Computability Theoretic Properties of Injection Structures

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Abstract

We study computability theoretic properties of computable injection structures and the complexity of isomorphisms between these structures. We prove that a computable injection structure is computably categorical if and only if it has finitely many infinite orbits. We also prove that a computable injection structure is Δ_2^0 categorical if and only if it has finitely many orbits of type ω or finitely many orbits of type Z. Furthermore, every computably categorical injection structure is relatively computably categorical, and every Δ_2^0 categorical injection structure is relatively Δ_2^0 categorical. We investigate analogues of these results for Σ_1^0, Π_1^0, and n-c.e. injection structures.

We study the complexity of the set of elements with orbits of a given type in computable injection structures. For example, we show that for every c.e. Turing degree $\mathbf{b}$, there is a computable injection structure $\mathcal{A}$ in which the set of all elements with finite orbits has degree $\mathbf{b}$ and, for every Σ_2^0 Turing degree $\mathbf{c}$, there is a computable injection structure $\mathcal{B}$ in which the set of elements with orbits of type ω has degree $\mathbf{c}$. We also study various index set results for infinite computable injection structures. For example, we show that the index set of infinite computably categorical injection structures is a Σ_3^0 complete set and that the index set of infinite Δ_2^0 categorical injection structure is a Σ_4^0 complete set.

We also explore the connection between the complexity of the character and the first-order theory of computable injection structures. We show that for an injection structure with a computable character, there is a decidable structure isomorphic to it. However, there are computably categorical injection structures with undecidable theories.

Keywords: computability theory, permutations, injections, effective categoricity, computable model theory

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1 Introduction

Computable model theory deals with the algorithmic properties of effective mathematical structures and the relationships among such structures. Perhaps the most basic kind of relationship between two structures is that of isomorphism. Thus, is natural to study the isomorphism problem in the context of computable mathematics by investigating the following question.

Given two effective structures that are isomorphic, what is the least complex isomorphism between them?

Let $\mathbb{N} = \{0, 1, 2, \ldots\}$ denote the natural numbers and $\mathbb{Z} = \{0, \pm 1, \pm 2, \ldots\}$ denote the integers. We let ω denote the order type of $\mathbb{N}$ under the usual ordering, and Z denote the order type of $\mathbb{Z}$ under the usual ordering. In this paper, we restrict our attention to countable structures for computable languages. Hence, if a structure is infinite, we can assume that its universe is $\mathbb{N}$. We recall some basic definitions. If $\mathcal{A}$ is a structure with universe A for a language $\mathcal{L}$, then $\mathcal{L}^A$ is the language obtained by expanding $\mathcal{L}$ by constants for all elements of A. The *atomic diagram* of $\mathcal{A}$ is the set of all atomic sentences and negations of atomic sentences from $\mathcal{L}^A$ which are true in $\mathcal{A}$. The *elementary diagram* of $\mathcal{A}$ is the set of all first-order sentences of $\mathcal{L}^A$ which are true in $\mathcal{A}$. A structure is *computable* if its atomic diagram is computable and a structure is *decidable* if its elementary diagram is computable. We call two structures *computably isomorphic* if there is a computable function that is an isomorphism between them. A computable structure $\mathcal{A}$ is *relatively computably isomorphic* to a (possibly noncomputable) structure $\mathcal{B}$ if there is an isomorphism between them which is computable in the atomic diagram of $\mathcal{B}$. A computable structure $\mathcal{A}$ is called *computably categorical* if every computable structure that is isomorphic to $\mathcal{A}$ is computably isomorphic to $\mathcal{A}$. A computable structure $\mathcal{A}$ is called *relatively computably categorical* if $\mathcal{A}$ is relatively computably isomorphic to every structure that is isomorphic to $\mathcal{A}$. Similar definitions arise for other naturally definable classes of structures and their isomorphisms. For example, for any $n \in \mathbb{N}$, a structure is Δ^0_n if its atomic diagram is Δ^0_n, two Δ^0_n structures are Δ^0_n *isomorphic* if there is a Δ^0_n isomorphism between them, and a computable structure $\mathcal{A}$ is Δ^0_n *categorical* if every computable structure isomorphic to $\mathcal{A}$ is Δ^0_n isomorphic to $\mathcal{A}$.

For a Turing degree $\mathbf{d}$, a computable structure $\mathcal{A}$ is called $\mathbf{d}$-*computably categorical* if for every computable structure $\mathcal{B}$ isomorphic to $\mathcal{A}$, there exists a $\mathbf{d}$-computable isomorphism from $\mathcal{B}$ onto $\mathcal{A}$. Hence, for example, $\mathbf{0}$-computably categorical structures are the same as computably categorical ones. In [10], Miller introduced and first studied $\mathbf{d}$-computable categoricity for computable algebraic fields. He defined the *degree of categoricity* of a computable structure $\mathcal{A}$, if it exists, to be the least Turing degree $\mathbf{d}$ for which $\mathcal{A}$ is $\mathbf{d}$-computably categorical. Since there are only countably many computable structures, most Turing degrees are not degrees of categoricity. In [8], Fokina, Kalimullin, and Miller investigated which Turing degrees can be the degrees of categoricity. Their investigation was further extended by Csima, Franklin, and Shore in [6].

Among the simplest nontrivial structures are equivalence structures, that is, structures of the form $\mathcal{A} = (A, E)$, where E is an equivalence relation on A. The complexity of isomorphisms between computable equivalence structures was recently studied by Calvert, Cenzer, Harizanov, and Morozov in [2] where they characterized computably categorical equivalence structures and also relatively Δ_2^0 categorical equivalence structures. Cenzer, LaForte, and Remmel [4] extended this work by investigating equivalence structures in the Ershov hierarchy. More recently, Cenzer, Harizanov, and Remmel [3] studied Σ_1^0 and Π_1^0 equivalence structures.

For any equivalence structure $\mathcal{A}$, we let $Fin(\mathcal{A})$ denote the set of elements of $\mathcal{A}$ that belong to finite equivalence classes. For equivalence structures, it is natural to consider the different sizes of the equivalence classes of the elements in $Fin(\mathcal{A})$ since such sizes code information into the equivalence relation. The *character* of an equivalence structure $\mathcal{A}$ is a subset of $(\mathbb{N} - \{0\}) \times (\mathbb{N} - \{0\})$ defined by:

$$\chi(\mathcal{A}) = \{(k, n) : n, k > 0 \text{ and } \mathcal{A} \text{ has at least } n \text{ equivalence classes of size } k\}.$$

This set provides a kind of skeleton for $Fin(\mathcal{A})$. Any set $K \subseteq (\mathbb{N} - \{0\}) \times (\mathbb{N} - \{0\})$ such that for all $n > 0$ and k, $(k, n + 1) \in K$ implies $(k, n) \in K$, is simply called a *character*. We say that a character K is *bounded* if there is some finite k_0 such that for all $(k, n) \in K$, we have $k < k_0$. Khisamiev [9] introduced the concepts of an s-function and an s_1-function in his work on Abelian p-groups with computable isomorphic copies. In their book [1], Ash and Knight investigated equivalence structures in the context of Khisamiev's results. For background on computable model theory and categoricity, see Ershov and Goncharov [7].

Definition 1.1. *Let $f : \mathbb{N}^2 \to \mathbb{N}$. The function f is an* s-function *if*
1. for every $i, s \in \mathbb{N}$, $f(i, s) \leq f(i, s + 1)$ and
2. for every $i \in \mathbb{N}$, the limit $m_i = \lim_{s\to\infty} f(i, s)$ exists.

We say that f is an s_1-function *if, in addition,*
3. for every $i \in \mathbb{N}$, $m_i < m_{i+1}$.

Calvert, Cenzer, Harizanov, and Morozov [2] gave conditions under which a given character K can be the character of a computable equivalence structure. In particular, they observed that if K is a bounded character and $\alpha \leq \omega$, then there is a computable equivalence structure with character K and exactly α infinite equivalence classes. To prove the existence of computable equivalence structures for unbounded characters K, they needed additional information given by s-functions and s_1-functions. They showed that if K is a Σ_2^0 character, $r \in \mathbb{N}$, and either
(a) there is an s-function f such that

$$(k, n) \in K \Leftrightarrow card(\{i : k = \lim_{s\to\infty} f(i, s)\}) \geq n, \text{ or}$$

(b) there is an s_1-function f such that for every $i \in \mathbb{N}$, $(\lim_{s\to\infty} f(i, s), 1) \in K$,
then there is a computable equivalence structure with character K and exactly r infinite equivalence classes.

In this paper, we study injection structures. Here, an injection is, as usual, a one-to-one function, and an *injection structure* $\mathcal{A} = (A, f)$ consists of a set A and an injection $f : A \to A$. The structure $\mathcal{A}$ is a *permutation structure* if f is a permutation of A. Given $a \in A$, the *orbit* $\mathcal{O}_f(a)$ *of* a *under* f is

$$\mathcal{O}_f(a) = \{b \in A : (\exists n \in \mathbb{N})[f^n(a) = b \vee f^n(b) = a]\}.$$

Clearly, the isomorphism type of a permutation structure $\mathcal{A}$ is determined by the number of orbits of size k for $k = 1, 2, \ldots, \omega$. By analogy with characters of equivalence structures, we define the *character* $\chi(\mathcal{A})$ of an injection structure $\mathcal{A} = (A, f)$ as

$$\chi(\mathcal{A}) = \{(k, n) \in (\mathbb{N} - \{0\}) \times (\mathbb{N} - \{0\}) : \mathcal{A} \text{ has at least } n \text{ orbits of size } k\}.$$

We let $Ran(f)$ denote the range of f.

An injection structure (A, f) may have two types of infinite orbits: Z-orbits, which are isomorphic to $(\mathbb{Z}, S)$ and in which every element is in $Ran(f)$, and ω-orbits, which are isomorphic to (ω, S) and have the form $\mathcal{O}_f(a) = \{f^n(a) : n \in \mathbb{N}\}$ for some $a \notin Ran(f)$. Thus, injection structures are characterized by the number of orbits of size k for each finite k and by the number of orbits of types Z and ω. We will examine the complexity of the set of elements with orbits of a given type in an injection structure $\mathcal{A} = (A, f)$. In particular, we will study the complexity of $Fin(\mathcal{A}) = \{a : \mathcal{O}_f(a) \text{ is finite}\}$. It is clear from the definitions above that any computable injection structure (A, f) will induce a Σ^0_1 equivalence structure (A, E) in which the equivalence classes are simply the orbits of (A, f).

The outline of this paper is as follows. In Section 2, we investigate algorithmic properties of computable injection structures and their characters, characterize computably categorical injection structures, and show that all computably categorical injection structures are relatively computably categorical. More specifically, we prove that a computable injection structure $\mathcal{A}$ is computably categorical if and only if it has finitely many infinite orbits. In Section 3, we characterize Δ^0_2 categorical injection structures as those with finitely many orbits of type ω or with finitely many orbits of type Z. We show that they coincide with the relatively Δ^0_2 categorical structures. Finally, we prove that every computable injection structure is relatively Δ^0_3 categorical.

In Section 4, we consider the *spectrum question*, which is to determine the possible sets (or degrees of sets) that can be the set $Fin(\mathcal{A})$ for some computable injection structure $\mathcal{A}$ of a given isomorphism type. For example, we show that for any c.e. degree $\mathbf{b}$, there is a computable injection structure $\mathcal{A}$ such that $Fin(\mathcal{A})$ has degree $\mathbf{b}$. In Section 5, we study the complexity of the theory $Th(\mathcal{A})$ of a computable injection structure $\mathcal{A}$, as well as the complexity of its elementary diagram $FTh(\mathcal{A})$. We prove that the character $\chi(\mathcal{A})$ and the theory $Th(\mathcal{A})$ have the same Turing degree. We also show that there is a computably categorical injection structure the theory of which is not decidable. In Section 6, we study index sets for infinite computable injection structures. That is, let

ϕ_e denote the eth partial computable function and let $\mathcal{A}_e = (\mathbb{N}, \phi_e)$. For any property $\mathcal{P}$ of injection structures, we let

$$Inj(\mathcal{P}) = \{e : \mathcal{A}_e \text{ has property } \mathcal{P}\}.$$

Then let $\mathcal{P}_{cc}$ be the property of being a computable categorial injection structure and $\mathcal{P}_{\Delta^0_2 c}$ be the property of being a computable Δ^0_2 categorial injection structure. We show that $Inj(\mathcal{P}_{cc})$ is Σ^0_3 complete and $Inj(\mathcal{P}_{\Delta^0_2 c})$ is Σ^0_4 complete. We also establish that for infinite computable injection structures, the isomorphism problem is Π^0_4 complete, while the computable isomorphism problem is Σ^0_3 complete.

In Section 7, we consider Σ^0_1 injection structures, that is, injection structures $\mathcal{A} = (A, f)$ where A is an infinite Σ^0_1 set and f is the restriction of a partial computable function to A. We show that every Σ^0_1 injection structure is computably isomorphic to a computable injection structure. Hence if two Σ^0_1 injection structures are isomorphic, then they are isomorphic via a Δ^0_3 isomorphism. In Section 8, we consider Π^0_1 injection structures, that is, injection structures $\mathcal{A} = (A, f)$ where A is an infinite Π^0_1 set and f is the restriction of a partial computable function to A. We show that there are Π^0_1 injection structures that have arbitrary non-trivial Σ^0_2 characters and hence there are Π^0_1 injections structures which are not isomorphic to any computable injection structure. In Section 9, we consider n-c.e. injection structures. We prove that for any n-c.e. injection structure $\mathcal{A}$, there exist a Π^0_1 structure $\mathcal{B}$ and a computable injection that maps $\mathcal{B}$ onto $\mathcal{A}$. The notions and notation of computability theory are standard and as in Soare [11].

2 Computably Categorical Structures

In this section, we first show that the characters of computable injection structures are exactly the c.e. characters. We then characterize computably categorical injections structures and show that they coincide with relatively computably categorical injection structures.

Lemma 2.1. *For any computable injection structure $\mathcal{A} = (A, f)$:*

(a) $\{(k, a) : a \in Ran(f^k)\}$ *is a* Σ^0_1 *set,*

(b) $\{(a, k) : card(\mathcal{O}_f(a)) \geq k\}$ *is a* Σ^0_1 *set,*

(c) $\{a : \mathcal{O}_f(a) \text{ is infinite}\}$ *is a* Π^0_1 *set,*

(d) $\{a : \mathcal{O}_f(a) \text{ has type } Z\}$ *is a* Π^0_2 *set,*

(e) $\{a : \mathcal{O}_f(a) \text{ has type } \omega\}$ *is a* Σ^0_2 *set, and*

(f) $\chi(\mathcal{A})$ *is a* Σ^0_1 *set.*

Proof. Parts (a) and (b) are straightforward.

(c) $\mathcal{O}_f(a)$ is infinite if and only if $(\forall n)[f^n(a) \neq a]$.

(d) $\mathcal{O}_f(a)$ has type Z if and only if $\mathcal{O}_f(a)$ is infinite and $(\forall n)(\exists b)[f^n(b) = a]$.

(e) $\mathcal{O}_f(a)$ has type ω if and only if $\mathcal{O}_f(a)$ is infinite and not of type Z.

(f) First notice that $card(\mathcal{O}_f(a)) = k$ if and only if $f^k(a) = a$ and $(\forall i < k)[f^i(a) \neq a]$. Hence the property that $card(\mathcal{O}_f(a)) = k$ is computable.

Then $(k, n) \in \chi(\mathcal{A})$ if and only if

$$(\exists x_1) \cdots (\exists x_n) \left(\bigwedge_{i=1}^{n} card(\mathcal{O}_f(x_i)) = k \ \& \ \bigwedge_{i \neq j} (\forall t < k)[f^t(x_i) \neq x_j] \right)$$

□

Proposition 2.2. *For any Σ^0_1 character K, there is a computable injection structure $\mathcal{A} = (A, f)$ with character K and with any specified finite or countabley infinite number of orbits of types ω and Z. Furthermore, $Fin(\mathcal{A})$ is computable and $Ran(f)$ is computable.*

Proof. First we build (A, f) with character K and no infinite orbits. If K is finite, this is trivial. If K is infinite, let $(k_0, n_0), (k_1, n_1), \ldots$ be a computable enumeration of K without repetition. For each $i > 0$, let $s_i = k_0 + k_1 + \cdots + k_{i-1}$. Let $s_0 = 0$. Define f to have orbits $\mathcal{O}_f(s_i) = \{s_i, s_i + 1, \ldots, s_i + k_i - 1\}$. Given $a \in \mathbb{N}$, let i be the least such that $s_i \leq a < s_{i+1}$. Then $f(a) = a + 1$ if $a < s_{i+1} - 1$ and $f(s_{i+1} - 1) = s_i$.

For any $m \leq \omega$ and $n \leq \omega$, we can easily create a computable injection structure consisting of exactly m orbits of type ω and n orbits of type Z. For example, to have ω orbits of type ω and 3 orbits of type Z, let the orbits of type Z be $\{0, 1, 2, 4, \ldots\}$, $\{3, 6, 12, \ldots\}$ and $\{5, 10, 20, \ldots\}$, and let the orbits of type ω have the form $\{2a + 7, 4a + 14, 8a + 28, \ldots\}$ for $a \in \mathbb{N}$. In the orbits of type ω, let $f(x) = 2x$. In the orbit of 0, let $f(0) = 1$, $f(2) = 0$ and for each i, let $f(2^{2i}) = 2^{2i+2}$ and $f(2^{2i+3}) = 2^{2i+1}$. For the other two orbits of type Z, let $f(a \cdot 2^{2i}) = a \cdot 2^{2i+2}$ and $f(a \cdot 2^{2i+3}) = 2^{2i+1}$.

If K is infinite, then let $\mathcal{A} = (A, f)$ have character K and no infinite orbits, and let $\mathcal{B} = (B, g)$ have the desired number of infinite orbits, and define the disjoint union $\mathcal{A} \oplus \mathcal{B}$ in the natural way by mapping A to the even numbers and B to the odd numbers. If K is finite of cardinality c, we may assume that (A, f) has universe $\{0, 1, \ldots, c - 1\}$, and then build a copy of (B, g) to have universe $\{c, c + 1, \ldots\}$ by mapping b to $b + c$. It is easy to construct $\mathcal{B}$ so that the range of f is computable. The only difficult case is when $\mathcal{B}$ has infinitely many orbits of type ω. For example, if there are no other orbits, then we can take a standard model such as $f(x) = 2x$ with universe $\mathbb{N} - \{0\}$. □

Proposition 2.2 shows that injection structures are simpler than equivalence structures in an important way. The characters are simpler, i.e., they are Σ^0_1 rather than Σ^0_2, and there is no distinction between characters that have or do not have s_1-functions.

Theorem 2.3. *If $\mathcal{A} = (\mathbb{N}, f)$ is a computable injection structure with finitely many infinite orbits, then $\mathcal{A}$ is relatively computably categorical.*

Proof. Assume $\mathcal{A}$ has m orbits of type ω and n orbits of type Z, where $m, n \in \mathbb{N}$. Let $a_1, \dots, a_m$ be elements of the m orbits of type ω, each not in $Ran(f)$. Let $b_1, \dots, b_n$ be representatives of the n orbits of type Z. A Scott formula for a finite sequence $c_0, \dots, c_r$ of elements distinct from $a_1, \dots, a_m, b_1, \dots, b_n$ is a conjunction of Δ^0_1 formulas of the following kinds. First, for each $t \leq r$, we have either

(1) $f^k(c_t) = c_t$ for some minimal k, or
(2) $f^k(a_i) = c_t$ for some unique i and k, or
(3) $f^k(b_i) = c_t$ for some unique i and k, or
(4) $f^k(c_t) = b_i$ for some unique i and k.

Nothing more needs to be said about the elements c_t which fall into cases (2), (3) or (4). For two elements c_s and c_t which fall into case (1) with the same value of k, we need to add either

(5) $f^j(c_s) = c_t$ for some unique $j < k$, or
(6) $(\forall j < k)[f^j(c_s) \neq c_t]$.

If two finite sequences satisfy the same Scott formula as defined above, then it is clear that there exists an automorphism of $\mathcal{A}$ mapping one sequence to the other, while preserving the infinite orbits. □

Our next result will show that these are the only computably categorical injection structures.

Theorem 2.4. *If a computable injection structure $\mathcal{A}$ has infinitely many infinite orbits, then it is not computably categorical.*

Proof. First, consider the cases where either $\mathcal{A}$ consists of an infinite number of orbits of type ω and no orbits of type Z, or $\mathcal{A}$ consists of an infinite number of orbits of type Z and no orbits of type ω.

Suppose first that $\mathcal{A} = (\mathbb{N} - \{0\}, f)$ consists of infinitely many orbits of type ω, where $f((2i+1)2^n) = (2i+1)2^{n+1}$ for each $i \geq 0$ and $n \geq 0$. Thus, $Ran(f)$ is a computable set. Now we build a computable structure $\mathcal{B} = (\mathbb{N} - \{0\}, g)$ isomorphic to $\mathcal{A}$ such that $Ran(g)$ is not computable, so that $\mathcal{A}$ cannot be computably isomorphic to $\mathcal{B}$.

Let C be a noncomputable c.e. set which does not contain 0 or 1. Let $C = \bigcup_s C_s$, where $\{C_s : s \in \mathbb{N}\}$ is a computable sequence of finite sets such that $C_s \subseteq \{2, \dots, s-1\}$ and $card(C_{s+1} - C_s) \leq 1$ for all s. The injection g is defined in stages. At stage s, we define a finite partial function g_s is such that
(i) if $i \in \{1, \dots, s\} - C_s$, then $g_s((2i+1)2^n) = (2i+1)2^{n+1}$ for $0 \leq i \leq s$,
(ii) if $i \in C_s$, then $(2i+1)$ will be in the orbit of 1 under g_s, and
(iii) $\{(2i+1)2^n : i, n \leq s\} \subseteq Dom(g_s)$.

Initially, we have $g_0(1) = 2$. Now assume that at stage s, that we have defined g_s so that it satisfies (i)-(iii). Then at stage $s+1$, we first extend g_s as follows. If $i \in \{1, \dots, s+1\} - C_{s+1}$, $g_{s+1}((2i+1)2^n) = (2i+1)2^{n+1}$ for $0 \leq i \leq s+1$. Next, add elements to the end of the current orbit of 1 so that we ensure that $\{(2i+1)2^n : i, n \leq s+1\} \subseteq Dom(g_{s+1})$ if necessary. Finally if $i \in C_{s+1} - C_s$, then let m be the unique element of the current orbit of 1 for which $g_{s+1}(m)$ is not defined and let $g_{s+1}(m) = 2i+1$. This has the effect of adding the current orbit of $2i+1$ at the end of the current orbit of 1. It is clear that if $g = \bigcup_{s \geq 0} g_s$, then $(\mathbb{N} - \{0\}, g)$ is a computable injection structure isomorphic to $\mathcal{A}$ and that $(\mathbb{N} - \{0, 1\}) - Ran(g) = \{2i+1 : i \notin C\}$. Thus $\mathcal{B} = (\mathbb{N} - \{0\}, g)$ is not computably isomorphic to $\mathcal{A}$.

Next, suppose that $\mathcal{A} = (\mathbb{N} - \{0\}, f)$ consists of infinitely many orbits of type Z, where every orbit of type Z is computable. We shall build a structure $\mathcal{B} = (\mathbb{N} - \{0\}, g)$ in which the orbit of 1 is not computable. The construction is similar to that given above with several modifications. First, to make the orbits have type Z, we extend the orbits at stage $s+1$ to the right when s is even and to the left when s is odd. Second, when i appears in $C_{s+1} - C_s$, we append the orbit of $2i+1$ to the orbit of 1. In this way, we have $i \in C$ if and only if $2i+1 \in \mathcal{O}_g(1)$, so that $\mathcal{O}_g(1)$ is not computable.

Now, let $\mathcal{A} = (\mathbb{N} - \{0\}, f)$ be a computable injection structure with infinitely many infinite orbits. Suppose first that $\mathcal{A}$ has infinitely many orbits of type ω and that $Ran(f)$ is computable. Let $\mathcal{A}_0$ be the restriction of $\mathcal{A}$ to the orbits of type ω, and let $\mathcal{B}_0$ be a computable structure isomorphic to $\mathcal{A}_0$, but not computably isomorphic to $\mathcal{A}_0$, as above. By Proposition 2.2, there is a computable injection structure $\mathcal{C}$ with $\chi(\mathcal{C}) = \chi(\mathcal{A})$ and such that $\mathcal{C}$ has the same number of orbits of type Z as $\mathcal{A}$. Let $\mathcal{B} = \mathcal{B}_0 \oplus \mathcal{C}$. Then $\mathcal{B}$ is a computable injection structure that is isomorphic to $\mathcal{A}$, but is not computably isomorphic to $\mathcal{A}$, since any isomorphism would have to map $\mathcal{A}_0$ to $\mathcal{B}_0$. Note that in $\mathcal{B}_0 \oplus \mathcal{C}$ the element 1 in $\mathcal{B}_0$ gets mapped to 2. Thus the orbit of 2 in $\mathcal{B}$ is of type ω and is Turing equivalent to C. The argument when $\mathcal{A}$ has infinitely many orbits of type Z is similar. □

The following corollary is immediate.

Corollary 2.5. *Let $\mathcal{A}$ be a computable injection structure.*

1. *The structure $\mathcal{A}$ is computably categorical if and only if $\mathcal{A}$ is relatively computably categorical.*

2. *The structure $\mathcal{A}$ is computably categorical if and only if $\mathcal{A}$ has finitely many infinite orbits.*

We also have the following corollary to the proof of Theorem 2.4.

Corollary 2.6. *Let* **d** *be a c.e. degree.*

1. *If $\mathcal{A} = (\mathbb{N}, f)$ is a computable injection structure that has infinitely many orbits of type ω, then there is a computable injection structure $\mathcal{B} = (\mathbb{N}, g)$ isomorphic to $\mathcal{A}$ in which $Ran(g)$ is a c.e. set of degree* $\mathbf{d}$ *and there is an $x \in \mathbb{N}$ such that $\mathcal{O}_g(x)$ is of type ω and is a c.e. set of degree* $\mathbf{d}$ *and, for all $y \in \mathbb{N} - \mathcal{O}_g(x)$, if $\mathcal{O}_g(y)$ is of type ω, then $\mathcal{O}_g(y)$ is computable.*

2. *If $\mathcal{A} = (\mathbb{N}, f)$ is a computable injection structure that has infinitely many infinite orbits of type Z, then there is a computable injection structure $\mathcal{B} = (\mathbb{N}, g)$ isomorphic to $\mathcal{A}$ such that and there is an $x \in \mathbb{N}$ such that $\mathcal{O}_g(x)$ is of type Z and is a c.e. set of degree* $\mathbf{d}$ *and, for all $y \in \mathbb{N} - \mathcal{O}_g(x)$, if $\mathcal{O}_g(y)$ is of type Z, then $\mathcal{O}_g(y)$ is computable.*

Proof. Parts (1) and (2) are immediate from the proof of Theorem 2.4. □

3 Δ_2^0 Categorical Structures

Theorem 3.1. *Suppose that a computable injection structure $\mathcal{A}$ does not have infinitely many orbits of type ω or does not have infinitely many orbits of type Z. Then $\mathcal{A}$ is relatively Δ_2^0 categorical.*

Proof. Recall from Lemma 2.1 that $\{a : \mathcal{O}(a) \text{ is infinite}\}$ is a Π_1^0 set. Under the assumption of our theorem, $\{a : \mathcal{O}(a) \text{ has type } \omega\}$ and $\{a : \mathcal{O}(a) \text{ has type } Z\}$ will be Δ_2^0 sets. Thus, given isomorphic computable structures $\mathcal{A} = (A, f)$ and $\mathcal{B} = (B, g)$, we can use an oracle for $\mathbf{0}'$ to partition A and B into three sets each: the orbits of finite type, the orbits of type ω, and the orbits of type Z.

First suppose that $\mathcal{A}$ consists of infinitely many orbits of type ω and only finitely many orbits of type Z. We shall construct an isomorphism $h : A \to B$, which is computable in $\mathbf{0}'$. First let $c_1 < \cdots < c_t$ be representatives of the orbits of type Z in $\mathcal{A}$, and $d_1 < \cdots < d_t$ be the representatives of the orbits of type Z in $\mathcal{B}$. Then define $h(c_i) = d_i$ for $i = 1, \ldots, t$ and extend h in the obvious way to map the orbits of $c_1, \ldots, c_t$ to the orbits of $d_1, \ldots, d_t$, respectively. This map will be computable in $\mathbf{0}'$ since the orbit of every c_i and d_i is computable in $\mathbf{0}'$.

Next, a simple back-and-forth argument will allow us to show that we can construct an isomorphism from $(Fin(\mathcal{A}), f)$ onto $(Fin(\mathcal{B}), g)$, which is computable in $\mathbf{0}'$. That is, at any given stage s, assume that we have defined an isomorphism h_s on a finite set of orbits of $(Fin(\mathcal{A}), f)$ onto a finite set of orbits of $(Fin(\mathcal{B}), g)$. Then let $a \in Fin(\mathcal{A})$ be the least element not in $Dom(h_s)$. We can compute its orbit $\{a, f(a), f^2(a), \ldots, f^{n-1}(a)\}$ in $(Fin(\mathcal{A}), f)$. Then search through the elements of $(Fin(\mathcal{B}), g)$ until we find a b not in $Ran(h_s)$ such that b has an orbit of size n, and define $h_{s+1}(f^i(a)) = g^i(b)$ for $i = 0, 1 \ldots, n-1$. Next, let d be the least element of $Fin(\mathcal{B})$, which is not in $Ran(h_s)$ and not in the orbit of b. We can compute the orbit of d, $\{d, g(d), g^2(d), \ldots, g^{m-1}(d)\}$, in $(Fin(\mathcal{B}), g)$. Then we search for a $c \in Fin(\mathcal{A})$ such that c is not in $Dom(h_s)$ nor in the orbit of a, and has an orbit of size m. Then we set $h_{s+1}(f^i(c)) = g^i(d)$ for $i = 0, 1 \ldots, m-1$. Since $Fin(\mathcal{A})$ and $Fin(\mathcal{B})$ are c.e. sets, it follows that h restricted to $Fin(\mathcal{A})$ is computable in $\mathbf{0}'$.

Let A_ω and B_ω denote the sets of elements that are in orbits of type ω in $\mathcal{A}$ and $\mathcal{B}$, respectively. It is easy to see that $A - A_\omega$ and $B - B_\omega$ are c.e. sets. Since $Ran(f)$ and $Ran(g)$ are c.e. sets, we may use an oracle for $\mathbf{0}'$ to compute a list $a_0, a_1, \dots$ of $A_\omega - Ran(f)$, and similarly a list $b_0, b_1, \dots$ of $B_\omega - Ran(g)$. Then we extend the isomorphism h by mapping $\mathcal{A}_\omega$ to B_ω as follows. Given $a \in \mathcal{A}_\omega$, compute the unique i and n such that $a = f^n(a_i)$, and let $h(a) = g^n(b_i)$.

Next, suppose that $\mathcal{A}$ consists of infinitely many orbits of type Z and only finitely many orbits of type ω. Then, again, we shall construct an isomorphism $h : A \to B$ that is computable in $\mathbf{0}'$. Let $c_1 < \cdots < c_t$ be the first elements of the orbits of type ω in $\mathcal{A}$, and let $d_1 < \cdots < d_t$ be the first elements of the orbits of type ω in $\mathcal{B}$. Then define $h(c_i) = d_i$ for $i = 1, \dots, t$, and extend h in the obvious way to map the orbits of $c_1, \dots, c_t$ to the orbits of $d_1, \dots, d_t$, respectively. This function will be computable in $\mathbf{0}'$ since $A - Ran(f)$ and $B - Ran(g)$ are computable in $\mathbf{0}'$. Then we can use the back-and-forth argument to define an isomorphism h from $Fin(\mathcal{A})$ onto $Fin(\mathcal{B})$, computable in $\mathbf{0}'$.

Let A_Z and B_Z be the sets of elements that lie in orbits of type Z in $\mathcal{A}$ and $\mathcal{B}$, respectively. In this case, it is easy to see that $A - A_Z$ and $B - B_Z$ are c.e. sets. Since the orbit of any element in $\mathcal{A}$ or $\mathcal{B}$ is computable in $\mathbf{0}'$, it follows that we can use an oracle for $\mathbf{0}'$ to compute a list $a_0, a_1, \dots$ of representatives for the orbits. That is, observe that $\{(x, y) : \mathcal{O}(x) = \mathcal{O}(y)\}$ is a c.e. set. Let a_0 be the least element in A_Z, and for each i, let a_{i+1} be the least $a \in A_Z$ that is not in the same orbit as any of $a_0, \dots, a_i$. Similarly, we can compute a list $b_0, b_1, \dots$ of representatives for the orbits of B_Z. Then an isomorphism h can be defined on A_Z as follows. Given $a \in A_Z$, compute the unique i and n such that either $a = f^n(a_i)$ or $a_i = f^n(a)$. In the first instance, let $h(a) = g^n(b_i)$, and in the second instance, let $h(a) = b$ for the unique b such that $g^n(b) = b_i$.

In each case, if $\mathcal{B}$ is not computable, we can nevertheless use an oracle for $(deg(\mathcal{B}))'$ to compute the isomorphisms described above. Hence $\mathcal{A}$ is, in fact, relatively Δ_2^0 categorical. □

Theorem 3.2. *Suppose that a computable injection structure $\mathcal{A}$ has infinitely many orbits of type ω and infinitely many orbits of type Z. Then $\mathcal{A}$ is not Δ_2^0 categorical.*

Proof. Clearly, it suffices to consider only the case when $\mathcal{A}$ has no finite orbits. Let $\mathcal{A} = (A, f)$ be an injection structure in which the union of the orbits of type ω is a computable subset of A. We will construct a computable injection structure $\mathcal{B} = (\mathbb{N} - \{0\}, g)$, which is isomorphic to $\mathcal{A}$, but in which the union of orbits of type ω is a Σ_2^0 set that is not Δ_2^0, so that $\mathcal{A}$ and $\mathcal{B}$ cannot be Δ_2^0 isomorphic.

It is well-known that $Fin = \{e : W_e \text{ is finite}\}$ is a Σ_2^0 complete set (see [11]). Thus, for any Σ_2^0 set C, there is a computable function $F : \mathbb{N} \times \mathbb{N} \to \{0, 1\}$ such that for all i,

$$i \in C \iff (\{s : F(i, s) = 1\} \text{ is finite}).$$

Fix an arbitrary Σ_2^0 set C and a corresponding F. We will define g such that

$$i \in C \Longleftrightarrow (\mathcal{O}_g(2i+1) \text{ has type } \omega).$$

The orbits of $\mathcal{B} = (B, g)$ will be exactly $\{\mathcal{O}_g(2i+1) : i \in \mathbb{N}\}$.

Initially, we have $g_0(1) = 2$. After stage s, the function g_s will have orbits $\mathcal{O}_{g_s}(2i+1) = \{(2i+1)2^n : n \leq 2s+2\}$ for $i \leq s$. At stage $s+1$, we first extend each orbit of g_s as follows. Fix $i \leq s$. Let a be the unique element of $\mathcal{O}(2i+1)$ not in $Ran(g_s)$, and let b be the unique element of $\mathcal{O}(2i+1)$ not in $Dom(g_s)$. In either case, we let $g_{s+1}(b) = (2i+1)2^{2s+3}$. If $F(i, s+1) = 0$, then $g_{s+1}((2i+1)2^{2s+3}) = (2i+1)2^{2s+4}$, and if $F(i, s+1) = 1$, then let $g_{s+1}((2i+1)2^{2s+4}) = a$. Next, we add a new orbit by letting $g((2s+3)2^n) = (2s+3)2^{n+1}$ for all $n \leq 2s+3$. Let $g = \bigcup_s g_s$. If $i \in C$, then $\{s : F(i, s) = 1\}$ is finite, say of cardinality m. Then, by the construction, there is no element b such that $g^{m+1}(b) = 2i+1$, and hence $\mathcal{O}(2i+1)$ has type ω. If $i \notin C$, then $\{s : F(i, s) = 1\}$ is infinite. Thus, for any m, there is an element b such that $g^{m+1}(b) = 2i+1$, and hence $\mathcal{O}(2i+1)$ has type Z. It is clear that $(\mathbb{N} - \{0\}, g)$ is a computable injection structure all orbits of which are infinite, and $\mathcal{O}_g(2i+1)$ is of type ω if and only if $i \in C$. If C is not a Δ_2^0 set, then $\mathcal{B}$ cannot be Δ_2^0 isomorphic to $\mathcal{A}$. □

The following corollary is immediate.

Corollary 3.3. *Let $\mathcal{A}$ be a computable injection structure.*

1. *The structure $\mathcal{A}$ is Δ_2^0 categorical if and only if $\mathcal{A}$ is relatively Δ_2^0 categorical.*
2. *The structure $\mathcal{A}$ is Δ_2^0 categorical if and only if $\mathcal{A}$ has finitely many orbits of type ω or finitely many orbits of type Z.*

We also have the following corollary to the proof of Theorem 3.2.

Corollary 3.4. *For any Σ_2^0 set C, there exists a computable injection structure $\mathcal{A} = (A, f)$ in which the set of elements with orbits of type ω is a Σ_2^0 set with Turing degree equal to $deg(C)$.*

The following theorem shows that there is a Δ_2^0 categorical injection structure the degree of categoricity of which is $\mathbf{0}'$.

Theorem 3.5. *Let $\mathcal{M}$ be a computable Δ_2^0 categorical injection structure, which is not computably categorical. Then there is a computable injection structure $\mathcal{A}$ isomorphic to $\mathcal{M}$ such that the degree of categoricity of $\mathcal{A}$ is $\mathbf{0}'$.*

Proof. Let $K \subseteq (\mathbb{N} - \{0, 1\})$ be a c.e. set such that $deg(K) = \mathbf{0}'$. First, consider the case when $\mathcal{M}$ has infinitely many orbits of type ω. Let $\mathcal{M}_0$ be the restriction of $\mathcal{M}$ to the orbits of type ω. Let $\mathcal{A}_0 = (\mathbb{N} - \{0\}, f)$ be isomorphic to $\mathcal{M}_0$ and such that $(\mathbb{N} - \{0\}) - Ran(f) = \{2i+1 : i \in \mathbb{N}\}$. Then, by the proof of Theorem 2.4, we obtain a computable structure $\mathcal{B} = (\mathbb{N} - \{0\}, g)$ isomorphic to $\mathcal{A}_0$, and an isomorphism $G : \mathcal{A}_0 \to \mathcal{B}$ such that

$$(\forall i)[i \in K \Leftrightarrow 2i+3 \in Ran(g) \Leftrightarrow 2i+3 \in G(Ran(f))]$$

Since $Ran(f)$ is a computable set, we have $K \leq_T G$, and thus $\mathbf{0}' \leq \deg(G)$. Hence the degree of categoricity of $\mathcal{A}_0$ is $\mathbf{0}'$. Let $\mathcal{C}$ be a computable injection structure with $\chi(\mathcal{C}) = \chi(\mathcal{M})$ and with the same number of orbits of type Z as $\mathcal{M}$. Let $\mathcal{A} = \mathcal{A}_0 \oplus \mathcal{C}$. Then $\mathcal{A}$ is a computable injection structure isomorphic to $\mathcal{M}$ such that the degree of categoricity of $\mathcal{A}$ is $\mathbf{0}'$.

Now, assume that $\mathcal{M}$ has infinitely many orbits of type Z. Let $\mathcal{M}_0$ be the restriction of $\mathcal{M}$ to the orbits of type Z. Assume that $\mathcal{A}_0$ is an injection structure isomorphic to $\mathcal{M}_0$ such that every orbit of $\mathcal{A}_0$ of type Z is computable and $\{2i+1 :\in \mathbb{N}\}$ are a set of representatives for the orbits of $\mathcal{A}_0$. Then, by the proof of Theorem 2.4, we obtain a computable structure $\mathcal{B} = (\mathbb{N} - \{0\}, g)$ isomorphic to $\mathcal{A}_0$, and an isomorphism $G : \mathcal{A}_0 \to \mathcal{B}$ such that

$$(\forall i)[i \in K \Leftrightarrow 2i+1 \in G(Orb_{\mathcal{A}}(a))],$$

where $G(a) = 1$. Since $Orb_{\mathcal{A}_0}(a)$ is a computable set, we have $K \leq_T G$, and thus $\mathbf{0}' \leq \deg(G)$. Hence the degree of categoricity of $\mathcal{A}_0$ is $\mathbf{0}'$. We now continue as in the previous case. □

Theorem 3.6. *Every computable injection structure $\mathcal{A}$ is relatively Δ_3^0 categorical.*

Proof. Let $\mathcal{A}$ be a computable injection structure and let $\mathcal{B}$ be isomorphic to $\mathcal{A}$. It follows from Lemma 2.1 that, using an oracle for $\mathbf{0}''$, we can partition A into three Δ_3^0 sets, that is, the orbits of finite type, the orbits of type ω, and the orbits of type Z. Using an oracle for $(deg(\mathcal{B}))''$, we can similarly partition B. It then follows from the proofs of Theorems 2.3 and 3.1 that we can compute from $(deg(\mathcal{B}))''$ isomorphisms between the three substructures of $\mathcal{A}$ and $\mathcal{B}$. □

The following theorem shows that there is a computable injection structure the degree of categoricity of which is $\mathbf{0}''$.

Theorem 3.7. *Let $\mathcal{M}$ be a computable injection structure, which is not Δ_2^0 categorical. Then there is a computable injection structure $\mathcal{A}$ isomorphic to $\mathcal{M}$ such that the degree of categoricity of $\mathcal{A}$ is $\mathbf{0}''$.*

Proof. Let $C \subseteq \mathbb{N}$ be a Σ_2^0 set such that $deg(C) = \mathbf{0}''$. Without loss of generality, based on the proof of Theorem 3.5, assume that $\mathcal{M}$ has no finite orbits. Then $\mathcal{M}$ has infinitely many orbits of type ω, and infinitely many orbits of type Z. Let $\mathcal{A}$ be an injection structure isomorphic to $\mathcal{M}$ such that A_ω (the set of all elements of $\mathcal{A}$ the orbits of which have type ω) is a computable set. Then, by the proof of Theorem 3.2, we obtain a computable structure $\mathcal{B}$ isomorphic to $\mathcal{A}$ and an isomorphism $G : \mathcal{A} \to \mathcal{B}$ such that

$$(\forall i)[i \in C \Leftrightarrow 2i+1 \in G(A_\omega)].$$

Since the set A_ω is computable, we have $C \leq_T G$, and thus, $\mathbf{0}'' \leq \deg(G)$. Hence the degree of categoricity of $\mathcal{A}$ is $\mathbf{0}''$. □

One of the most common finer classifications of the class of Δ_2^0 objects is the difference hierarchy of Ershov. Recall that a set $A \subseteq \mathbb{N}$ is said to be *n-c.e.* for $n \in \mathbb{N}$, if there is a computable function $f : \mathbb{N} \times \mathbb{N} \to \{0,1\}$ such that for every $x \in \mathbb{N}$:

1. $f(x, 0) = 0$;
2. $lim_s f(x, s) = A(x)$;
3. $card(\{s : f(x, s+1) \neq f(x, s)\}) \leq n$.

2-c.e. sets are also called d.c.e., for difference of c.e. sets. The set A is *ω-c.e.* if instead of property 3 above there is a computable function h such that for all x, $|\{x : A_{s+1}(x) \neq A_s(x)\}| \leq h(x)$.

Definition 3.8. *(a) Let* $\mathrm{g(x) = lim_s f(x, s)}$*, where* f *is a computable function.*
1. The function g *is an* n-c.e. function *if for all* $\mathrm{x} \in \mathbb{N}$*,*

$$\mathrm{card(\{s : f(x, s) \neq f(x, s+1)\}) < n.}$$

2. The function g *is an* ω-c.e. function *if there is a computable function* h *such that for all* $\mathrm{x} \in \mathbb{N}$*,*

$$\mathrm{card(\{s : f(x, s) \neq f(x, s+1)\}) \leq h(x).}$$

(b) For $\alpha \leq \omega$*, a function* p *is* graph-α-c.e. *if the graph of* p *is an* α*-c.e. set.*

In [4], Cenzer, LaForte, and Remmel investigated α-c.e. and graph-α-c.e. functions.

Theorem 3.9. *There exist two computable injection structures that are not ω-c.e. isomorphic.*

Proof. We will diagonalize against all possible ω-c.e. isomorphisms h_e from a standard computable injection structure $\mathcal{A}$ with domain $\mathbb{N}$, consisting of infinitely many orbits of type Z, to another isomorphic computable structure $\mathcal{B} = (\mathbb{N}, g)$ that we shall construct. That is, we diagonalize over all pair of partial recursive functions $h_e = (f_e, g_e)$. Then we have to ensure that if f_e and g_e are total functions and for all $x \in \mathbb{N}$, $\mathrm{card}(\{s : f_e(x,s) \neq f_e(x, s+1)\}) \leq g_e(x)$, then $f^{(e)}$ is not an isomorphism from $\mathcal{A}$ onto $\mathcal{B}$ where $f^{(e)}(x) = \lim_s f_e(x, s)$ for all x. We then have two sets of requirements.

N_e: $\mathcal{B}$ has at least e different Z orbits and

P_e: Either f_e is not total, g_e is not total, there is an x such that $\mathrm{card}(\{s : f_e(x,s) \neq f_e(x, s+1)\}) > g_e(x)$, or there exists a pair a_e and b_e such that a_e and b_e are in different Z orbits in $\mathcal{A}$, but $f^{(e)}(a_e)$ and $f^{(e)}(b_e)$ are in the same Z orbit in $\mathcal{B}$.

We construct g in stages. Initially, we let $g(0) = 1$ and our intention is to use $\{(2i+1)2^n : n \in \mathbb{N}\}$ to build a Z orbit by adding elements to the end of chains at even stages and elements at the beginning of chains at odd stages. To meet the negative requirements N_e, at any stage $s \geq e$, we shall have e Γ_e markers on elements of the form $(2i+1)$ which are currently in different orbits relative to g_s. We then allow two orbits with Γ_e markers on an element in an orbit to be merged only for the sake of one meeting requirements $P_0, \ldots, P_{e-1}$.

To meet requirement P_e, at any stage $s \geq e$, we note that there can be at most Γ_j markers for $j \leq e$ on at most $1 + 2 + \cdots + e = \binom{e+1}{2}$ different orbits. Then we specify $2 + \binom{e+1}{2}$ elements from pairwise distinct Z orbits in $\mathcal{A}$ and place Δ_e markers on these elements. Then if a_e and b_e are two elements with Δ_e markers on them such that $f_e(a_e, s)$ and $f_e(b_e, s)$ are defined and they are currently in two different g_s orbits which do not have any Γ_j markers on them for some $j \leq e$, then we will define g_{s+1} so that we merge the orbits of $f_e(a_e, s)$ and $f_e(b_e, s)$.

A standard finite injury priority argument will show that we can meet all of the requirements N_e and P_e to construct the desired computable injection structure $\mathcal{B}$. □

4 Spectrum Problems

Corollaries 2.6 and 3.4 are results about the *spectra* of natural relations on computable injection structures. For a computable injection structure $\mathcal{A} = (A, f)$ and any cardinal $k \leq \omega$, we may consider the possible Turing degrees of $\{a : card(\mathcal{O}_f(a)) = k\}$ as well as of $\{a : \mathcal{O}_f(a) \text{ has type } \omega\}$ and $\{a : \mathcal{O}_f(a) \text{ has type } Z\}$. For example, we know that for any computable injection structure $\mathcal{A}$, $Fin(\mathcal{A})$ is a c.e. set. Thus, a natural question is to ask whether for any computable injection structure $\mathcal{A}$ and any c.e. Turing degree $\mathbf{c}$, there exists a computable injection structure $\mathcal{B}$, which is isomorphic to $\mathcal{A}$, such that $Fin(\mathcal{B})$ has degree $\mathbf{c}$. Clearly, this is not possible for any computable injection structure. For example, if $\mathcal{A}$ has only finitely many infinite orbits, then $A - Fin(\mathcal{A})$ is c.e., so $Fin(\mathcal{A})$ must be computable. Similarly, if $\mathbf{c} \neq \mathbf{0}$, then $Fin(\mathcal{A})$ must be infinite if we are to have a computable injection structure $\mathcal{B}$, which is isomorphic to $\mathcal{A}$, such that $Fin(\mathcal{B})$ of degree $\mathbf{c}$. However, we can prove the following theorem.

Theorem 4.1. *Let $\mathbf{c}$ be a c.e. Turing degree. Let $\mathcal{A}$ be a computable injection structure such that $\mathcal{A}$ has infinitely many orbits of size k for every $k \in \mathbb{N}$, and also has infinitely many infinite orbits. Then there is a computable structure $\mathcal{B}$ isomorphic to $\mathcal{A}$ such that $Fin(\mathcal{B})$ is of degree $\mathbf{c}$.*

Proof. Without loss of generality, assume that $\mathcal{A} = (\mathbb{N}, f)$ is such that we can find a computable set of representatives $a_0, a_1, \ldots$ of the finite orbits of $\mathcal{A}$, a computable set of representatives $b_0, b_1, \ldots$ of the orbits of type Z of $\mathcal{A}$, and a computable set $c_0, c_1, \ldots$ of the first elements of all orbits of type ω in $\mathcal{A}$.

First, consider the case when $\mathcal{A}$ has infinitely many orbits of type ω. Thus, in this case, the list $b_0, b_1, \ldots$ may be finite. Let C be a noncomputable c.e. set, and let $C = \bigcup_s C_s$, where $\{C_s : s \in \mathbb{N}\}$ is a computable sequence of finite sets such that for every s, $C_s \subseteq \{0, 1, \ldots, s-1\}$ and $card(C_{s+1} - C_s) \leq 1$. The construction of $\mathcal{B} = (\mathbb{N}, g)$ is quite simple. First, define g so that the orbits of each a_i and b_i in $\mathcal{B}$ is the same as the orbits of a_i and b_i in $\mathcal{A}$, respectively, by having f and g agree on such orbits. Next, we define g so that orbit of each c_{2i+1} in $\mathcal{B}$ is the same as the orbit of c_{2i+1} in $\mathcal{A}$ by having f and g agree on the orbit of c_{2i+1}.

For each $\in \mathbb{N}$, we define g on the orbit of c_{2i} in stages. At stage s, if $i \notin C_s$, then we define $g^k(c_{2i}) = f^k(c_{2i})$ for $k \leq s$. If $i \in C_{t+1} - C_t$, then we define $g(f^{t+1}(c_{2i})) = c_{2i}$ and set $g(f^r(c_{2i})) = f^{r+1}(c_{2i})$ for $r > t+1$. This will make the orbit of c_{2i} split into two orbits, one which is finite and one which is of type ω in $\mathcal{B}$. If $i \notin C$, then f and g will agree on the orbit of c_{2i}. It then easily follows that $c_{2i} \in Fin(\mathcal{B})$ if and only if $i \in C$. Thus, $C \leq_T Fin(\mathcal{B})$. However, if we know C, then we can easily compute $Fin(\mathcal{B})$. Hence $Fin(\mathcal{B}) \equiv_T C$. Finally, it is easy to see that our assumptions ensure that $\mathcal{A}$ is isomorphic to $\mathcal{B}$.

The construction when $\mathcal{A}$ has only finitely many orbits of type ω, but infinitely many orbits of type Z is similar. In that case, we split each orbit of b_{2i} into two orbits, one finite containing b_{2i} and one infinite of type Z if $i \in C$, and otherwise we will have g and f agree on the orbit of b_{2i}. We also have that f and g agree on the orbits of each a_i, each c_i, and each b_{2i+1}. $\square$

Note that in the proof the Theorem 4.1, we used the assumption that $\mathcal{A}$ had infinitely many orbits of size k for each k to ensure that $\mathcal{B}$ we constructed is isomorphic to $\mathcal{A}$, since in such a situation adding an extra finite orbit does not change the isomorphism type. However, in general, we cannot do this for an arbitrary injection structure $\mathcal{A}$. We can, however, easily modify the construction of Theorem 4.1 to prove a similar result in the case when $\mathcal{A}$ has infinitely many orbits of different sizes. Similarly, it easy to see that we can modify the construction to produce that a computable injection structure $\mathcal{A} = (\mathbb{N}, f)$ such that $Fin(\mathcal{A})$ has degree c and the sizes of the finite orbits of elements of $\mathcal{A}$ are pairwise distinct. A more general question is to characterize the possible characters of computable injection structures $\mathcal{A}$ which have $Fin(\mathcal{A})$ of degree c for any given c.e. degree c.

A more refined question is what sets can be realized as $Fin(\mathcal{A})$ for a computable injection structure. An easy observation here is that $Fin(\mathcal{A})$ can never be a simple c.e. set. That is, if $Fin(\mathcal{A})$ is not computable, then $Fin(\mathcal{A})$ must be infinite and also there must be some infinite orbits. But then each infinite orbit of $\mathcal{A}$ is a c.e. set in the complement of $Fin(\mathcal{A})$. Similarly, no infinite orbit of $\mathcal{A}$ can be a simple c.e. set. We next show that there is a non-simple c.e. set that cannot be $Fin(\mathcal{A})$ for any computable injection structure $\mathcal{A}$.

Theorem 4.2. *Let C be a simple c.e. set. Then there is no computable injection structure $\mathcal{A} = (\mathbb{N}, f)$ such that $2C = Fin(\mathcal{A})$ or $2C$ is an orbit of $\mathcal{A}$ where $2C = \{2c : c \in C\}$.*

Proof. Suppose, for a contradiction, that $\mathcal{A} = (\mathbb{N}, g)$ is a computable injection structure such that $2C = Fin(\mathcal{A})$ or that $2C$ is an orbit of $\mathcal{A}$. Then for $c \in C$, the orbit $\mathcal{O}_f(2c) \subseteq 2C$ and hence every element of $\mathcal{O}_f(2c)$ is even. Let $D = \{n : (\exists m)(2m+1 \in \mathcal{O}_f(2n))\}$. D a c.e. set disjoint from C, and is therefore finite. Hence there are only finitely many even numbers $2a_1, \ldots, 2a_n$ which have odd numbers in their orbits. It follows that if $2x \notin \{2a_1, \ldots, 2a_n\}$, then $\mathcal{O}_f(2x)$ consists entirely of even numbers.

Now suppose that $2C = Fin(\mathcal{A})$. If there is an $x \notin C \cup \{a_1, \ldots, a_n\}$, such that $\mathcal{O}_f(2x)$ is infinite, then $\{m : 2m \in \mathcal{O}_f(2x)\}$ would be an infinite c.e. outside of C which would violate our assumption that C is simple. But then it follows that if $x \notin C \cup \{a_1, \ldots, a_n\}$, $\mathcal{O}_f(2x)$ is finite. But this would imply that $2C = \{2n : n \in N\} - \{2a_1, \ldots, 2a_n\}$ which would imply $C = \mathbb{N} - \{a_1, \ldots, a_n\}$ which again violates our assumption that C is simple.

Finally suppose that $2C = \mathcal{O}_f(2m)$. Again if there is an $x \notin C \cup \{a_1, \ldots, a_n\}$ such that $\mathcal{O}_f(2x)$ is infinite, then $\{m : 2m \in \mathcal{O}_f(2x)\}$ would be an infinite c.e. outside of C which would violate our assumption that C is simple. Thus we must assume that for all $x \notin C \cup \{a_1, \ldots, a_n\}$, $\mathcal{O}_f(2x)$ is finite. But then $\{n : 2n \in Fin(\mathcal{A})\}$ is an infinite c.e. set disjoint from C which again violates our assumption that C is simple. $\square$

Now, for a computable injection structure $\mathcal{A} = (\mathbb{N}, g)$, $Ran(g)$ is always a c.e. set so a natural question to ask is whether $Ran(g)$ can be *any* c.e. set. There are some obvious restrictions, namely, if $\mathcal{A}$ has only finitely many orbits of type ω, then $Ran(g)$ will be computable.

Theorem 4.3. *Let C be any infinite, co-infinite c.e. set and let K be any c.e. character.*

(i) There is a computable injection structure $(\mathbb{N}, g)$ with $Ran(g) = C$.

(ii) If C is not simple, then there is a computable injection structure $(\mathbb{N}, h)$ with character K, with $Ran(h) \equiv_T C$, and with an arbitrary number of orbits of type Z.

Proof. Let C be an infinite, co-infinite c.e. set and let $c_0, c_1, \ldots$ be a computable enumeration $c_0, c_1, \ldots$ without repetitions.

(i) The function g is defined in stages g_s as follows. Initially, $g_0 = \emptyset$. At stage 1, there are three cases. If $c_0 \neq 0$, then we let $g_1 = \{(0, c_0)\}$. If $c_0 = 0$ and $c_1 \neq 1$, then we let $g_1(0) = c_1$ and $g_1(1) = 0$. If $c_0 = 0$ and $c_1 = 1$, then we let $g_1(0) = 1$ and $g(1) = 0$.

Suppose that after s steps, we have defined a finite function g_s such that $\{0, 1, \ldots, s-1\} \subseteq Dom(g_s)$ and $\{c_0, c_1, \ldots, c_{s-1}\} \subseteq Ran(g_s) \subset C$. At stage $s+1$, let i be the least number not in $Dom(g_s)$, and let m be the least number such that $c_m \notin Ran(g_s)$. If $i \neq c_m$, let $g_{s+1}(i) = c_m$. If $i = c_m$, let j be the least number such that $j > i$ and $j \notin Dom(g_s)$, and let n be the least number such that $n > m$ and such that $c_n \notin Ran(g_s)$. Now define $g_{s+1}(j) = c_m$ and $g_{s+1}(i) = c_n$. In this way we ensure that $s \in Dom(g_{s+1})$ and $c_s \in Ran(g_{s+1})$.

(ii) Let A be an infinite computable set, disjoint from C. Then both A and $\mathbb{N} - A$ are computably isomorphic to $\mathbb{N}$, so we can build a computable structure

$(\mathbb{N}-A, g)$ with range computably isomorphic to C. Then, by Proposition 2.2, we can build a computable structure (A, f) with character K and with the desired number of orbits of type Z. Now, $(\mathbb{N}, h)$ is defined so that $h(x) = f(x)$ if $x \in A$, and $h(x) = g(x)$ if $x \notin A$. □

Next, we consider the infinite orbits of an injection structure. Since each infinite orbit is a c.e. set, and also the union of all finite orbits forms a c.e. set, any noncomputable orbit will have a c.e. set in its complement. We can show that every c.e. degree contains a c.e. set which is an orbit of some computable injection structure.

Theorem 4.4. *For every c.e. set C and every computable injection structure $\mathcal{A}$ with infinitely many infinite orbits, there is a computable injection structure $\mathcal{B}$ isomorphic to $\mathcal{A}$ such that C is* 1-1 *reducible to an orbit of $\mathcal{A}$.*

Proof. First, suppose that $\mathcal{A}$ has infinitely many orbits of type ω. Clearly, it suffices to build $\mathcal{B} = (\mathbb{N}, g)$ consisting exactly of infinitely many orbits of type ω, and having an orbit, say the orbit of 0, such that C is 1-1 reducible to it. The construction of $\mathcal{B}$ is as follows. For each n, we start to build an orbit of type ω with the first element $2n$, using the odd numbers to extend the orbits. When a number n appears in C, we attach the chain beginning with $2n$ to the end of the chain beginning with 0. Thus, we see that for any n, $n \in C$ if and only if $2n \in \mathcal{O}_g(0)$.

If $\mathcal{A}$ has infinitely many orbits of type Z, then we can similarly define $\mathcal{B}$ consisting of infinitely many orbits of type Z, and such that $(n \in C \iff 2n \in \mathcal{O}(0))$, by extending each chain both forward and backward. □

5 Decidability of Structures and Theories

Recall that for any structure $\mathcal{A}$, $Th(\mathcal{A})$ denotes the first-order theory of $\mathcal{A}$ and $FTh(\mathcal{A})$ denotes the elementary diagram of $\mathcal{A}$. In the case of equivalence structures, Cenzer, Harizanov, and Remmel [3] showed that the character of an equivalence structure together with the number of infinite classes effectively determine its theory. Similarly, they showed that the character together with the function mapping any element to the size of its equivalence class effectively determine its elementary diagram.

Proposition 5.1. *For any injection structure $\mathcal{A}$, the character $\chi(\mathcal{A})$ is computable from the theory $Th(\mathcal{A})$. Hence if $Th(\mathcal{A})$ is decidable, then $\chi(\mathcal{A})$ is computable.*

Proof. It follows from the proof of Lemma 2.1 that for all finite n and k, there are formulas $\psi_k(x)$ and sentences $\phi_{n,k}$ (in the language of injection structures) such that:

1. For any injection structure $\mathcal{A}$ and any element $a \in \mathcal{A}$, $\mathcal{A} \models \psi_k(a)$ if and only if $card(\mathcal{O}_f(a)) = k$;

2. For any injection structure $\mathcal{A}$, $\mathcal{A} \models \phi_{n,k}$ if and only if $(n,k) \in \chi(\mathcal{A})$.

Furthermore, the formulas $\psi_k(x)$ and sentences $\phi_{n,k}$ are computable from k and n.

It follows from the argument above that, in fact, $\chi(\mathcal{A})$ is many-one reducible to $Th(\mathcal{A})$. □

Given an injection structure $\mathcal{A} = (A, f)$, let the relation $R(\mathcal{A}) \subseteq \mathbb{N} \times A$ be defined by

$$(n, a) \in R(\mathcal{A}) \Leftrightarrow (\exists x)[f^n(x) = a].$$

Theorem 5.2. *For any injective structure $\mathcal{A}$, the elementary diagram of $\mathcal{A}$ is Turing reducible to the join of the relation $R(\mathcal{A})$ with the atomic diagram of $\mathcal{A}$.*

Proof. The proof is by quantifier elimination and is a straightforward modification of the classical proof for the decidability of the theory of successor. We first expand the language by adding the relation symbols γ_n such that $\mathcal{A} \models \gamma_n(a)$ if and only if $(n, a) \in R(\mathcal{A})$. Let $\psi(x, t_1, \ldots, t_k)$ be any conjunction of literals in this expanded language, where $t_1, \ldots, t_k$ are either variables or elements of $\mathcal{A}$, and let θ be $(\exists x)\psi$. Atomic formulas contain the basic terms $x, t_1, \ldots, t_k$ as well as terms $f^j(t)$ for one of those basic terms. The atomic formulas have one of two forms: $\gamma_n(s)$ for some term s or $s = t$ for some terms s and t. Since f is an injection, we can ensure that all occurrences of x in any equality have the form $f^j(x)$ for the same j, by applying f to each equality. Similarly, the formula $\gamma_n(f^i(x))$ is equivalent to $\gamma_{n+1}(f^{i+1}(x))$, so that each such occurrence of x is equivalent to some $\gamma_m(f^j(x))$. Now, the formula $(\exists x)\psi(f^j(x))$ is equivalent to the formula $(\exists y)[\gamma_j(y) \ \& \ \psi(y)]$. We may assume that for every term s of ψ, either $y = s$ or $y \neq s$ occurs among the literals of ψ.

Now, there are two cases. First, suppose that one of the literals has the form $y = s$. Then the formula $(\exists y)\psi(y)$ is equivalent to the formula $\psi(s)$. Second, suppose that for all terms s, the literal $y \neq s$ occurs in ψ. Then the fact that $\mathcal{A}$ is infinite means that such y must exist, so $(\exists y)\psi$ is equivalent to the formula obtained by deleting the quantifier $\exists y$ and all literals containing y.

At the end of quantifier elimination, we can determine whether the reduced, quantifier-free formula ϕ holds in $\mathcal{A}$ by consulting the atomic diagram of $\mathcal{A}$ as well as $R(\mathcal{A})$. □

Theorem 5.3. *For any injection structure $\mathcal{A}$, there is a structure $\mathcal{B}$ isomorphic to $\mathcal{A}$, such that $\mathcal{B}$ and $R(\mathcal{B})$ are computable from $\chi(\mathcal{B})$.*

Proof. We may assume, without loss of generality, that $\mathcal{A}$ has no infinite orbits, since, if needed, we can simply adjoin either infinitely many or some fixed finite number of computable infinite orbits of either type. The structure $\mathcal{B}$ will contain a distinct orbit $\mathcal{O}([\langle k, n\rangle])$ for each $(k, n) \in \chi(\mathcal{A})$, where we let $\langle k, n\rangle = 2^{k+1} \cdot 3^{n+1}$. Let $\chi(\mathcal{A})$ be enumerated numerically as $\langle k_0, n_0\rangle, \langle k_1, n_1\rangle, \ldots$ and let $b_0, b_1, \ldots$ enumerate $\mathbb{N} - \chi(\mathcal{A})$. Then the function $f = f^{\mathcal{B}}$ is defined by using the elements $b_0, b_1, \ldots$ to fill out the orbits $[\langle k_0, n_0\rangle], [\langle k_1, n_1\rangle], \ldots$ in order as needed. It is easy to see that $\mathcal{B}$ and $R(\mathcal{B})$ are computable from $\chi(\mathcal{B})$. □

Putting the previous results together, we have the next two theorems along with some immediate corollaries.

Theorem 5.4. *For any injection structure $\mathcal{A}$, $Th(\mathcal{A})$ and $\chi(\mathcal{A})$ have the same Turing degree.*

Proof. It follows from Proposition 5.1 that $\chi(\mathcal{A})$ is Turing reducible to $Th(\mathcal{A})$. Conversely, let $\mathcal{A}$ be an injection structure, and let $\mathcal{B}$, isomorphic to $\mathcal{A}$, be given by Theorem 5.3 so that $\mathcal{B}$ and $R(\mathcal{B})$ are both computable from $\chi(\mathcal{B})$ (which, of course, equals $\chi(\mathcal{A})$). It follows from Theorem 5.2 that $FTh(\mathcal{B})$ is computable from $\chi(\mathcal{A})$. Now, $Th(\mathcal{B})$, which equals $Th(\mathcal{A})$, is computable from $FTh(\mathcal{B})$, and hence $Th(\mathcal{A})$ is computable from $\chi(\mathcal{A})$, as desired. □

Corollary 5.5. *For any injection structure $\mathcal{A}$, $Th(\mathcal{A})$ is decidable if and only if $\chi(\mathcal{A})$ is computable.*

Theorem 5.6. *For any injection structure $\mathcal{A}$ with computable character $\chi(\mathcal{A})$, there is a decidable structure $\mathcal{B}$ isomorphic to $\mathcal{A}$. (Hence $Th(\mathcal{A})$ is decidable.)*

Proof. Again, it suffices to assume that $\mathcal{A}$ has no infinite orbits. By Theorem 5.3, there is a structure $\mathcal{B}$ isomorphic to $\mathcal{A}$, which is computable from $\chi(\mathcal{B})$, and hence $\mathcal{B}$ and $R(\mathcal{B})$ are also computable. It now follows from Theorem 5.2 that $FTh(\mathcal{B})$ is decidable, and hence $Th(\mathcal{B})$, which equals $Th(\mathcal{A})$, is decidable. □

Clearly, any bounded character is computable.

Corollary 5.7. *If the injection structure $\mathcal{A}$ has bounded character, then $Th(\mathcal{A})$ is decidable.*

In contrast to the result for equivalence structures [3] that every computably categorical structure is decidable, here we have the following negative result.

Proposition 5.8. *There is a computably categorical injection structure $\mathcal{A}$ such that $Th(\mathcal{A})$ is not decidable.*

Proof. Simply, let W be a non-computable c.e. set, and let $\mathcal{A}$ have character $\{(n, 1) : n \in W\}$ and no infinite orbits, by Proposition 2.2. Then $\chi(\mathcal{A})$ is not computable and hence, by Proposition 5.1, $Th(\mathcal{A})$ is also not decidable. □

Proposition 5.9. *For any computable character K, there is a decidable injection structure $\mathcal{A}$ with character K and with any specified countable number of orbits of types ω and Z. Furthermore, $Fin(\mathcal{A})$ is computable.*

Proof. It suffices to construct $\mathcal{A}$ with no infinite orbits. By Proposition 2.2, there is a computable injection structure $\mathcal{B}$ with character K. Now, by Theorem 5.6, there is a decidable structure $\mathcal{A}$ isomorphic to $\mathcal{B}$. □

Corollary 5.10. *If $\mathcal{B}$ is a computable injection structure with computable character and with no infinite orbits, then $\mathcal{B}$ is decidable.*

Proof. It follows from Proposition 5.9 that there is a decidable structure $\mathcal{A}$ that is isomorphic to $\mathcal{B}$. Since $\mathcal{A}$ has no infinite orbits, it is computably categorical and hence $\mathcal{A}$ and $\mathcal{B}$ are computably isomorphic. Hence $\mathcal{B}$ is also decidable. □

6 Index Sets for infinite injection structures

In this section, we develop a theory of index sets for infinite computable injection structures. Fix an enumeration of structures $\mathcal{A}_e = (\mathbb{N}, \phi_e)$, where ϕ_e is the usual e-th partial computable function. Clearly, this list includes every computable injection structure with universe $\mathbb{N}$. We avoid consideration of structures with computable universe $A \subset \mathbb{N}$ since we want to focus on the properties of the structure and not on the universe. Of course every computable structure with an infinite universe $A \subset \mathbb{N}$ is computably isomorphic to a structure with universe $\mathbb{N}$.

We say that a set A is D^0_n if A is the difference of two Σ^0_n sets or equivalently A is the intersection of Σ^0_n and Π^0_n set. We say A is D^0_n-complete if A is D^0_n and for any D^0_n set B, there is a computable function f such that $e \in B \iff f(e) \in A$. If $A \subseteq B \subseteq \mathbb{N}$, then we say that A is Σ^0_n *within* B if it is the intersection of B with a Σ^0_n set. We say that A is Σ^0_n *complete within* B if for any Σ^0_n set C, there is a computable function f such that, for every e, $f(e) \in B$ and $e \in C \iff f(e) \in A$. Similar definitions apply for Π^0_n and other notions of complexity.

We want to consider properties of computable injection structures which indicate the number of orbits of types ω and Z. Let Inj_m ($Inj_{\leq m}$) be the set of indices e such that $\mathcal{A}_e$ is a computable injection structure with exactly m ($\leq m$) orbits of type ω. The sets $Inj_{<m}$, $Inj_{\geq m}$ and $Inj_{>m}$ are defined similarly. Let Inj^n ($Inj^{\leq n}$) be the set of indices e such that $\mathcal{A}_e$ is a computable injection structure with exactly n ($\leq n$) orbits of type Z. The sets $Inj^{<n}$, $Inj^{\geq n}$ and $Inj^{>n}$ are defined similarly. Let Inj^n_m be the set of indices e such that $\mathcal{A}_e$ is a computable injection structure with exactly m orbits of type ω and n orbits of type Z. We can replace m by either $\leq m$, $< m$, $\geq m$, $> m$ and n by either $\leq n$, $< n$, $\geq n$, $> n$ in the definition of Inj^n_m to define similar sets. For example, the set $Inj^{>n}_{\leq m}$ is the set of all indices e such that $\mathcal{A}_e$ is an injection structure with no more than m orbits of type ω and more than n orbits of type Z.

Theorem 6.1. *The set $Inj = \{e : \mathcal{A}_e$ is a computable injection structure$\}$ is Π^0_2 complete and the set Inj^0_0 of injection structures with no infinite orbits is Π^0_2 complete and also Π^0_2 complete within Inj.*

Proof. It is easy to see that Inj is a Π^0_2 set. That is, $e \in Inj$ if and only if ϕ_e is total and injective which is a Π^0_2 condition. It follows from part (c) of Lemma 2.1 that Inj^0_0 is also a Π^0_2 set.

For the completeness of Inj and also of $Inj_{0,0}$, we define a reduction f of the Π^0_2 complete set $Inf = \{e : W_e$ is infinite$\}$ to Inj. Define the structure $\mathcal{A}_{f(e)} = (\mathbb{N}, \phi_{f(e)})$ as follows. Let $s_0 < s_1 < \ldots$ enumerate the possibly finite set of stages at which a new elements comes new elements appear in the standard enumeration of W_e. Here we assume that at most one element enters W_e at any stage. To define $\phi = \phi_{f(e)}$, wait until s_0 appears and then let $\phi(0) = 1, \phi(1) = 2, \ldots, \phi(s_0 - 1) = s_0$ and $\phi(s_0) = 0$. Next wait until s_1 appears and let $\phi(s_0+1) = s_0+2, \ldots, \phi(s_1-1) = s_1$ and $\phi(s_1) = s_0+1$. If W_e is finite and s_k is the last stage at which an element enters W_e, then $\phi_{f(e)}(s_k + 1)$ is undefined, so that $f(e) \notin Inj$. If W_e is infinite, then $\phi_{f(e)}$ is total and $Fin(\mathcal{A}_{f(e)}) = \mathbb{N}$

For the Π^0_2 completeness of Inj^0_0 within Inj, modify the construction above to produce a computable function g as follows. Begin by letting $\phi_{g(e)}(0) = 1, \phi_{g(e)}(1) = 2, \ldots$ until s_0 appears and then let $\phi_{g(e)}(s_0) = 0$. Next let $\phi_{g(e)}(s_0 + 1) = s_0+2, \phi_{g(e)}(s_0+2) = s_0+3, \ldots$ again until s_1 appears and then $\phi_{g(e)}(s_1) = s_0+1$. If s_k is the last stage at which an element enters W_e, then $\phi_{g(e)}(s_k+n+1) = s_k+n+2$ for all n. It follows from this construction that if W_e is infinite, then all orbits of $\mathcal{A}_{g(e)}$ are finite, whereas if W_e is finite, then $\mathcal{A}_{g(e)}$ has a finite number of finite orbits together with one orbit of type ω. Then g shows that Inj^0_0 is Π^0_2 complete in Inj.

□

Before giving further results for the above index sets, we first observe that the property of being in the same orbit under f, that is, the relation "$\mathcal{O}_f(a) = \mathcal{O}_f(b)$", is Σ^0_1.

Theorem 6.2. *For each $m \geq 0$, $Inj_{\leq m}$ is Π^0_2 complete, and the relations $Inj_{>m}$ and Inj_{m+1} are D^0_2 complete.*

Proof. The set $Inj_{>m}$ is Σ^0_2 *within* Inj, that is, it is the intersection of the Σ^0_2 class S with Inj, where S is the set of indices e such that there exist at least $m+1$ elements $x \notin Ran(\phi_e)$. The other upper complexity bounds follow from this fact.

Let $Fin = \{e : W_e \text{ is finite}\}$ and $Inf = \{e : W_e \text{ is infinite}\}$. Then Fin is a complete Σ^0_2 set, Inf is a complete Π^0_2 set and $D = \{\langle a, b\rangle : a \in Fin \ \& \ b \in Inf\}$ is a complete D^0_2 set.

For the Π^0_2 completeness of $Inj_{\leq 0}$, we can use the same proof that we used to prove that Inj^0_0 was Π^0_2 complete.

For the Π^0_2 completeness of $Inj_{\leq m}$ where $m \geq 1$, first consider the computable function g that we used to show that Inj^0_0 is Π^0_2 complete in Inj. That is, we constructed a computable function g such that if W_e is finite, then $\mathcal{A}_{g(e)}$ is a computable injection structure with one infinite orbit of type ω and finitely many finite orbits and if W_e is infinite, then all orbits of $\mathcal{A}_{g(e)}$ are finite. Then let $g_{m+1}(e)$ be the computable function such that $A_{g_{m+1}(e)}$ is a disjoint union of $m+1$ computable copies of $A_{g(e)}$. It follows that $a \in Inf \iff g_{m+1}(e) \in Inj_{\leq m}$ so that $Inj_{\leq m}$ is Π^0_2 complete.

For D^0_2 completeness of Inj_{m+1}, we can define a computable function h such that $\mathcal{A}_{h(\langle a,b\rangle)} = \mathcal{A}_{g_{m+1}(a)} \oplus (\mathcal{A}_{g_{m+1}(b)} \oplus \mathcal{A}_{g_{m+1}(b)})$. Now if $\langle a, b\rangle \in D$, then $\mathcal{A}_{g_{m+1}(a)}$ has exactly $m+1$ infinite orbits of type ω and $\mathcal{A}_{g(b)}$ has no infinite orbits so that $\mathcal{A}_{h(\langle a,b\rangle)}$ will have exactly $m+1$ orbits of type ω. If $a \in Inf$, then $\mathcal{A}_{h(\langle a,b\rangle)}$ has no infinite orbits if $b \in Inf$ and will have $2m+2$ orbits of type ω if $b \in Fin$. Thus h shows that D is many-one reducible to Inj_{m+1}.

For D^0_2 completeness of $Inj_{>m}$, we can define a computable function q such that $\mathcal{A}_{q(\langle a,b\rangle)} = \mathcal{A}_{g_{m+1}(a)} \oplus (\mathcal{A}_{f(b)} \oplus \mathcal{A}_{f(b)})$ where f is the computable function constructed in Theorem 6.1 such that $\mathcal{A}_{f(e)}$ is a computable injection structure with no infinite orbits if $e \in Inf$ and $\phi_{f(e)}$ is not total if $e \in Fin$. Now if $\langle a,b\rangle \in D$, then $\mathcal{A}_{g_{m+1}(a)}$ has exactly $m+1$ infinite orbits of type ω and $\mathcal{A}_{g(b)}$ has no infinite orbits so that $\mathcal{A}_{q(\langle a,b\rangle)}$ will be a computable injection structure with exactly $m+1$ orbits of type ω. If $b \in Fin$, then $\mathcal{A}_{q(\langle a,b\rangle)}$ will not be a computable injection structure. Thus q shows that $Inj_{>m}$ is D^0_2 complete. □

Theorem 6.3. *For each $n \geq 0$, $Inj^{\leq n}$ is Π^0_3 complete, $Inj^{>n}$ is Σ^0_3 complete, and Inj^{n+1} is D^0_3 complete.*

Proof. The set $Inj^{>n}$ is Σ^0_3, since it is the set of indices e such that there exist $n+1$ elements $x_0, \ldots, x_n$, each having an orbit of type Z and no two being in the same orbit. The other upper complexity bounds follow from this fact.

First we show that $Inj^{>0}$ is Σ^0_3 complete and $Inj^{\leq 0}$ is Π^0_3 complete. Let $Cof = \{e : W_e \text{ is co-finite}\}$. Cof is a Σ^0_3 complete set.

We define a computable function r such that $\mathcal{A}_{r(e)}$ is defined by the following construction. We start to build ω chains going forward from each number $2m+1$ by mapping x to $2x$. When a number m comes into W_e at stage $s+1$, we find the longest sequence $k, k+1, \ldots, m, m+1, \ldots, n$ including m which is contained in $W_{e,s+1}$. We take the chains from $2(m+1)+1$ to $2n+1$ and put them at the end of the $2m+1$ chain, removing them from any further activity. If $k < m$, then we put the newly expanded $2m+1$ chain at the end of the $2k+1$ chain. Finally, we add an element to the beginning of the $2k+1$ chain.

If W_e is co-finite, there will be a least m such that every $n \geq m$ belongs to W_e. In that case, the orbit of $2m+1$ will be a Z chain, all of the $2n+1$ chains for $n > m$ will be included in this orbit, and there will be finitely many orbits of type ω for the numbers $k < m$. If W_e is co-infinite, then W_e will be an infinite union of finite consecutive sequences, each of which will have an orbit of type ω and these orbits will include all elements. Thus $e \in Cof \iff r(e) \in Inj^{>0}$ so that $Inj^{>0}$ is Σ^0_3 complete and $Inj^{\leq 0}$ is Π^0_3 complete.

To show that $Inj^{>m}$ is Σ^0_3 complete and $Inj^{\leq m}$ is Π^0_3 complete for $m \geq 1$, let $r_m(e)$ be the computable function such that $\mathcal{A}_{r_m(e)}$ is $\mathcal{A}_{r(e)} \oplus \mathcal{B}_m$ where B_m is a computable injection structure that has m Z chains and no other orbits. Then $e \in Cof \iff r_m(e) \in Inj^{>m}$ so that $Inj^{>m}$ is Σ^0_3 complete and $Inj^{\leq m}$ is Π^0_3 complete.

To show that Inj^m is D^0_3 complete for $m \geq 1$, we reduce the D^0_3 complete $E = \{\langle a,b\rangle : a \in Cof \text{ and } b \notin Cof\}$. Let k be the computable function such that $\mathcal{A}_{k(\langle a,b\rangle)} = \mathcal{A}_{r_{m-1}(a)} \oplus (\mathcal{A}_{r_{m-1}(b)} \oplus \mathcal{A}_{r_{m-1}(b)})$ Then it is easy to check that $\mathcal{A}_{k(\langle a,b\rangle)}$ has exactly m Z chains if and only if $a \in Cof$ and $b \notin Cof$.

□

Theorem 6.4. *The property of being a computable categorical computable injection structure over $\mathbb{N}$ is Σ^0_3 complete. That is,*

$$CCI = \{e : \mathcal{A}_e \textit{ is an injection structure with finitely many infinite orbits}\}$$

is a Σ^0_3 complete set.

Proof. The set CCI is Σ^0_3 since $\mathcal{A}_e$ has finitely many infinite orbits if and only if there exists a finite sequence $a_0, \ldots, a_k$ such that for every b, if $b \notin \mathcal{O}(a_i)$ for any $i \leq k$, then $\mathcal{O}(b)$ is finite. The details follow from Lemma 2.1.

For completeness, consider the Σ^0_3 complete set $Cof = \{e : W_e \text{ is co-finite}\}$. We define a computable function f such that for every e, $\mathcal{A}_{f(e)}$ will have finitely many infinite orbits if and only if W_e is co-finite. The orbits of $\mathcal{A}_{f(e)}$ will be exactly the orbits $\mathcal{O}(2i+1)$ for $i \in \mathbb{N}$, and the even numbers will be used to fill out the orbits. The injection $\phi_{f(e)}$ will be defined in stages, as $\phi^s_{f(e)}$, so that for $i \leq s$, we have defined $\phi^t_{f(e)}(2i+1)$ for all $t \leq s$.

At stage $s+1$, we do the following. For $i \leq s$, we have three cases. If $i \notin W_{e,s+1}$, then at stage $s+1$ we simply define $\phi^{s+1}_{f(e)}(2i+1)$ to be the next available even number. If $i \in W_{e,s+1} - W_{e,s}$, then we define $\phi^{s+1}_{f(e)}(2i+1) = 2i+1$. If we already have $i \in W_{e,s}$, there is nothing to do. That is, if $i \in W_{e,s}$, then $\phi^t_{f(e)}(2i+1) = 2i+1$ for some $t \leq s$ and hence $\phi^s_{f(e)}(2i+1)$ is automatically defined for all $s \geq t$. For $i = s+1$, we choose the next $s+1$ available even numbers, say $2n_1, 2n_2, \ldots 2n_{s+1}$, and let $\phi^t_{f(e)}(2s+3) = 2n_t$ for all $t \leq s+1$.

The following facts are immediate from the construction.

1. The orbit $\mathcal{O}_{f(e)}(2i+1)$ is finite if and only if $i \in W_e$.

2. The function $\phi_{f(e)}$ is total and one-to-one.

It follows from Fact 2 that $f(e) \in Inj$ for all e. It follows from Fact 1 that $\mathcal{A}_{f(e)}$ has finitely many infinite orbits if and only if W_e is co-finite. $\square$

Theorem 6.5. *The property of being Δ^0_2 computable categorical injection structure is Σ^0_4 complete. That is, the set DCI of indices e such that $\mathcal{A}_e$ is an injection structure with finitely many orbits of type ω or finitely many orbits of type Z, is a Σ^0_4 complete set.*

Proof. The set DCI is Σ^0_4 since $\mathcal{A}_e$ has finitely many infinite orbits of type ω or of type Z if and only if one of the following two cases holds.

Case I: There exists a finite sequence $a_0, \ldots, a_k$ such that, for every b, if $b \notin \mathcal{O}(a_i)$ for any $i \leq k$, then $\mathcal{O}(b)$ does not have type ω.

Case II: There exists a finite sequence $a_0, \ldots, a_k$ such that, for every b, if $b \notin \mathcal{O}(a_i)$ for any $i \leq k$, then $\mathcal{O}(b)$ does not have type Z.

It follows from Lemma 2.1 that the condition in Case I is Σ^0_3, and the condition in Case II is Σ^0_4. For completeness, let C be any Π^0_4 set. Then there is a Π^0_2 relation Q such that for every e, $e \in C$ if and only if $\{n : Q(e,n)\}$ is infinite. Since Q is Π^0_2, there is a computable relation R such that $Q(e,n)$ if and only if $\{r : R(e,n,r)\}$ is infinite. That is, $e \in C$ if and only if there are infinitely many n such that there are infinitely many r for which $R(e,n,r)$ holds. We may assume that R is defined in stages, as R_s, so that $R_{s+1} - R_s$ contains at most one element (e,n,r) for each s and that $e+n+r \leq s$.

We now define a reduction g such that for any e, $\mathcal{A}_{g(e)}$ has only infinite orbits, and has infinitely many orbits of type Z if and only if $e \in C$. We first define a reduction f so that $\mathcal{A}_{f(e)}$ has finitely many orbits of type Z if and only if $e \in C$ and then modify the construction to obtain $\mathcal{A}_{g(e)}$ as the disjoint union of $\mathcal{A}_{f(e)}$ with a computable structure consisting of an infinite number of orbits of type ω.

As in the proof of Theorem 6.4, the orbits of $\mathcal{A}_{f(e)}$ will be exactly the orbits $\mathcal{O}(2i+1)$ for $i \in \mathbb{N}$ and the even numbers will be used to fill out the orbits. The injection $\phi_{f(e)}$ will be defined in stages, as $\phi^s_{f(e)}$, so that for $i \leq s$, we have defined $\phi^t_{f(e)}(2i+1)$ for all $t \leq s$.

At stage $s+1$, we do the following. For $i \leq s$, we extend each orbit $\mathcal{O}(2i+1)$ by defining $\phi^{s+1}_{f(e)}(2i+1)$ to be the next available even number. We begin the orbit $\mathcal{O}(2s+3)$ by defining $\phi^t_{f(e)}(2s+3)$ for $t \leq s+1$ as the next $s+1$ even numbers.

Finally, if (e,n,r) enters R at stage $s+1$, then we add a new element to the front of $\mathcal{O}(2n+1)$. That is, let b be the unique element of $\mathcal{O}(2n+1)$ at stage s which is not in $Ran(\phi_{f(e),s})$, let $2a$ be the next available even number, and let $\phi^{s+1}_{f(e)}(2a) = b$.

The following facts are immediate from the construction.

1. For every n, $\mathcal{O}_{f(e)}(2n+1)$ is infinite and has type Z if and only if $\{r : R(e,n,r)\}$ is infinite.
2. The function $\phi_{f(e)}$ is total, one-to-one, and onto.

It follows from Fact 2 that $f(e) \in Inj$ for all e. It follows from Fact 1 that $\mathcal{A}_{f(e)}$ has infinitely many orbits of type Z if and only if $e \in C$.

Finally, let $\mathcal{B}$ be a computable structure consisting of an infinite number of orbits each of type ω and let $\mathcal{A}_{g(e)}$ be the disjoint union of $\mathcal{B}$ with $\mathcal{A}_{f(e)}$. It follows that $g(e) \in DCI \Leftrightarrow e \notin C$. Therefore, DCI is Σ^0_4 complete. □

Next, we consider the complexity of the *isomorphism problem* for infinite computable injection structures.

Theorem 6.6. *The set $\{(i,j) : \mathcal{A}_i$ is isomorphic to $\mathcal{A}_j\}$ is Π^0_4 complete.*

Proof. For the upper bound on the complexity, observe that $\mathcal{A}_i$ is isomorphic to $\mathcal{A}_j$ if and only if they have the same character and the same number of infinite orbits of each type. In particular, for the most complicated case, it must be that for every n, if $\mathcal{A}_i$ has at least n orbits of type Z, then in $\mathcal{A}_j$, there must exist at least n elements, all in different orbits of type Z. Since being in an orbit of type Z is Π^0_2, it is easy to see that the overall definition is Π^0_4.

For completeness, fix a structure $\mathcal{A}_j$ with infinitely many orbits of type ω and infinitely many orbits of type Z. Let Q be an arbitrary unary Π^0_4 relation, so that there is a computable ternary relation R such that for every e,

$$e \in Q \Longleftrightarrow (\exists^\infty n)(\exists^\infty k)R(n,k,e),$$

where the quantifier $\exists^\infty$ stands for "there exist infinitely many."

Define a computable function f by constructing $\mathcal{A}_{f(e)}$ as follows. Each orbit will be $\mathcal{O}(4n)$ for some n. Odd numbers will be used to fill out the orbits. At each stage s, we will have finite chains of length $\geq s$ containing $4n$ for each $n < s$. At stage $s+1$, we do the following:

(i) start a new chain for $4s$ of length $s+1$ by letting $t_1 < \cdots < t_s$ be the least odd number which have not been used in the construction up to this point and define $f_e(4s) = t_1$ and $f_e(t_i) = t_{i+1}$ for $1 \leq i < s$,

(ii) add a new odd number at the end of each previous chain starting at $4n$ for $n < s$, and,

(iii) for every $n < s$, if $R(n, s+1, e)$ holds, then add a new odd number at the beginning of the finite chain containing $4n$.

Observe that for each n, if there exist infinitely many k such that $R(n, k, e)$, then in the construction we will add a new element to the front of $\mathcal{O}(4n)$ infinitely often, so that $\mathcal{O}(4n)$ will have type Z. Finally, use the numbers which are equivalent to 2 mod 4 to build infinitely many orbits of type ω.

Suppose first that $e \in Q$. Then by the argument above, $\mathcal{A}_{f(e)}$ will have infinitely many orbits of type Z, in addition to having infinitely many orbits of type ω. It follows that $\mathcal{A}_{f(e)}$ will be isomorphic to $\mathcal{A}_j$. On the other hand, if $e \notin Q$, then only finitely many of the orbits $\mathcal{O}(4n)$ will have type Z, so that $\mathcal{A}_{f(e)}$ will not be isomorphic to $\mathcal{A}_j$. $\square$

Next, we investigate the complexity of the *computable isomorphism problem* for injection structures.

Theorem 6.7. *The set $\{(i,j) : \mathcal{A}_i$ is computably isomorphic to $\mathcal{A}_j\}$ is Σ_3^0 complete.*

Proof. As usual, this set is Σ_3^0 since the condition asks for the existence of a total computable function that is an isomorphism. For completeness, let $\mathcal{A}_j = (\mathbb{N}, f)$ be a fixed structure with infinitely many orbits of type ω and infinitely many orbits of type Z, in which $Ran(f)$ is a computable set, the set of all elements belonging to orbits of type Z is computable, and the set $\{(a,b) : a, b$ are in the same orbit$\}$ is also computable. Note that any two injection structures with these properties will be computably isomorphic.

Recall that the set $Comp = \{e : W_e$ is computable$\}$ is Σ_3^0 complete. We shall define a computable function f so that for every $e \in \mathbb{N}$, $e \in Comp$ if and only if $\mathcal{A}_{f(e)}$ is computably isomorphic to $\mathcal{A}_j$. We begin with a copy $\mathcal{A}'$ of $\mathcal{A}_j$ containing the odd numbers as elements and with the nice properties described above. Next, we construct $\mathcal{B}_{f(e)}$ as follows. Every orbit of $\mathcal{A}_{f(e)}$ will be $\mathcal{O}(4n)$ for some n. Numbers of the form $4n+2$ will be used to fill out the orbits. At stage s, each orbit will extend forward and backward from $4n$. At stage $s+1$, if n comes into W_e, then we stop adding elements to the front of the orbit $\mathcal{O}(4n)$. Thus, if $n \in W_e$, then $\mathcal{O}(4n)$ will have type ω, and otherwise it will have type Z. The structure $\mathcal{A}_{f(e)}$ is defined to be the disjoint union of the structures $\mathcal{A}'$ and $\mathcal{B}_{f(e)}$. Notice that for every n, $\mathcal{O}(4n)$ has type Z if and only if $n \in W_e$. Thus, if $e \notin Comp$, then $\mathcal{A}_{f(e)}$ cannot be computably isomorphic to $\mathcal{A}_j$. On the other hand, if $e \in Comp$, then $\mathcal{A}_{f(e)}$ will have the properties described above, and hence will be computably isomorphic to $\mathcal{A}_j$. $\square$

7 Σ_1^0 injection structures

Recall that an injection structures (A, f) is said to be a Σ_1^0 injection structure if A is an infinite Σ_1^0 set and f is the restriction of a partial computable function to A. The complexities of orbits and the character for Σ_1^0 injection structures are unchanged from those for computable structures.

Lemma 7.1. *For any Σ_1^0 injection structure $\mathcal{A} = (A, f)$:*

(a) $\{(k, a) : a \in Ran(f^k)\}$ is a Σ_1^0 set,

(b) $\{(a, k) : card(\mathcal{O}_f(a)) \geq k\}$ is a Σ_1^0 set,

(c) $\{a : \mathcal{O}_f(a)$ is infinite$\}$ is a D_1^0 set, in fact, the intersection of a Π_1^0 set with A.

(d) $\{a : \mathcal{O}_f(a)$ has type $Z\}$ is a Π_2^0 set,

(e) $\{a : \mathcal{O}_f(a)$ has type $\omega\}$ is a Σ_2^0 set, and

(f) $\chi(\mathcal{A})$ is a Σ_1^0 set.

Proof. It is a straightforward extension of the proof of Lemma 2.1. $\square$

Since the character of any Σ_1^0 structure is Σ_1^0, the following result is immediate from Proposition 2.2.

Proposition 7.2. *Any Σ_1^0 injection structure is isomorphic to a computable injection structure.*

In fact, there is always a computable isomorphism.

Proposition 7.3. *For any Σ_1^0 injection structure $\mathcal{A}$, there exists a computable injection structure $\mathcal{B}$ and a computable isomorphism from $\mathcal{B}$ onto $\mathcal{A}$.*

Proof. Let A be a Σ_1^0 set and let f be a partial computable function such that the restriction of f to A is an injection. Let ϕ be a computable one-to-one enumeration of A. Then define $\mathcal{B} = (\mathbb{N}, g)$ by letting $g(b) = \phi^{-1}(f(\phi(b)))$. The mapping $\phi : \mathcal{B} \to \mathcal{A}$ is the desired computable isomorphism. The inverse mapping ϕ^{-1} maps $\mathcal{A}$ onto $\mathcal{B}$ and is partial computable with domain A. $\square$

Theorem 7.4. *If $\mathcal{A}_1$ and $\mathcal{A}_2$ are isomorphic Σ_1^0 injection structures with finitely many infinite orbits, then there is an isomorphism $\psi : \mathcal{A}_1 \to \mathcal{A}_2$ such that both ψ and ψ^{-1} are partial computable.*

Proof. Let the computable structures $\mathcal{B}_1$ and $\mathcal{B}_2$ be given by Proposition 7.3 together with partial computable isomorphisms $\phi_i : \mathcal{B}_i \to \mathcal{A}_i$ for $i = 1, 2$. Since $\mathcal{B}_1$ is computably categorical, there is a computable isomorphism $g : \mathcal{B}_1 \to \mathcal{B}_2$. Now define $\psi : \mathcal{A}_1 \to \mathcal{A}_2$ by $\psi(a) = \phi_2(g(\phi_1^{-1}(a)))$. □

The following result is proved similarly, using Theorem 3.1.

Theorem 7.5. *If $\mathcal{A}_1$ and $\mathcal{A}_2$ are isomorphic Σ^0_1 injection structures with finitely orbits of type Z or finitely many orbits of type ω, then there is a Δ^0_2 isomorphism $\psi : \mathcal{A}_1 \to \mathcal{A}_2$.*

Now we can use Theorem 3.6 and Proposition 7.3 to establish the following results.

Theorem 7.6. *If $\mathcal{A}_1$ and $\mathcal{A}_2$ are isomorphic Σ^0_1 injection structures, then there is a Δ^0_3 isomorphism $\psi : \mathcal{A}_1 \to \mathcal{A}_2$.*

Proposition 7.7. *For any d.c.e. set B, there is a Σ^0_1 injection structure $\mathcal{A} = (A, f)$ such that B is* 1-1 *reducible to $A - Fin(\mathcal{A})$.*

Proof. Let $B = C - D$, where C and D are c.e. sets and $D \subset C$. Let $A = \{2n+1 : n \in C\} \cup \{2n : n \in \mathbb{N}\}$. For each n, we begin to define the orbit of $2n+1$ in $\mathcal{A}$ by setting $f(2n+1) = 2(2n+1)$, $f(2(2n+1)) = 2^2(2n+1)$, and so on, until we see that $n \in D$ at some stage $s+1$. Then let $f(2^s(2n+1)) = 2n+1$ and for $t > s$, let $f(2^t(2n+1)) = 2^{t+1}(2n+1)$. It follows that for each n, $n \in B \Leftrightarrow 2n+1 \in A - Fin(\mathcal{A})$. □

8 Π^0_1 Injection Structures

Recall that an injection structure (A, f) is a Π^0_1 injection structure if A is an infinite Π^0_1 set and f is the restriction of a partial computable function to A. The complexities of orbits and the character for Π^0_1 injection structures are higher than those for computable structures.

Lemma 8.1. *For any Π^0_1 injection structure $\mathcal{A} = (A, f)$:*

(a) $\{(k, a) : a \in Ran(f^k)\}$ is a Σ^0_2 set, although each orbit of $\mathcal{A}$ is a d.c.e. set,

(b) $\{(a, k) : card(\mathcal{O}_f(a)) \geq k\}$ is a Π^0_1 set,

(c) $\{a : \mathcal{O}_f(a)$ is finite$\}$ is a D^0_1 set, the intersection of A with a c.e. set,

(d) $\{a : \mathcal{O}_f(a)$ has type $Z\}$ is a Π^0_3 set,

(e) $\{a : \mathcal{O}_f(a)$ has type $\omega\}$ is a Σ^0_3 set, and

(f) $\chi(\mathcal{A})$ is a Σ^0_2 set.

Proof. In the first part of (a) and in (d), (e) and (f), we are quantifying over the Π^0_1 set A, which increases the complexity. For the second part of (a), the orbit of any $a \in A$ is

$$\{x : (\exists n)[f^n(a) = x] \vee [x \in A \ \&\ (\exists n)(f^n(x) = a)]\}.$$

□

Next we show that there are Π^0_1 injection structures that have arbitrary non-trivial Σ^0_2 characters so that there are Π^0_1 injection structures which are not isomorphic to a computable injection structure.

Theorem 8.2. *For any Σ^0_2 character K, which is both infinite and co-infinite, there is a Π^0_1 injection structure with character K and with any finite or countably infinite number of orbits of type Z and type ω.*

Proof. Let K be a Σ^0_2 character, which is both infinite and co-infinite. Clearly, it suffices to construct a Π^0_1 injection structure $\mathcal{A} = (A, f)$ with character K and with no infinite orbits. Let R be a computable relation so that for all $n, k > 0$, $(n, k) \in K$ if and only if $\{t : R(n, k, t)\}$ is finite. We may assume that for each t, there is at most one pair (n, k) such that $R(n, k, t)$. We begin with a computable injection structure $\mathcal{A} = (\mathbb{N}, f)$ consisting of infinitely many orbits of size k for each k. For example, for all k, let $f(\langle k, nk + k + i\rangle) = \langle k, nk + k + i + 1\rangle$ for $i < k$ and let $f(\langle k, nk + 2k\rangle) = \langle k, nk + k\rangle$. The Π^0_1 set A will be defined in stages A_s, so that $A_{s+1} \subseteq A_s$ and $A = \bigcap_s A_s$. At each stage s, there will be uniformly computable families of infinitely many orbits $\mathcal{O}^s_{k,n}$ of size k, for each $k \geq 1$.

At stage $s + 1$, we look for the unique $\langle n, k\rangle$ such that $R(n, k, s + 1)$ and delete the orbit $\mathcal{O}_{k,n}$ from the set A_s to obtain the set A_{s+1}. The remaining orbits are renamed so that $\mathcal{O}^{s+1}_{k,i} = \mathcal{O}^s_{k,i}$ for $i < n$, and $\mathcal{O}^{s+1}_{k,i} = \mathcal{O}^s_{k,i+1}$ for $i \geq n$.

Suppose that $(n, k) \in K$. Then for all $m \leq n$, there are just finitely many t such that $R(m, k, t)$ so that after some stage s, we will retain the mth orbit of size k at each successive stage. Hence $\mathcal{A}$ will have at least n orbits of size k.

Suppose that $(n, k) \notin K$. Then there are infinitely many t such that $R(n, k, t)$ so that the nth orbit of size k at any stage s is eventually deleted from A. Since K is a character, this also happens for any $i > n$, so $\mathcal{A}$ has at most $n - 1$ orbits of size k. □

Now, we will consider the complexity of isomorphisms for Π^0_1 injection structures. Since computable injection structures with finitely many infinite orbits are relatively computably categorical, we obtain the following theorem.

Theorem 8.3. *If $\mathcal{A}$ and $\mathcal{B}$ are isomorphic Π^0_1 injection structures with only finitely many infinite orbits, then $\mathcal{A}$ and $\mathcal{B}$ are Δ^0_2 isomorphic.*

In contrast to the previous positive results for Π^0_1 equivalence structures in [3], the results for Π^0_1 injection structures that are isomorphic to Δ^0_2 categorical structures are mixed.

Lemma 8.4. *Let $\mathcal{A} = (A, f)$ be a Π^0_1 injection structure and define the relation $R_{\mathcal{A}}(a, b)$ to hold if and only if a and b are in the same orbit. Then $\{\langle a, b\rangle : R_{\mathcal{A}}(a, b)$ holds$\}$ is a d.c.e. set.*

Proof. Just observe that

$$R(a, b) \Leftrightarrow [a \in A \wedge b \in A \wedge (\exists n)(f^n(a) = b \vee f^n(b) = a)].$$

□

Theorem 8.5. *Let $\mathcal{A} = (A, f)$ be a Π^0_1 injection structure such that $\chi(\mathcal{A})$ is a Σ^0_1 set and which has only finitely many orbits of type ω. Then $\mathcal{A}$ is Δ^0_2 isomorphic to a computable structure.*

Proof. Suppose that $\mathcal{A}$ has n orbits of type ω and $a_1, \ldots, a_n$ are the unique elements in these orbits which are in A but not in the range of f restricted to A. It follows from Lemma 8.1 that if we are given an oracle for $0'$, we can effectively list $Fin(\mathcal{A}) = \{b_0 < b_1 < \cdots\}$ and we can effectively list the elements $c_0 < c_1 < \cdots$ which are in $A - (Fin(\mathcal{A}) \cup \mathcal{O}_{\mathcal{A}}(a_1) \cup \cdots \cup \mathcal{O}_{\mathcal{A}}(a_n))$. Now let $\mathcal{B} = (\mathbb{N}, g)$ be a computable injection structure which is isomophic to $\mathcal{A}$ such that $Fin(\mathcal{B})$ is computable and the set of elements which lie in Z orbits is computable. Let $x_1, \ldots x_n$ be the representatives of the ω orbits which are not in $Ran(g)$, let $Fin(\mathcal{B}) = \{y_0 < y_1 < \cdots\}$, and let $z_0 < z_1 < \cdots$ be an effective list of the elements in $\mathbb{N}$ which are in $\mathbb{N} - (Fin(\mathcal{B}) \cup \mathcal{O}_{\mathcal{B}}(x_1) \cup \cdots \cup \mathcal{O}_{\mathcal{B}}(x_n)$.

Then using the lists $b_0 < b_1 < \ldots$ and $y_0 < y_1 < \ldots$ and that fact that f and g are partial computable, we can effectively construct an isomorphism $h_2 : (Fin(\mathcal{A}), f) \to (Fin(\mathcal{B}), g)$. Thus h_1 will be computable in $0'$. We can extend h_1 to include the ω orbits by defining $h_1(f^m(a_i)) = g^m(x_i)$ for all $1 \leq i \leq n$ and $m \geq 0$. Finally we can use the lists $c_0 < c_1 < \ldots$ and $z_0 < z_1 < \ldots$ to construct an isomorphism h_2 which maps the Z orbits of $\mathcal{A}$ onto the Z orbits of $\mathcal{B}$ as follows. First if $\mathcal{A}$ has only finitely many Z orbits, then we let $r_1, \ldots, r_p$ be a list of representatives from the Z orbits of $\mathcal{A}$ and $s_1, \ldots, s_p$ be a list of representatives of the Z orbits of $\mathcal{B}$. Then we define $h_2(r_i) = s_i$ for $i = 1, \ldots, p$ and extend h_2 the Z orbits in the obvious way. If $\mathcal{A}$ has infinitely many Z orbits, then using the $0'$ oracle and Lemma 8.4, we can effectively construct a sequence $i_0 < i_1 < \cdots$ such that $i_0 = 0$, i_1 is the least i such that $c_i \notin \mathcal{O}_{\mathcal{A}}(c_{i_0})$, and for all $k \geq 1$, i_{k+1} is the least i such that $c_i \notin \bigcup_{s=0}^{k} \mathcal{O}_{\mathcal{A}}(c_{i_s})$. Similarly, using the $0'$ oracle, we can effectively construct a sequence $j_0 < j_1 < \cdots$ such that $j_0 = 0$, j_1 is the least i such that $z_i \notin \mathcal{O}_{\mathcal{B}}(z_{i_0})$, and for all $k \geq 1$, j_{k+1} is the least i such that $z_i \notin \bigcup_{s=0}^{k} \mathcal{O}_{\mathcal{B}}(z_{j_s})$. Then we can define $h_2(c_{i_s}) = z_{j_s}$ for all $s \geq 0$ and extend h_2 so that it maps the $\mathcal{O}_{\mathcal{A}}(c_{i_s})$ to $\mathcal{O}_{\mathcal{B}}(z_{j_s})$ in the obvious way.

Then $h = h_1 \cup h_2$ will be computable in $0'$ so that h will be the desired Δ^0_2 isomorphism from $\mathcal{A}$ onto $\mathcal{B}$. □

Note that in the proof of Theorem 8.5, if $\mathcal{A}$ and $\mathcal{B}$ are isomorphic Π^0_1 injection structures, then h will still be a Δ^0_2 isomorphism because we can use the $0'$ oracle to effectively construct the lists $y_0 < y_1 < \ldots$ and $z_0 < z_1 \ldots$. Thus we have the following corollary.

Corollary 8.6. *If $\mathcal{A}$ and $\mathcal{B}$ are isomorphic Π^0_1 injection structures with only finitely many orbits of type ω, then $\mathcal{A}$ and $\mathcal{B}$ are Δ^0_2 isomorphic.*

For Π^0_1 injection structures with infinitely many orbits of type ω, it is not always possible to have a Δ^0_2 isomorphism.

Proposition 8.7. *For any infinite, co-infinite Σ^0_2 set C, there is a Π^0_1 injection structure $\mathcal{A} = (A, f)$ consisting of infinitely many orbits of type ω and such that $C \leq_T Ran(f)$.*

Proof. Let C be a Σ^0_2 set, which is both infinite and co-infinite. Let R be a computable relation so that for all n, $n \in C$ if and only if $\{t : R(n,t)\}$ is finite. We construct $\mathcal{A}$ so that for each n, $2(2n+1)$ belongs to the nth orbit and will be the first element of its orbit if and only if $n \notin C$. At each stage s of the construction, every orbit will begin with an odd number, but this odd number will be deleted from A whenever a new t is found so that $R(n,t)$. Thus if $n \in C$, then $2(2n+1) = f(a)$ for some fixed odd number a. If $n \notin C$, then every a such that $f(a) = 2(2n+1)$ at some stage s is eventually removed from A, so that $2(2n+1) \notin Ran(f)$.

Here are the details of the construction. The Π^0_1 set A will be defined in stages A_s, so that $A_{s+1} \subseteq A_s$ and $A = \bigcap_s A_s$. The computable function f will be defined in stages, as f_s, such that (A_s, f_s) has exactly s orbits of size s. At stage 0, $A = \mathbb{N}$ and f_0 is the empty function.

There are two parts to the construction at stage $s+1$. First, we add a new orbit of size $s+1$ as follows. Select the next available odd number $2a+1$, and let $f_{s+1}(2a+1) = 2(2(s+1)+1)$, and $f_{s+1}(2^i(2(s+1)+1) = 2^{i+1}(2(s+1)+1)$ for $i = 1, 2, \ldots, s-1$. Next, we consider for $i \leq s$, whether $R(i, s+1)$ holds. If it does, then we remove the initial odd number from the orbit of $2i+1$ and replace it with the next available odd number b, so that $f_{s+1}(b) = 2(2i+1)$. Finally, we add the $(s+1)$st element $2^s(2i+1)$ to the orbit of $2(2i+1)$, and let $f_{s+1}(2^{s-1}(2i+1)) = 2^s(2i+1)$.

Let $A = \cap_s A_s$. Then, clearly, A is a Π^0_1 set and f is a partial computable function with domain A so that (A, f) is a Π^0_1 injection structure. As indicated above, we have $n \in C \Leftrightarrow 2(2n+1) \in Ran(f)$, so $C \leq_T Ran(f)$. □

Theorem 8.8. *For any Σ^0_1 character K, there is a Π^0_1 injection structure $\mathcal{B}$ with character K, with infinitely many orbits of type ω, and any finite number of orbits of type Z, such that $\mathcal{B}$ is not Δ^0_2 isomorphic to any Σ^0_1 injection structure.*

Proof. Let C be a Σ^0_2 set, which is not Δ^0_2, and let $\mathcal{B}_0 = (B, g_0)$ be given by Proposition 8.7 so that $Ran(g)$ is not Δ^0_2. Now build $\mathcal{B} = (B, g)$ by adjoining a natural computable structure with character K and with the desired number of orbits of type Z. Then the range of g is the disjoint union of the range of g_0 with a computable set, and is still not a Δ^0_2 set. Let $\mathcal{A} = (A, f)$ be a computable structure given by Proposition 2.2 so that $\mathcal{A}$ is isomorphic to $\mathcal{B}$, but $Ran(f)$

is computable. If ϕ were a Δ_2^0 isomorphism from $\mathcal{A}$ onto $\mathcal{B}$, then $Ran(g) = \phi[Ran(f)]$ and is, therefore, a Σ_2^0 set. However, $B - Ran(g) = \phi[\mathbb{N} - Ran(f)]$ and is, therefore, also a Σ_2^0 set. It would then follow that $Ran(g)$ is, in fact, a Δ_2^0 set. Thus $\mathcal{A}$ and $\mathcal{B}$ cannot be Δ_2^0 isomorphic.

By Theorem 7.5, any Σ_1^0 structure $\mathcal{D}$ which is isomorphic to $\mathcal{A}$ is in fact Δ_2^0 isomorphic to $\mathcal{A}$. It follows that $\mathcal{D}$ cannot be Δ_2^0 isomorphic to $\mathcal{B}$. □

Proposition 8.9. *For any d.c.e. set B, there is a Π_1^0 injection structure $\mathcal{A} = (A, f)$ such that B is 1-1 reducible to $Fin(\mathcal{A})$.*

Proof. Let $B = C \cap D$, where C is a Π_1^0 set, D is a c.e. set, and $D \cup C = \mathbb{N}$. Let $A = \{2n+1 : n \in C\} \cup \{2n : n \in \mathbb{N}\}$. For each n, start to define the orbit of $2n+1$ in $\mathcal{A}$ by having $f(2n+1) = 2(2n+1)$, $f(2(2n+1)) = 4(2n+1)$, and so on, until we see that $n \in D$ at some stage $s+1$. Then let $f(2^s(2n+1)) = 2n+1$, and for $t > s$, let $f(2^t(2n+1)) = 2^{t+1}(2n+1)$. It follows that for each n, $n \in B \Leftrightarrow 2n+1 \in Fin(\mathcal{A})$. □

9 Injection Structures in the Ershov Hierarchy

In this section, we consider injection structures (A, f), where A is an infinite n-c.e. set and f is the restriction of a partial computable function to A, which defines an injection on A. The complexities of orbits for n-c.e. injection structures are almost the same as that of Π_1^0 structures.

Lemma 9.1. *Let $\mathcal{A} = (A, f)$ be an n-c.e. injection structure for $n \geq 2$.*

(a) $\{(k, a) : a \in Ran(f^k)\}$ *is a Σ_2^0 set, although each orbit of $\mathcal{A}$ is an n- c.e. set.*

(b) $\{(a, k) : card(\mathcal{O}_f(a)) \geq k\}$ *is an n-c.e. set, that is, the intersection of A with a computable set.*

(c) $\{a : \mathcal{O}_f(a)$ *is infinite*$\}$ *is the intersection of A with a Π_1^0 set, and is therefore n-c.e. if n is even, and $(n+1)$-c.e. if n is odd.*

(d) $\{a : \mathcal{O}_f(a)$ *has type* $Z\}$ *is a Π_3^0 set,*

(e) $\{a : \mathcal{O}_f(a)$ *has type* $\omega\}$ *is a Σ_3^0 set, and*

(f) $\chi(\mathcal{A})$ *is a Σ_2^0 set.*

Proof. We only consider parts (a) and (c) since all other proofs are as before. Part (a) easily follow by writing out the definition of $\{(k, a) : a \in Ran(f^k)\}$ and using the fact that A is a Δ_2^0 set. Note that since f is partial computable, $O_f(a) = \{x : (\exists k)((f^k(a) = x) \vee (\exists m)(f^m(x) = a\}$ is a c.e. set. Thus $\mathcal{O}_{\mathcal{A}}(a) = A \cap O_f(a)$ is n-c.e. since the intersection of an n-c.e. set and a c.e. set is n-c.e. set. The result in (c) follows in a similar manner where we use the fact that for even n, the family of n-c.e. sets is closed under intersection with Π_1^0 sets, while for odd n, the intersection of a Π_1^0 set with an n-c.e. set is an $(n+1)$-c.e. set. □

Lemma 9.2. *For any $n \geq 1$ and any infinite n-c.e. set B, there is a Π^0_1 set A and a (total) computable* 1-1 *function ϕ mapping A onto B.*

Proof. The proof is by induction on n. Certainly, for $n = 1$, every infinite c.e. set has a 1-1 enumeration. Now, let B be an infinite $(n+1)$-c.e. set. By induction, there are a Π^0_1 set A_1 and a computable 1-1 function ϕ_1 mapping A_1 onto B_1. There are two cases. If n is odd, then $B = B_1 \cup E$, where E is a c.e. set, and we may assume that E is infinite and disjoint from B_1. Let ϕ_2 be a computable 1-1 enumeration of E. Now, let $A = \{2n : n \in A_1\} \cup \{2n+1 : n \in \mathbb{N}\}$, and define $\phi(2n) = \phi_1(n)$ and $\phi(2n+1) = \phi_2(n)$. Note that ϕ is a computable 1-1 function.

If n is even, then $B = B_1 \cap A_2$, where A_2 is a Π^0_1 subset of B_1. Now let $A = \{n : \phi_1(n) \in A_2\}$. Then ϕ_1 maps A onto B, as desired. □

This lemma leads to the following proposition.

Proposition 9.3. *For every n-c.e. injection structure $\mathcal{A}$, there exist a Π^0_1 structure $\mathcal{B}$ and a computable injection $\phi : \mathbb{N} \to \mathbb{N}$ that maps $\mathcal{B}$ onto $\mathcal{A}$.*

Proof. Let $\mathcal{A} = (A, f)$ be an n-c.e. structure, so that f is a partial computable function such that the restriction of f to A is an injection. Let B be a Π^0_1 set, and let ϕ be a computable 1-1 function mapping B onto A. Then define $\mathcal{B} = (B, g)$ by letting $g(b) = \phi^{-1}(f(\phi(b)))$. Observe that for $b \in B$, $\phi(b) \in A$, so $f(\phi(b)) \in A$, and thus $\phi^{-1}(f(\phi(b)))$ is defined and can be computed, since ϕ is a computable 1-1 function. Thus, the mapping $\phi : \mathcal{B} \to \mathcal{A}$ is the desired computable isomorphism. The inverse mapping ϕ^{-1} maps $\mathcal{A}$ onto $\mathcal{B}$ and is partial computable with domain A. □

Combining Proposition 9.3 with Corollary 8.6, we obtain the following result.

Corollary 9.4. *If $\mathcal{A}$ and $\mathcal{B}$ are isomorphic n-c.e. injection structures with only finitely many orbits of type ω, then $\mathcal{A}$ and $\mathcal{B}$ are Δ^0_2 isomorphic.*

Theorem 9.5. *Let K be a Σ^0_2 character.*

(a) There is a 2-c.e. injection f such that $(\mathbb{N}, f)$ has character K and infinitely many infinite orbits.

(b) If K has an s_1-function, then there is a 2-c.e. injection f such that $(\mathbb{N}, f)$ has character K and no infinite orbits.

Proof. (a) By a previous result from [2], there exists a computable equivalence structure $\mathcal{B} = (\mathbb{N}, E)$ with character K and with infinitely many infinite equivalence classes. Define the injection structure $\mathcal{A} = (\mathbb{N}, f)$ as follows. At each stage s, there will be a finite injection $f_s : \{0, 1, \ldots, s\} \to \{0, 1, \ldots, s\}$ forming a collection of finite orbits. In each orbit, the largest element will have a value $f_s(m)$, which has never changed since being defined. At stage 0, we have $f_0(0) = 0$. At stage $s+1$, there are two cases. If $E(i, s+1)$ holds for some $i \leq s$, we insert $s+1$ into the orbit of i by taking the largest element m of this orbit and making $f_{s+1}(m) = s+1$ and $f_{s+1}(s+1) = f_s(m)$. If $E(i, s+1)$ does not hold for all $i \leq s$, let $f_{s+1}(s+1) = s+1$.

It is clear that if the equivalence class of a in $\mathcal{B}$ is of finite size k, then $\mathcal{O}(a)$ in $\mathcal{A}$ will also have size k.

If the equivalence class of a is infinite, then, by the construction, if $[a] = \{a_0 < a_1 < \dots\}$, then in $\mathcal{A}$, $f(a_i) = a_{i+1}$ for all i, so that $\mathcal{O}(a)$ is infinite.

(b) In this case, $\mathcal{B} = (\mathbb{N}, E)$ with no infinite orbits exists, so the construction will produce $\mathcal{A}$ with character K and no infinite orbits. $\square$

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