doubles every 6 hours. Find the population $P(t)$ given that $P(0) = 200$. Find the rate of growth k such that $P' = kP$.
$P(t) = 200 2^{t/6} = 200e^{tln\ 2/6}$. Thus the rate $k = \frac{ln\ 2}{6}$.

17. The element Gatrion decays subject to the differential equation $G' = -.002P$. Find a formula for $P(t)$ given $P(0) = 5mg$. Find the half-life for Gatrion.
$P(t) = 5e^{-.002t}$.
Half-life occurs when $e^{-.002t} = 1/2$, so that $t = 500 ln\ 2$.