Differential Equations Final Exam Solutions

(7) 1. Find the integrating factor μ and solve explicitly the linear differential equation

$$dy/dx - \frac{2y}{x} = x^2\ cos\ x$$

$P = -2/x$ and $\mu = exp(\int P) = e^{-2ln\ x} = x^{-2}$
$y = x^2 \int cos\ x = x^2[C + sin\ x] = x^2 sin\ x + Cx^2$.
Final Answer: $y = x^2 sin\ x + Cx^2$.

(8) 2. Use the Test for Exactness to check the differential equation

$$(2x + y^3\ sec^2 x)dx + (1 + 3y^2 tan\ x)dy = 0$$

Then integrate to find the general (implicit) solution.
Finally, solve the initial value problem when $x_0 = \pi$ and $y_0 = 2$.

$M_y = 3y^2 sec^2 x = N_x$ so the equation is exact.
Integrating, we have
$(x^2 + y^3 tan\ x) \bigcup (y + y^3 tan\ x) = x^2 + y^3 tan\ x + y = C$.
For $x_0 = \pi$ and $y_0 = 2$, we have $\pi^2 + 8(0) + 2 = C$
Final Answer: $x^2 + y^3 tan\ x + y = 2 + \pi^2$

(8)3. Find the steady-state solution y_p of a spring-mass system subject to the differential equation

$$y'' + 4y' + 20y = sin\ 2t.$$

(Hint: Use Undetermined Coefficients.)

$y_p = Asin\ 2t + Bcos\ 2t$.
$y_p'' + 4y_p' + 20y_p = (16A - 8B)sin\ 2t + (8A + 16B)cos\ 2t = sin\ 2t$.
$(sin\ 2t):\quad 16A - 8B = 1$
$(cos\ 2t):\quad 8A + 16B = 0$
Then $A = -2B$, so $-40B = 1$, thus $B = -.025$ and $A = .05$.
Final Answer: $y_p = .05 sin\ 2t - .025 cos\ 2t$.

(7) 4. Use Variation of Parameters to find a particular solution to

$$y'' + y = sec\ t$$

$y_1 = cos\ t$ and $y_2 = sin\ t$.
Wronskian $W[y_1, y_2] = 1$.
$y_p = v_1 y_1 + v_2 y_2$
$v_1' = -sin\ t sec\ t = -tan\ t$, so $v_1 = ln\ cos\ t$
$v_2' = cos\ t sec\ t = 1$, so $v_2 = t$.
Final Answer: $y_p = cos\ t ln\ cos\ t + t\ sin\ t$.

(8) 5. Use Laplace Transforms and Partial Fractions to solve

$$y'' + 4y = 8t; \qquad y(0) = 3; \quad y'(0) = 0.$$

Transform: $s^2 Y - 3s + 4Y = 8/s^2$

Solve: $Y = \frac{3s^3+8}{s^2(s^2+4)}$.

Partial Fractions: $Y = A/s + B/s^2 + (Cs + D)/(s^2 + 4)$.
So $y = A + Bt + C cos\ 2t + \frac{1}{2} D sin\ 2t$.

$3s^3 + 8 = As(s^2 + 4) + B(s^2 + 4) + Cs^3 + Ds^2$.

$(s^3): \quad 3 = A + C$
$(s^2): \quad 0 = B + D$
$(s): \quad 0 = 4A$, so $A = 0$
$(1): \quad 8 = 4B$, so $B = 2$
Then $C = 3$ and $D = -2$.
Final Answer: $y = 2t + 3cos\ 2t - sin\ 2t$.

(7) 6. Use Laplace transforms to solve the differential equation

$$y' + 5y = 10\delta(t - 3); \qquad y(0) = 100$$

Express the solution using step functions.

Transform: $sY - 100 + 5Y = 10e^{-3s}$
Solve: $Y = \frac{100+10e^{-3s}}{s+5} = 100F(s) + 10e^{-3s}F(s)$
where $F(s) = \frac{1}{s+5}$, so $f(t) = e^{-5t}$
Now $y = 100f(t) + 10u(t - 3)f(t - 3)$
Final Answer: $y = 100e^{-5t} + 10u(t - 3)e^{-5(t-3)}$
For $t < 3$, $y = 10e^{-5t}$ and for $t > 3$, $y = 100e^{-5t} + 10e^{-5(t-3)}$

(8) 7. Use any method to find the first four nonzero terms in the Taylor polynomial approximation for the initial value problem

$$x' + (sin\ t)x = 0; \quad x(0) = 1$$

$x = a_0 + a_1 t + a_2 t^2 + \ldots$
$a_0 = x(0) = 1$
$a_1 = x'(0) = -sin\ 0 \cdot 0 = 0$
$x'' = -x' sin\ t - x cos\ t$, so $x''(0) = -1$ and $a_2 = -\frac{1}{2}$.
$x''' = -x'' sin\ t - 2x' cos\ t + x\ sin\ t$, so $x'''(0) = 2$ and $a_3 = 2/6 = 1/3$
$x^{iv} = -x''' sin\ t - 3x'' cos\ t + 3x' sin\ t + x\ cos\ t$, so $x^{(iv)}(0) = 4$, so $a_4 = \frac{4}{24}$

Final Answer: $x = 1 - \frac{1}{2}t^2 + \frac{1}{3}t^3 + \frac{1}{6}t^4$.

(7) 8.Find the indicial equation and solve the Cauchy-Euler differential equation

$$x^2 y'' + 7xy' + 8y = 0; \qquad y(1) = 3 \quad y'(1) = 5$$

$r(r - 1) + 7r + 8 = r^2 + 6r + 8$
Roots are $r = -2$ and $r = -4$.
Solution is $y = c_1 x^{-2} + c_2 x^{-4}$.
$3 = y(1) = c_1 + c_2$
$5 = y'(1) = -2c_1 - 4c_2$
So $c_1 = 17/2$ and $c_2 = -11/2$.
Final Answer: $y = 8.5x^{-2} - 5.5x^{-4}$.