Proof-Theoretic Strength of the Stable Marriage Theorem and Other Problems

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Abstract

We study the proof theoretic strength of several infinite versions of finite combinatorial theorem with respect to the standard Reverse Mathematics hierarchy of systems of second order arithmetic. In particular, we study three infinite extensions of the stable marriage theorem of Gale and Shapley. Other theorems studied include some results on partially ordered sets due to Dilworth and to Dushnik and Miller.

1 Introduction

In this paper, we shall study the proof theoretic strength of several combinatorial theorems in the standard Reverse Mathematics hierarchy of systems of second order arithmetic, RCA_0, WKL_0, ACA_0, ATR_0, etc. as introduced by Friedman and Simpson [20]. In section 2, we shall study the proof theoretic strengths of several infinite extensions of the so-called stable marriage problem introduced by Gale and Shapley [5]. We will show that there are three natural infinite extensions of the Gale-Shapley theorem for the existence of stable marriages that are equivalent to WKL_0, ACA_0, and ATR_0, respectively, over RCA_0. We should note that there are a number of papers devoted to studying the logical strength of various infinite extensions of finite combinatorial theorems in Reverse Mathematics, see for example [1, 7, 8, 9, 10, 11, 12, 19]. In particular, a closely related problem, the Philip Hall Marriage Theorem has been studied by Hirst [8]. It is rare to have one combinatorial problem where there are natural infinite versions that are equivalent to WKL_0, ACA_0, and ATR_0 over RCA_0. We should note that the results of [1, 19] show that another related theorem, namely the countable version of König's duality theorem, is equivalent to Π^1_1-CA_0 over RCA_0 and hence is proof theoretically stronger than any of the countable versions of stable marriage problem presented in this paper.

In section 3, we shall show how many standard combinatorial theorems can code up separating sets for pairs of disjoint r.e. sets. These codings allow for simple proofs that these combinatorial theorems are equivalent to WKL_0 over RCA_0. Furthermore, many of the combinatorial problems that we shall consider have an even stronger property. That is, for any recursively bounded tree T, there is an instance I_T of the problem P such that there is an effective 1:1 correspondence between the set of infinite paths through T and the set of solutions to I_T. However, there are other combinatorial problems that do not have this stronger property. For example, we show that there is a combinatorial matching problem which is equivalent to WKL_0 over RCA_0 and which has the property that for any instance of the problem, there are either finitely many or $2^{\aleph_0}$ solutions; hence such a matching problem cannot encode the set of paths through a recursive tree which has a countably infinite set of paths. Thus there are combinatorial problems which have the same proof theoretic strength over RCA_0, but the structure of the set of solutions to instances of the problems are radically different. These results show that there are inherent limitations in the equivalence relation of equivalent proof theoretic strength over RCA_0. Namely, such an equivalence relation groups together combinatorial problems that can be easily distinguished via natural properties of the structure of the possible sets of solutions to instances of the problems.

Before proceeding with our analysis of various combinatorial problems, we shall state three theorems concerning the equivalence of various results over RCA_0 that we will use in this paper. The proofs of all these results can be found in Simpson's book [20].

Theorem 1.1 *The following are equivalent over RCA_0.*

1. ACA_0.
2. *If $f : \mathbb{N} \to \mathbb{N}$ is an injection, then the range of f is a set.*
3. *(Konig's Lemma) If T is an infinite, finitely branching tree, then there is an infinite path through T.*

Theorem 1.2 *The following are equivalent are equivalent over RCA_0*

1. WKL_0*, i.e. every infinite tree $T \subset \{0,1\}^{<\mathbb{N}}$ has an infinite path.*
2. *(Bounded Konig's Lemma) If $T \subset \mathbb{N}^{\mathbb{N}}$ is an infinite tree and there is a function g such that for all $\tau \in T$ and all $m < |\tau|$, $\tau(m) < g(m)$, then T has an infinite path.*
3. *The completeness theorem for propositional logic with countably many variables.*
4. *The compactness theorem for propositional logic with countable many variables.*

5. *(Σ^0_1 separation) Let $\phi_i(n)$, $i = 0, 1$ be Σ^0_1 formulas in which X is does not occur freely. If $\neg\exists n(\phi_0(n) \wedge \phi_1(n))$, then*

$$\exists X \forall n((\phi_0(n) \rightarrow n \in X) \wedge (\phi_1(n) \rightarrow n \notin X)).$$

As pointed out in [20], we can define the notion of a *well-ordering* in RCA_0 as follows. Let $X \subset \mathbb{N}^2$ be reflexive. We say X is *well founded* if it has no infinite descending sequence. That is, there is no $f : \mathbb{N} \rightarrow field(X)$ such that $f(n+1) <_X f(n)$ for all $n \in \mathbb{N}$. We say that X is a *countable linear ordering* if it is a reflexive linear ordering of its field. That is, if the following hold.

1. $\forall i \forall j \forall k(i \leq_X j \wedge j \leq_X k \rightarrow i \leq_X k)$,
2. $\forall i \forall j(i \leq_X j \wedge j \leq_X i \rightarrow i = j)$ and
3. $\forall i \forall j(i, j \in field(X) \rightarrow i \leq_X j \vee j \leq_X i)$.

We say that X is a *countable well ordering* if it is both well founded and a countable linear ordering. We let $WO(X)$ be the formula that expresses that X is a well ordering. The following is proved in [20].

Theorem 1.3 *The following are equivalent over RCA_0.*

1. *Arithmetic transfinite recursion, ATR_0.*
2. *CWO, the comparability of countable well orderings, i.e. the statement*

$$\forall X \forall Y((WO(X) \wedge WO(Y)) \rightarrow ((|X| \leq |Y|) \vee (|Y| \leq |X|)$$

   where $|X| \leq |Y|$ mean that there is an order preserving isomorphism from $(field(X), X)$ onto an initial segment of $(field(Y), Y)$.

3. *Σ^1_1 Separation. Let $\phi_i(n)$, $i = 0, 1$ be Σ^1_1 formulas in which X is does not occur freely. If $\neg\exists n(\phi_0(n) \wedge \phi_1(n))$, then*

$$\exists X \forall n((\phi_0(n) \rightarrow n \in X) \wedge (\phi_1(n) \rightarrow n \notin X)).$$

2 The Proof-Theoretic Strength of Infinite Extensions of the Stable Marriage Theorem

The *Stable Marriage Problem* was introduced by Gale and Shapley [5] in 1962 and is related to the problem of college admissions. They gave an algorithm for solving the finite problem, which was later discovered to have been used in the matching of graduate medical students with hospitals since 1952. Other variants of the problem have been studied in computer science, economics, game theory and operations research. For example, Knuth [16] related the stable marriage problem to finding the shortest path on a graph and to searching a table by hashing.

An instance of the stable marriage problem of size n is consists of two disjoint finite sets $B = \{b_1, b_2, \ldots, b_n\}$ (the set of *boys*) and $G = \{g_1, g_2, \ldots, g_n\}$ (the set of *girls*). In addition, each boy b_i has a ranking or a linear ordering $<_i$ of G which reflects his preference for the girls that he wants to marry. That is, if $g_j <_i g_k$, then b_i would prefer to marry g_j over g_k. Similarly each girl g_j has a ranking or linear ordering $<^j$ of B which reflects her preferences in the boys she would like to marry. A *matching* (or *marriage*) M is a 1:1 correspondence between B and G. We say that b and g are *partners* in M if they are matched in M and write $p_M(b) = g$ and also $p_M(g) = b$. A matching M is *unstable* if there is a pair (b, g) from $B \times G$ such that b and g are not partners in M but b prefers g to $p_M(b)$ and g prefers b to $p_M(g)$. Such a pair (b, g) is said to *block* the matching M and is called a *blocking pair* for M. A matching M is *stable* if there is no blocking pair for M.

One can also consider a stable marriage problem where the two finite sets B and G have a different cardinalities. For example, suppose $|B| < |G|$. In this case, a matching M is a 1:1 correspondence between B and some subset G' of G of cardinality $|B|$. We say (b, g) a blocking pair for M if b prefers g over $p_M(b)$ and either $g \notin G'$ or g prefers b over $p_M(g)$. Once again a matching is stable if there is no blocking pair for M. The definition of blocking pairs and stable marriages in the case where $|G| < |B|$ are defined similarly.

The result of Gale and Shapley is that any finite marriage problem has a solution. In fact, they give an algorithm which produces a solution in n stages and takes $\leq o(n^3)$ steps. As we want to extend this algorithm to the infinite case, we will give some details here. Let $B = \{b_1, b_2, \ldots\}$ and $G = \{g_1, g_2, \ldots\}$ and assume that $|B| \leq |G|$.

The matching M is produced in stages M_s so that b_t always has a partner at each stage $s \geq t$ and $p_{M_t}(b_t) \leq_t p_{M_{t+1}}(b_t) \leq_t \cdots$. On the other hand, for each $g \in G$, if g has a partner at stage t, then g will have a partner at each stage $s \geq t$ and $p_{M_t}(g) \geq^t p_{M_{t+1}}(g) \geq^t \cdots$. Thus as s increases, the partners of b_t become less preferable and the partners of g become more preferable.

The Gale-Shapley Algorithm

Stage 1. At stage 1, b_1 chooses the first girl g in his preference list and we set $M_1 = \{(b_1, g)\}$.

Stage s+1. At the end of stage s, assume that we have produced a matching $M_s = \{(b_1, g_{i(1,s)}), \ldots, (b_s, g_{i(s,s)})\}$. We will say that partners in M_s are "engaged". The idea is that at stage $s+1$, b_{s+1} will try to get a partner by "proposing" to the girls in G in his order of preference. When b_{s+1} proposes to a girl g_j, g_j accepts his proposal if either g_j is not currently engaged or is currently engaged to a boy b such that $b_{s+1} <^j b$. In the case where g_j prefers b_{s+1} over her current partner b, then g_j breaks off an engagement with b and b then has to search for a new partner. To be more precise, we begin stage $s+1$ by letting $M = M_s$ and letting $b^* = b_{s+1}$. Then we apply the following routine. We have b^* propose to the girls in order of his preference until one accepts. Here

g will accept the proposal as long as she is either not engaged or prefers b^* to her current partner $p_M(g)$. Then we add (b^*, g) to M and proceed according to one of the following two cases.

Case I: If g was not engaged, then we terminate the procedure and let $M_{s+1} = M \cup \{(b^*, g)\}$.

Case II: If g was engaged to b, then we set $M = (M - \{(b, g)\}) \cup \{(b^*, g)\}$ and $b^* = b$ and we continue.

It is easy to prove that there is exactly one girl that was not engaged at step s but is engaged at stage $s+1$ and that, for each girl g_j that is engaged in M_s, g_j will be engaged in M_{s+1} and that $p_{M_{s+1}}(g_j) \leq^j p_{M_s}(g_j)$. That is, for any girl g_j, once she becomes engaged, she will remain engaged and her partners will only gain in preference as the stages proceed. Moreover, it is easy to see that each b need only propose at most once to each g during stage $s+1$, which gives an upper bound of $(s+1)^2$ steps in the procedure.

We note that if $|B| = n$, that at the end of stage n, M_n will be a stable marriage. That is, suppose (b_i, g_j) is a blocking pair for M_n. We claim that it must be the case that b_i proposed to g_j during the procedure. That is, b_i proposes to the girls in order of his preference and hence if b_i never proposed to g_j, then b_i must prefer $p_M(b_i)$ over g_j which would violate the assumption that (b_i, g_j) is blocking pair. But now consider the time at which b_i proposed to g_j. Then either g_j first accepted and then moved to a more preferred partner or g_j did not accept because she preferred her current partner to b_i. Since every time g_j changes partners, she moves to a boy which is more preferred than her current partner, it would follow that $p_M(g_j) <^j b_i$ which again contradicts that fact that (b_i, g_j) is a blocking pair for M_n. Thus there can be no such blocking pair and hence M_n is a stable matching.

In the case where $|G| < |B|$, we can simply reverse the roles of the girls and boys in the above algorithm. Finally, it is easy to check that this proof only requires Σ^0_1 induction and hence can be carried out in RCA_0.

Next we will formulate several infinite versions of the Gale-Shapley Theorem for the existence of stable matchings. Throughout this paper, our concern will be with countable societies so that when we say that a society is infinite, we mean that either the set of boys B, the set of girls G or both are countably infinite.

An infinite instance of the stable marriage problem consists of a countably infinite set of boys $B = \{b_0, b_1, b_2, \dots\}$, a countably infinite set of girls $G = \{g_0, g_1, g_2, \dots\}$, preference orders $\leq_i$ for each b_i among G, and preference orders $\leq^j$ for each g_j among B. The instance is *recursive* if B and G are recursive sets of natural numbers ω and the orderings $\leq_i$ and $\leq^j$ are uniformly recursive. The instance is *listed* if each ordering $\leq_i$ and $\leq^j$ has order type ω and is effectively listed if there are functions $P(i, n)$ and $Q(j, n)$ such that for all i and j, $g_{P(i,0)} <_i g_{P(i,1)} <_i g_{P(i,2)} <_i \dots$ enumerates the preference order of b_i and, similarly,

$b_{Q(i,0)} <^j b_{Q(i,1)} <^j b_{Q(i,2)} <^j \ldots$ enumerates the preference order of g_j.

For an infinite instance of the stable marriage problem, a *stable matching* M consists of either a 1:1 mapping $M : B \to G$ such that there is no blocking pair for M or a 1:1 mapping $M : G \to B$ such that there is no blocking pair for M. Thus in a stable matching, either all the boys have a partner or all the girls have a partner. A stable matching in which every boy has a partner and every girl has a partner is called a *symmetric stable matching.*

Theorem 2.1 *Suppose that $\langle B, G, \{<_i\}_{b_i \in B}, \{<^j\}_{g_j \in G}\rangle$ is an infinite instance of the stable marriage problem where for all i and j, the order type induced by the orderings $<_i$ and $<^j$ is ω. Then there is a stable matching for M.*

Proof: We claim that we can simply extend the Gale-Shapley algorithm to the infinite case. That is, consider the following construction.

Stage 0. At stage 0, b_0 chooses the first girl g in his preference list and we let $M_0 = \{(b_0, g)\}$.

Stage s+1. At the end of stage s, assume that we have produced a matching $M_s = \{(b_1, g_{i(1,s)}), \ldots, (b_s, g_{i(s,s)})\}$. We start stage $s+1$ by letting $M = M_s$ and letting $b^* = b_{s+1}$. Then we apply the following routine. We have b^* propose to the girls in order of his preference until one accepts. Here g will accept the proposal as long as she is either not engaged or prefers b^* to her current partner $p_M(g)$. We then have two cases.

Case I: If g was not engaged, then we terminate the procedure and let $M_{s+1} = M \cup \{(b^*, g)\}$.

Case II: If g was engaged to b, then we set $M = (M - \{(b, g)\}) \cup \{(b^*, g)\}$ and $b^* = b$ and we continue.

It is easy to prove by induction that there is exactly one girl who is not engaged at stage s but becomes engaged at stage $s+1$ and that, for each girl g_j that is engaged in M_s, g_j will be engaged in M_{s+1} and that $p_{M_{s+1}}(g_j) \leq^j p_{M_s}(g_j)$. Thus, for any girl g_j, once she becomes engaged, she will remain engaged and her partners will only gain in preference as the stages proceed. Hence there will be some finite stage t_{g_j} and a boy $b(g_j)$ such that for all $s \geq t_{g_j}$, $(b(g_j), g_j) \in M_s$. Thus we let

$$M = \{(b(g_j), g_j) : \text{there is a stage } s \text{ such that } g_j \text{ is engaged at stage } s\}.$$

We claim that M is a stable matching. First observe that if there is a boy b_i such that b_i has no partner relative to M, then there must be infinitely many stages $s \geq i$ such that b_i was engaged to $p_{M_s}(b_i)$ at the end of stage s, but is not engaged to $p_{M_s}(b_i)$ at the end of stage $s+1$. It follows that b_i must have proposed to every girl in G since every time b_i loses a partner, b_j proposes to the next girl on his preference list. But this means that every girl

g_j eventually becomes engaged and hence every girl G has a partner under M. It thus follows that either (i) every boy has a partner under M or (ii) every girl has a partner under M. M is a 1:1 correspondence because at each stage s, M_s is a 1:1 correspondence. Finally to see that M is a stable matching, suppose for a contradiction that (b_i, g_j) is a blocking pair for M. There are two cases. First if b_i has no partner under M, then we can argue as above that there must have been a stage s such that b_i proposed to g_j. But then either g_j preferred her current partner to b_i at the time b_i proposed or g_j accepted b_i's proposal and then latter switched to a new partner which she preferred over b_i. In either case, there is a stage t such that g_j preferred $p_{M_t}(g_j)$ over b_i. But then we know that $p_{M_t}(g_j) \geq^j p_{M_{t+1}}(g_j) \geq^j p_{M_{t+2}}(g_j) \geq^j \ldots$. Thus $b_i >^j p_{M_t}(g_j) \geq^j p_M(g_j)$ and hence (b_i, g_j) could not be a blocking pair for M. Hence it must be the case that b_i has a partner relative to M. However, by the same argument above, there can not be a stage at which b_i proposed to g_i since the fact that b_i is not matched to g_j, implies that either g_j rejected b_i's proposal or g_j accepted b_i's proposal but later switched partners. But then we would be able to conclude that g_j prefers $p_M(g_j)$ over b_i. Thus it must be the case that b_i never proposed to g_j. But this means that for all stages $s \geq i$, $p_{M_s}(b_i) <_i g_j$. Hence $p_M(b_i) <_i g_j$ so that (b_i, g_j) is not a blocking pair for M.

Thus there can be no blocking pair for M and M is stable matching. $\Box$

It is important to note that even a recursive infinite instance of the stable marriage problem may have no symmetric solution. For example, suppose that each b_{n+1} prefers g_n first and that each g_n prefers b_{n+1} first. Then any stable marriage must match all of these pairs which leaves no partner for b_0. In this example, the Gale-Shapley algorithm will have $p_{M_s}(b_0) = g_s$ for each s and thus will not converge. In fact, this is a more general phenomenon, as indicated by the following result.

Theorem 2.2 *Let T be a recursive tree contained in $\omega^{<\omega}$. Then there is a recursive instance of the stable marriage problem $\langle B, G, \{<_i\}_{b_i \in B}, \{<^j\}_{g_j \in G}\rangle$ such that there is an effective 1:1 correspondence between the set of infinite paths through T and the set of symmetric stable matchings of $\langle B, G, \{<_i\}_{b_i \in B}, \{<^j\}_{g_j \in G}\rangle$.*

Proof: We let $\emptyset$ denote the empty sequence which is in every tree $T \subseteq \omega^{<\omega}$ by definition. We let $B = \{b_\eta : \eta \in T\}$ and $G = \{g_\eta : \eta \in T - \{\emptyset\}\}$. Given two sequences $\alpha, \beta \in \omega^{<\omega}$, we write $\alpha^\frown\beta$ for the concatenation of α and β and we write $\alpha^\frown i$ for $\alpha^\frown(i)$. For any $\alpha, \beta \in \omega^{<\omega}$, we let $|\alpha|$ denote the length of α and we write $\alpha \sqsubseteq \beta$ if α is an initial segment of β.

To define the preference orderings $<_\eta$ and $<^\eta$, we first fix some recursive ω ordering $\prec$ of the nodes of T. For example, we say that $\emptyset \prec \eta$ for all $\eta \in T - \{\emptyset\}$ and define $\eta = (\eta_0, \ldots, \eta_k) \prec \gamma = (\gamma_0, \ldots, \gamma_n)$ if and only if either (i) $\sum_{i=0}^k \eta_i + 1 < \sum_{i=0}^n \gamma_i + 1$ or (ii) $\sum_{i=0}^k \eta_i + 1 = \sum_{i=0}^n \gamma_i + 1$ and η is lexicographically less than γ. We shall specify a preference ordering $<_\eta$ either by giving a recursive sequence $S_\eta = \alpha^0, \alpha^1, \ldots$ of nodes of T without repetitions or by giving a pair $\langle S_\eta, \delta\rangle$ where $S_\eta = \alpha^0, \alpha^1, \ldots$ is a recursive sequence of nodes of T without repetitions and δ is a node of T which is not in S_η. That is, in the first case, $<_\eta$ is the ordering defined by setting $g_\beta <_\eta g_\gamma$ if and only if either (a) $\beta \in S_\eta$ and $\gamma \in T - S_\eta$, (b) $\beta = \alpha_s$ and $\gamma = \alpha_t$ and $s < t$, or (c) $\beta, \gamma \in T - S_\eta$ and $\beta \prec \gamma$. In the second case, $<_\eta$ is the ordering defined by setting $g_\beta <_\eta g_\gamma$ if and only if either (a) $\beta \in S_\eta \cup \{\delta\}$ and $\gamma \in T - (S_\eta \cup \{\delta\})$, (b) $\beta = \alpha_s$ and $\gamma = \alpha_t$ and $s < t$, (c) $\beta = \alpha_s$ and $\gamma = \delta$ or (c) $\beta, \gamma \in T - (S_\eta \cup \{\delta\})$ and $\beta \prec \gamma$. We shall specify the preference ordering $<^\eta$ in a similar manner.

This given, we define the orderings $<_\eta$ and $<^\eta$ as follows.

1. We let $<_\emptyset$ be the ordering determined by $S_\emptyset = (i_0), (i_1), \ldots$ where $i_0 < i_1 < \ldots$ consists of the set of all i such that (i) is in T. (Thus $b_\emptyset$'s preference order starts out with the girls $g_{(i_0)}, g_{(i_1)}, \ldots$, followed by the rest of girls in G in the standard order induced by $\prec$.)

2. If $\eta \neq \emptyset$, then we let $<_\eta$ be the ordering determined by the pair $\langle S_\eta, \eta \rangle$ where $S_\eta = \eta^\frown i_0, \eta^\frown i_1, \ldots$ and $i_0 < i_1 < \ldots$ consists of the set of all i such that $\eta^\frown i$ are in T. (Thus b_η's preference order starts out with the girls $g_{\eta^\frown i_0}, g_{\eta^\frown i_1}, \ldots$ followed by g_η and then followed by the rest of girls in G in the standard order induced by $\prec$.)

3. For all $\eta^\frown i \in T$, we let $S^{\eta^\frown i} = \eta^\frown i, \eta$. Thus $g_{\eta^\frown i}$'s preference order starts out with the boys $b_{\eta^\frown i}, b_\eta$ and then is followed by the rest of the boys in B in the standard order induced by $\prec$.)

It is easy to see that since T is a recursive tree, $\langle B, G, \{<_i\}_{b_i \in B}, \{<^j\}_{g_j \in G} \rangle$ is a recursive instance of the stable marriage problem. Moreover, we can assume that T is an infinite tree since otherwise, $|B| > |G|$ so that automatically there can be no symmetric stable matching.

Let $\pi = (\pi_0 = \emptyset, \pi_1, \pi_2, \ldots)$ be some infinite path through T. That is, for each i, $|\pi_i| = i$ and for all $i < j$, $\pi_i \sqsubset \pi_j$. Then let M_π be the matching

$$\{(b_{\pi_n}, g_{\pi_{n+1}}) : n \geq 0\} \cup \{(b_\eta, g_\eta) : \eta \in T - \pi\}.$$

We claim that M_π is symmetric stable matching. That is, suppose for a contradiction that (b_α, g_β) is a blocking pair for M_π. It cannot be that $\beta \notin \pi$ since otherwise $(b_\beta, g_\beta) \in M_\pi$ and b_β is the first choice of g_β. Thus it must be the case that $\beta = \pi_n$ for some $n > 0$. But in that case, $g_\beta = g_{\pi_n}$ is married to her second most preferred partner $b_{\pi_{n-1}}$. Thus the only way that (b_α, g_β) can be a blocking pair is if $b_\alpha = b_{\pi_n}$. However b_{π_n} is matched to $g_{\pi_{n+1}}$ in M_π which he prefers over g_{π_n}. Thus there can be no blocking pair and for each infinite path π through T, M_π is a stable matching.

We claim that every symmetric stable matching M is of the form M_π for some infinite path through T. That is, suppose that M is a symmetric stable matching for $\langle B, G, \{<_i\}_{b_i \in B}, \{<^j\}_{g_j \in G} \rangle$. First we claim that $b_\emptyset$ must be married to $g_{(i)}$ for some $i \in T$. That is, suppose for a contradiction that $p_M(b_\emptyset) = g_{(\eta_1, \ldots, \eta_k)}$ for some $k > 1$. Then we claim that for all $j < k$, $b_{(\eta_1, \ldots, \eta_j)}$ must be married to $g_{(\eta_1, \ldots, \eta_j)}$. That is, since $g_{(\eta_1)}$'s preference list starts out $b_{(\eta_1)}, b_\emptyset$ and $b_\emptyset$ prefers $g_{(\eta_1)}$ over $g_{(\eta_1, \ldots, \eta_k)}$, $(b_\emptyset, g_{(\eta_1)})$ would be a blocking pair of M unless

$(b_{(\eta_1)}, g_{(\eta_1)}) \in M$. Next assume by induction that $p_M(b_{(\eta_1,\ldots,\eta_i)}) = g_{(\eta_1,\ldots,\eta_i)}$ for $i < s$. Then the preference list of $g_{(\eta_1,\ldots,\eta_s)}$ starts out with $b_{(\eta_1,\ldots,\eta_s)}, b_{(\eta_1,\ldots,\eta_{s-1})}$ and $b_{(\eta_1,\ldots,\eta_{s-1})}$ prefers $g_{(\eta_1,\ldots,\eta_s)}$ over $g_{(\eta_1,\ldots,\eta_{s-1})}$. Hence $(b_{(\eta_1,\ldots,\eta_{s-1})}, g_{(\eta_1,\ldots,\eta_s)})$ is blocking pair for M unless $p_M(b_{(\eta_1,\ldots,\eta_s)}) = g_{(\eta_1,\ldots,\eta_s)}$. Thus it follows by induction that $b_{(\eta_1,\ldots,\eta_{k-1})}$ must be matched with $g_{(\eta_1,\ldots,\eta_{k-1})}$ in M. But then $b_{(\eta_1,\ldots,\eta_{k-1})}$ prefers $g_{(\eta_1,\ldots,\eta_k)}$ over $g_{(\eta_1,\ldots,\eta_{k-1})}$ and $g_{(\eta_1,\ldots,\eta_k)}$ prefers $b_{(\eta_1,\ldots,\eta_{k-1})}$ over $b_\emptyset$ which means that $(b_{(\eta_1,\ldots,\eta_{k-1})}, g_{(\eta_1,\ldots,\eta_k)})$ is a blocking pair for M. Thus $b_\emptyset$ must be married to some $g_{(i)}$ for some $(i) \in T$.

Next, we claim that every boy b_η such that $\eta \neq \emptyset$ must be married to g_η or to a girl $g_{\eta^\frown i}$ for some $\eta^\frown i$ in T. That is, if b_η is not married to g_η, then b_η is g_η's first choice and hence (b_η, g_η) would be a blocking pair for M unless b_η is married to a girl g which he prefers over g_η. However the only girls that b_η prefers over g_η are of the form $g_{\eta^\frown i}$ where $\eta^f rowni$ is in T. Thus since each boy b_η is married to either g_η or some girl $g_{\eta^\frown i}$ and M is symmetric, then each girl $g_{\beta^\frown i}$ must be married to either $b_{\beta^\frown i}$ or to b_β. Now let π be the set of nodes η such that b_η is married to a girl of the form $g_{\eta^\frown i}$ for some i. We claim that π is an infinite path through T and hence $M = M_\pi$. That is, suppose that there exist distinct nodes α and β in π such that neither $\alpha \sqsubset \beta$ nor $\beta \sqsubset \alpha$. Then let γ be the longest common initial segment of α and β and suppose that $\alpha = \gamma^\frown \mu$ and $\beta = \gamma^\frown \nu$. Then M cannot be symmetric since the $|\gamma| + |\mu| + |\nu| - 1$ girls of the form g_η where $\eta \sqsubseteq \alpha$ or $\eta \sqsubseteq \beta$ can only marry the $|\gamma| + |\mu| + |\nu| - 2$ boys b_δ such that $\delta \sqsubset \alpha$ or $\delta \sqsubset \beta$. Thus all the nodes in π must be comparable with respect to $\sqsubseteq$. Now suppose that $\eta \in \pi$ but there is no node of the form $\eta^\frown i$ in π. But this means that $(b_{\eta^\frown i}, g_{\eta^\frown i}) \in M$ for all i such that $\eta^\frown i \in T$. But this contradicts the fact that b_η must be matched to some $g_{\eta^\frown i}$ for some i. Thus it must be the case that π is an infinite path through T. $\Box$

Let ϕ_e be the partial recursive function computed by the e-th Turing machine and $W_e = \{n : \phi_e(n) \text{ is defined}\}$. Then we have the following corollary of Theorem 2.2.

Corollary 2.3 *The set U of all $\langle a, b, c, d\rangle$ such that $\langle W_a, W_b, \{W_{\phi_c(n)}\}_{n \in W_a}, \{W_{\phi_d(n)}\}_{n \in W_b}\rangle$ is a recursive instance of the stable marriage problem which has a symmetric stable matching is Σ^1_1 complete.*

Proof: It is easy to see that U is a Σ^1_1 set. On the other hand, it is well known that the set $InfPath$ of indices of primitive recursive functions which are the characteristic functions of trees in $\omega^{<\omega}$ which have an infinite path through them is Σ^1_1 complete. Our proof of Theorem 2.2 shows that $InfPath$ is 1:1 reducible to U so that U is Σ^1_1 complete. $\Box$

We note that the orderings used in Theorem 2.2 are all well orderings of either order type ω or $\omega + \omega$. If fact, our next results will show that as long as the preference orderings are well orderings, then any infinite instance of the stable marriage problems has a stable matching. We start with a very simple version of this result which will be relevant for our results on Reverse Mathematics.

Theorem 2.4 *Suppose that $\langle B, G, \{<_i\}_{b_i \in B}, \{<^j\}_{g_j \in G}\rangle$ is an infinite instance of the stable marriage problem where for all i, $<_i$ and $<^i$ are countable well orderings and for all i and j, $<_i = <_j$ and $<^i = <^j$. Then there is a unique stable matching for M.*

Proof: Suppose that $<_i$ has order type β and $<^i$ has order type α for all i. Since α and β are countable ordinals, then either $\alpha \leq \beta$ or $\beta \leq \alpha$. Suppose that $\alpha \leq \beta$. Then we can relabel the boys in $B = \{b_\eta : \eta \in \alpha\}$ so that for all γ and δ in α, $b_\gamma <^i b_\delta \iff \gamma < \delta$. Similarly we can relabel the girls in $G = \{g_\eta : \eta \in \beta\}$ so that for all γ and δ in β, $g_\gamma <_i g_\delta \iff \gamma < \delta$.

This given, it is then easy to see that $M_\alpha = \{(b_\eta, g_\eta) : \eta \in \alpha\}$ is a stable matching and that it is the only stable matching. That is, suppose that M is a stable matching. Now if $(b_0, g_0) \notin M$, then (b_0, g_0) will be a blocking pair for M since g_0 is b_0's first preference and b_0 is g_0's first preference. Now assume by induction that for all $\gamma < \delta$, $(b_\gamma, g_\gamma) \in M$. Then we claim that (b_δ, g_δ) must be in M since otherwise (b_δ, g_δ) would be a blocking pair for M. That is, if (b_δ, g_δ) is not in M, then either b_δ has no partner in M or is married to some g_η such that $\delta < \eta$. Similarly, either g_δ has no partner in M or is married to some b_η such that $\delta < \eta$. It is then easy to see that under such circumstances, (b_δ, g_δ) would be a blocking pair for M. Thus it follows that $M = M_\alpha$.

In the case where $\beta < \alpha$, we can show that $M_\beta = \{(b_\eta, g_\eta) : \eta \in \beta\}$ is the unique stable matching. $\square$

Finally, our next result will show that if the orderings $<_i$ and $<^i$ are well orderings for every i, then we can show that there is a transfinite version of the Gale-Shapley algorithm that will produce a stable matching.

Theorem 2.5 *Suppose that $\langle B, G, \{<_i\}_{b_i \in B}, \{<^j\}_{g_j \in G}\rangle$ is an infinite instance of the stable marriage problem where for all i and j, the orderings $<_i$ and $<^j$ are well-orderings. Then there is a stable matching for M.*

Proof: Let $B = \{b_0, b_1, b_2, \dots\}$ and $G = \{g_0, g_1, g_2, \dots\}$.

We define the stable matching M by transfinite induction. For any ordinal α, we will have a set $M^\alpha \subset B \times G$ of partners at stage α and a set $R^\alpha \subset B \times G$ of *rejected proposals*. At stage 0, $R^0 = \emptyset$ and $M^0 = \{(b_0, g)\}$, where g is the least girl relative to the ordering $<_0$. In general, at any successor stage $\alpha+1$, we check whether there is a $b \in B$ without a partner at stage α such that there is some $g \in G$ such that $(b, g) \notin R^\alpha$. If not, then we will terminate the procedure and M^α will be the desired stable matching. For the sake of completeness, we let $M^{\alpha+1} = M^\alpha$ and $R^{\alpha+1} = R^\alpha$ and terminate the procedure. If there is such a boy, then let $b = b_i$ where i is the least k such that b_k has no partner at stage M^α and there is some $g \in G$ such that $(b_k, g) \notin R^\alpha$. Then let $g = g_j$ be the $\leq_i$-least member of $G - \{g : (b, g) \in R^\alpha\}$. Then at stage $\alpha + 1$, b proposes to g with three possible outcomes. First if g has no partner in M_α, then we set $M_{\alpha+1} = M_\alpha \cup \{(b, g)\}$ and $R_{\alpha+1} = R_\alpha$. Second, if g has a partner in M_α but $p_{M_\alpha}(g) <^j b$, then we set $M_{\alpha+1} = M_\alpha$ and $R_{\alpha+1} = R_\alpha \cup \{(b, g)\}$. Third, if $b' = p_{M_\alpha}(g) >^j b$, then we set $M^{\alpha+1} = (M^\alpha \cup \{(b, g)\}) - \{(b', g)\}$

and $R^{\alpha+1} = R^\alpha \cup \{(b', g)\}$. For limit ordinals λ, we let $R^\lambda = \cup_{\alpha<\lambda} R^\alpha$ and $M^\alpha = lim_{\alpha \to \lambda} M^\alpha$. By this limit, we mean that $(b, g) \in M^\lambda$ if and only if there exists $\beta < \lambda$ such that $(b, g) \in M^\alpha$ for all $\alpha > \beta$.

Next observe that for $\alpha < \beta$, $Range(M^\alpha) \subset Range(M^\beta)$ and for any $g_j \in Range(M^\alpha)$, $p_{M_\beta}(g_j) \leq^j p_{M_\alpha}(g_j)$. It then easily follows by cardinality arguments, that there must be a countable ordinal γ such that $Dom(M^{\gamma+1}) = Dom(M^\gamma)$, $Range(M^{\gamma+1}) = Range(M^\gamma)$ and $R^{\gamma+1} = R^\gamma$.

We claim that $M = M^\gamma$ is a stable matching. It is clear from the construction that M is a partial matching, that is, each $b \in B$ is paired with at most one $g \in G$ and vice versa. Suppose by way of contradiction that there is a blocking pair (b_i, g_j) for M. If b_i has no partner in M, then it must be the case that $(b_i, g_j) \in R_\gamma$. But the only way that (b_i, g_j) can turn up in some R_γ is if there is some stage $\alpha + 1$ with $\alpha < \gamma$ and $(b_i, g_j) \in R_{\alpha+1} - R_\alpha$. But this means that g_j has a partner in $M_{\alpha+1}$ and $p_{M_{\alpha+1}}(g_j) <^j b_i$. It follows that $p_{M_\gamma}(g_j) \leq^j p_{M_{\alpha+1}}(g_j) <^j b_i$. Thus b_i must have a partner in M_α. But then we can argue exactly as above that it cannot be the case the $(b_i, g_j) \in R_\gamma$. However if $(b_i, g_j) \notin R_\gamma$, then it is easy to see that $p_{M_\gamma}(b_i) <_i g_j$. Thus, in fact, (b_i, g_j) is not a blocking pair for M. □

Next we want to consider the proof theoretic strengths of the Theorems 2.1, 2.4 and 2.5 in the standard systems of second order arithmetic considered in Reverse Mathematics.

First we observe that we can express the concept of having order type ω in a weak second order system by just mimicking the axioms for the natural numbers. To be more precise, the preference order of $<_i$ has order type ω if the following are all true.

1. $WO(<_i)$, i.e., $<_i$ is a well ordering, and
2. if g_k is not the most preferred partner of b_i, then there exists some $g_j <_i g_k$ such that g_k is the most preferred partner of b_i after g_j.

We shall say that an infinite instance of the stable marriage problem is $\mathbb{N}$-*listed* if $B, G \subseteq \mathbb{N}$ and all the orderings $<_i$ for $b_i \in B$ and all the orderings $<^i$ for $g_i \in G$ are of order type ω. Moreover, we assume that we can uniformly compute B, G, and the orderings $<_i$ and $<^j$ for all $i \in B$ and $j \in G$. That is, we assume that there is a single recursive set $D \subseteq \mathbb{N}^2$ such that $B = \{n : \langle 0, n\rangle inD\}$, $G = \{n : \langle 1, n\rangle \in D\}$, and for all $i \in B$ and $j \in G$, $<_i = \{\langle a, b\rangle : \langle\langle a, b\rangle, 2(i+1)\rangle \in D\}$ and $<^j = \{\langle a, b\rangle : \langle\langle a, b\rangle, 2(j+1)+1\rangle \in D\}$.

Theorem 2.6 *(RCA_0) The following are equivalent:*

1. *ACA_0;*
2. *Any $\mathbb{N}$-listed stable marriage problem has a solution.*

Proof: To prove that (1) implies (2), we use the Gale-Shapley algorithm. Given that the stable marriage problem is $\mathbb{N}$-listed, we use ACA_0 to find a listing for each $\leq_i$ and $\leq^j$. That is, $P(i, 1)$ is the unique g_j such that $(\forall k) g_j \leq_i g_k$ and then $P(i, n+1)$ is the unique g_j such that for all g_k other than $P(i, 0), \ldots, P(i, n)$, $g_j \leq_i g_k$. Thus the Gale-Shapley algorithm can be applied. Now let $p_{M_s} : \{b_1, b_2, \ldots, b_s\} \rightarrow G$ be the mapping defined by stage s and define a partial matching M by $M(b_i, g)$ if $lim_{s\rightarrow\infty} p_{M_s}(b_i) = g$. This definition is arithmetical and hence can be done in ACA_0. Our proof of Theorem 2.1 shows that M is a stable matching.

For the reverse direction, suppose that every $\mathbb{N}$-listed stable marriage problem has a solution. By Theorem 1.1, it suffices to show that the range of an arbitrary function f mapping $\mathbb{N}$ one-to-one into $\mathbb{N}$ is defined. We construct an instance of the stable marriage problem such that f may be defined from any solution. The construction is a variation of one that given by Hirst [8].

First we let $B = G = \mathbb{N}$. To avoid confusion, we shall use b_i in place of i when we are considering boys and use g_i in place of i when we are considering girls. There are several cases in the definition of the orderings $<_i$ and $<^j$ for each i and j. Much like in the proof of Theorem 2.2, we shall fix a standard ordering of the girls and boys to be the usual ordering of $\mathbb{N}$. We shall then specify a preference ordering for b_i by giving a finite sequence of girls $S_i = g_{j_0}, g_{j_1}, \ldots, g_{j_n}$. That is, we define $g <_i g'$ if and only if either

1. $g \in S_i$ and $g' \in G - S_i$,
2. $g, g' \in G - S_i$ and $g < g'$ or
3. $g = g_{j_k}$ and $g' = g_{j_l}$ and $k < l$.

We shall specify a preference order for g_j in a similar manner by specifying a sequence $S^j = b_{i_0}, b_{i_1}, \ldots, b_{i_n}$.

Case 1. For $i = 2m$, $S_i = g_{2n+1}, g_{2m}$ if there is an n such that $m < n$ and $f(n) = m$ and $S_i = g_{2m}$ if there is no n such that $n > m$ and $f(n) = m$. That is, if there is an n such that $n > m$ and $f(n) = m$, then g_{2n+1} is the most preferred girl of b_i, followed by g_{2m}, who are then followed by the rest of the girls in standard order. If there is no n such that $m < n$ and $f(n) = m$, then g_{2m} is the first girl preferred by b_i who is followed by the rest of the girls in standard order.

Case 2 If $i = 2n + 1$, $S_i = g_{2m}, g_{2n+1}$ if $f(n) = m < n$ and $S_i = g_{2n+1}$ if $f(n) \geq n$. Thus if $f(n) = m < n$, then g_{2m} is the most preferred girl of b_i, followed by g_{2n+1}, who are then followed by the rest of the girls in standard order. If $f(n) \geq n$, then g_{2n+1} is the first girl preferred by b_i and then the rest of the girls follow in standard order.

Case 3 If $j = 2m$, then $<^j$ is defined by setting $S^j = b_{2n+1}, b_{2m}$ if there is an n such that $m < n$ and $f(n) = m$ and setting $S^j = b_{2m}$ if there is no n such that $n > m$ and $f(n) = m$. Thus if $f(n) = m < n$, then b_{2n+1} is the most preferred boy of g_j, who is followed by b_{2m}, who are then followed by the rest of the boys in standard order. If there is no n such that $n > m$ and $f(n) = m$, then then b_{2m} is the first boy preferred by g_j who is then followed by the rest of the boys in standard order.

Case 4 If $j = 2n+1$, then $<^j$ is determined by the sequence $S^j = b_{2m}, b_{2n+1}$, if $f(n) = m < n$ and is determined by the sequence $S^j = b_{2n+1}$ if $f(n) \geq n$. Thus if $f(n) = m < n$, then b_{2m} is the most preferred boy of g_j, who is followed by b_{2n+1}, who are then followed by the rest of the boys in standard order. If $f(n) \geq n$, then b_{2m} is the first boy preferred by g_j who is then followed by the rest of the boys in standard order.

It is easy to see that our definitions yield an $\mathbb{N}$-listed stable marriage problem. Hence, by assumption, there must be a solution. Let M be any stable marriage for this problem. We claim that m is not in the range of f if and only if $(b_{2m}, g_{2m}) \in M$ and there does not exist an p such that $p \leq m$ and $f(p) = m$. That is, suppose there is no n such that $n > m$ and $f(n) = m$. Then g_{2m} is the most preferred girl of b_{2m} and b_{2m} is the most preferred boy of g_{2m}. Since either b_{2m} or g_{2m} must have a partner in M, (b_{2m}, g_{2m}) would be a blocking pair for M unless $(b_{2m}, g_{2m}) \in M$. One the other hand, if there is an n such that $m < n$ and $f(n) = m$, then g_{2n+1} is the most preferred girl of b_{2m} and b_{2m} is the most preferred boy of g_{2n+1} so that by the same argument, $(b_{2m}, g_{2n+1}) \in M$. Thus $(b_{2m}, g_{2m}) \in M$ if and only if there is no n such that $n > m$ and $f(n) = m$. Thus if $(b_{2m}, g_{2m}) \in M$, then m in is the range of f if and only if $m \in \{f(0), \ldots, f(m)\}$. It follows that we can define the range of f from any solution M of our stable marriage problem. $\Box$

Next we consider results about stable marriage problems which are equivalent to ATR_0. Let us say that an instance of the stable marriage problem $P = \langle B, G, \{<_i\}_{b_i \in B}, \{<^j\}_{g_j \in G}\rangle$ is *ordered* if $B, G \subseteq \mathbb{N}$ and for all i and j, $<_i = <_j$ is a well ordering and $<^i = <^j$ is a well ordering. We say that P has *levels* if $B, G \subseteq \mathbb{N}$ and there are countable well-orderings $\leq_B$ of order type β on B and $\leq_G$ of order type γ on G such that

(i) for each limit ordinal $\lambda < \gamma$, every boy $b \in B$ prefers any girl $g \in G$ with $|g|_G < \lambda$ to any girl g' with $|g'|_G \geq \lambda$ and vice versa, for each limit ordinal $\lambda < \beta$, every girl $g \in G$ prefers any boy $b \in B$ with $|b|_B < \lambda$ to any boy b' with $|b'|_B \geq \lambda$ where $|b|_B$ is the order type of $\{b' \in B : b' <_B b\}$ and $|g|_G$ is the order type of $\{g' \in G : g' <_G g\}$,

(ii) for each girl g_j and each limit ordinal λ, the restriction of the preference order $<^j$ to the set of boys in $\{b : \lambda \leq |b|_B < \lambda + \omega\}$ is of order type ω, and

(iii) for each boy b_i and each limit ordinal λ, the restriction of the preference order $<_i$ to the set of girls in $\{g : \lambda \leq |g|_G < \lambda + \omega\}$ is of order type ω.

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Theorem 2.7 *The following are equivalent over RCA_0.*

(i) ATR_0.

(ii) Any stable marriage problem with levels has a solution.

(iii) Any ordered stable marriage problem has a solution.

Proof: The implication from (ii) to (iii) is immediate.

To show that (iii) implies (i), we use the equivalence of ATR_0 with the CWO principle that any two well-orderings are comparable. Let $<_{(1)}$ and $<_{(2)}$ be two well-orderings of $\mathbb{N}$ and define the stable marriage problem P so that $B = G = \mathbb{N}$ and each $b_i \in B$ has order $<_i=<_{(2)}$ and each $g_j \in G$ has order $<^j=<_{(1)}$. This is an ordered stable marriage problem. Hence, by assumption, there must be a solution. Our proof of Theorem 2.4 shows that there is a unique stable matching in this case, which either induces an order isomorphism from $(\mathbb{N}, <_{(1)})$ onto an initial segment of $(\mathbb{N}, <_{(2)})$, or induces an order isomorphism from $(\mathbb{N}, <_{(2)})$ onto an initial segment of $(\mathbb{N}, <_{(1)})$.

To prove that (i) implies (ii), suppose that we are given

$$P = \langle B, G, \{<_i\}_{b_i \in B}, \{<^j\}_{g_j \in G}\rangle$$

satisfying the hypothesis of (ii). Let $\leq_B$ and $\leq_G$ be the associated well-orderings of B and G respectively. Suppose that β is the order type of $\leq_B$ and γ is the order type of $\leq_G$. We can assume that both B and G are infinite since otherwise, the result follows by applying a finite version of the Gale-Shapley algorithm. We shall also assume, without loss of generality, that $\beta \leq \gamma$. Then for each $\alpha < \beta$, let b_α be the boy $b \in B$ such that $|b|_B = \alpha$. Similarly for each $\delta < \gamma$, let g_δ be the girl $g \in G$ such that $|g|_G = \delta$. Finally for each λ, we let $B(\lambda) = \{b_\alpha : \alpha < \lambda\}$ and $G(\lambda) = \{g_\delta : \delta < \lambda\}$.

We then use a modified version of the Gale-Shapley algorithm to construct a stable matching for the given instance of the stable marriage problem.

Stage 0. Consider the society

$$\langle B(\omega), G(\omega), \{<_i \upharpoonright G(\omega)\}_{b_i \in B(\omega)}, \{g_j \upharpoonright B(\omega)\}_{g_j \in G(\omega)}\rangle.$$

This is a society where all the orderings are of order type ω so that we can run the infinite version of the Gale-Shapley algorithm of Theorem 2.1 to produce a matching M_0. Then by the proof of Theorem 2.1, we know that either $Dom(M_0) = B(\omega)$ or $Range(M_0) = G(\omega)$. We set $\beta_0 = 1$ if $Dom(M_0) = B(\omega)$ and $\beta_0 = 0$ otherwise. Similarly, we set $\gamma_0 = 1$ if $Range(M_0) = G(\omega)$ and $\gamma_0 = 0$ otherwise.

There are a couple of crucial properties about M_0 that follow from our proof of Theorem 2.1. Namely, if $Dom(M_0) \neq B(\omega)$, then every girl $g \in Range(M_0)$ prefers her partner in M_0 over all boys in $B(\omega) - Dom(M_0)$. Similarly, if $Range(M_0) \neq G(\omega)$, then every boy in $Dom(M_0)$ prefers his partner in M_0 over all girls in $G(\omega) - Range(G)$. Finally we claim that M_0 is in fact a stable matching for P in the sense that there can be no blocking pair (b, g) where either $b \in Dom(M_0)$ or $g \in Range(M_0)$. That is, we know that there is no blocking pair in $B(\omega) \times G(\omega)$ so that either (i) $b \in B - B(\omega)$ and $g \in Dom(M_0) \subseteq G(\omega)$

or (ii) $b \in Dom(M_0) \subseteq B(\omega)$ and $g \in G - G(\omega)$. But case (i) cannot hold because $p_{M_0}(g) \in B(\omega)$ and, by assumption, g prefers any boy in $B(\omega)$ over any boy in $B - B(\omega)$. Similarly, (ii) cannot hold since then $p_{M_0}(b) \in G(\omega)$ and, by assumption, b prefers any girl in $G(\omega)$ over any girl in $G - G(\omega)$.

Stage $\sigma + 1$. We assume that we have constructed a matching M_σ such that there are ordinals β_σ and γ_σ such that their ordinal products with ω, $\beta_\sigma\omega$ and $\gamma_\sigma\omega$ are such that either

1. $Dom(M_\sigma) = B(\beta_\sigma\omega)$ and $Range(M_\sigma) = G(\gamma_\sigma\omega)$,
2. $Dom(M_\sigma) = B(\beta_\sigma\omega) \cup B^*_\sigma$ where $B^*_\sigma \subset B(\beta_\sigma\omega + \omega) - B(\beta_\sigma\omega)$ and $Range(M_\sigma) = G(\gamma_\sigma\omega)$ or
3. $Range(M_\sigma) = G(\gamma_\sigma\omega) \cup G^*_\sigma$ where $G^*_\sigma \subset G(\gamma_\sigma\omega + \omega) - G(\gamma_\sigma\omega)$ and $Dom(M_\sigma) = B(\beta_\sigma\omega)$

Moreover we assume that M_σ is stable for P in the sense that there is no blocking pair (b, g) where either $b \in Dom(M_\sigma)$ or $g \in Range(M_\sigma)$. This implies that for any $b \in Dom(M)$, b prefers his partner in M_σ over all girls in $G - Range(M_\sigma)$. Similarly any $g \in Range(M)$, g prefers her partner in M_σ over all boys in $B - Dom(M)$. Then we have several cases.

Case 0. Either $B = B(\beta_\sigma\omega)$ or $G = G(\gamma_\sigma\omega)$.
In this case, we terminate the construction since M_σ is a stable matching for $\langle B, G, \{<_i\}_{b_i \in B}, \{<^j\}_{g_j \in G}\rangle$.

Case 1. $B - Dom(M_\sigma)$ and $G - Range(M_\sigma)$ are both infinite.

There are three subcases here.

Subcase 1A. $B(\beta_\sigma\omega + \omega)) - Dom(M_\sigma)$ and $G(\gamma_\sigma\omega + \omega) - Range(M_\sigma)$ are both infinite.

In this case, we consider the society where the boys are $B' = B(\beta_\sigma\omega + \omega) - Dom(M_\sigma)$ and the girls are $G' = G(\gamma_\sigma\omega + \omega) - Range(M_\sigma)$ and the preference orders are the restrictions of the preference orders from P. Our assumptions ensure that the restricted preference orders are all of order type ω so that we can apply the infinite version of the Gale-Shapley algorithm of Theorem 2.1. This will produce a relative stable matching M such that either $Dom(M) = B'$ or $Range(M) = G'$. We then let $M_{\sigma+1} = M_\sigma \cup M$. We also let $\beta_{\sigma+1} = \beta_\sigma + 1$ if $Dom(M) = B'$ and $\beta_{\sigma+1} = \beta_\sigma$ otherwise. Similarly, we let $\gamma_{\sigma+1} = \gamma_\sigma + 1$ if $Range(M) = G'$ and we let $\gamma_{\sigma+1} = \gamma_\sigma$ if otherwise.

Next we observe that $M_{\sigma+1}$ is stable in the sense that there is no blocking pair (b_i, g_j) for $M_{\sigma+1}$ such that either $b_i \in Dom(M_{\sigma+1})$ or $g_j \in Range(M_{\sigma+1})$. That is, suppose for a contradiction that there is such a blocking pair (b_i, g_j). First we claim that it cannot be the case that $b_i \in Dom(M_\sigma)$. For if $b_i \in Dom(M_\sigma)$, then it cannot be the case that $g_j \in Range(M_\sigma)$ since otherwise (b_i, g_j) would be a blocking pair for M_σ. However it cannot be the case that $g_j \in G - Range(M_\sigma)$ because b_i prefers his partner to all girls in $G - Range(M_\sigma)$. Next suppose that $b_i \in Dom(M)$. Thus $p_{M_{\sigma+1}}(b_i)$ is in $G' = G(\gamma_\sigma\omega + \omega) - Range(M_\sigma)$. Then it cannot be that $g_j \in G - G(\gamma_\sigma\omega + \omega)$ since by the definition of levels, b_i prefers every girl in $G(\gamma_\sigma\omega+\omega)$ to every girl in $G - G(\gamma_\sigma\omega+\omega)$. Similarly it cannot be that $g_j \in G'$ since M is a stable matching in our restricted society. Thus it must be the case that $g_j \in Range(M_\sigma)$. But then by our assumption, we know that g_j prefers her partner b in M_σ to b_i. Thus (b_i, g_j) cannot be a blocking pair. The argument that $g_j \notin Range(M_{\sigma+1})$ is similar.

Subcase 1B. $B(\beta_\sigma\omega + \omega) - Dom(M_\sigma)$ is finite.

Then (2) holds where $(B(\beta_\sigma\omega + \omega) - B(\beta_\sigma\omega)) - B^*_\sigma$ is finite and $Range(M_\sigma) = G(\gamma_\sigma\omega)$. In this case, we consider the restricted matching problem on $B' = B(\beta_\sigma\omega + 2\omega) - Dom(M_\sigma)$ and $G' = G(\gamma_\sigma\omega + \omega) - G(\gamma_\sigma\omega)$. Again we see that each preference order has order type ω. In particular the preference order of each g_j begins with a finite sequence from $B(\beta_\sigma\omega + \omega) - B^*_\sigma$ and is followed by an ω-sequence from $B(\beta_\sigma\omega + 2\omega) - B(\beta_\sigma\omega + \omega)$. The Gale-Shapley algorithm can thus be applied to produce a relative stable matching M such that either $Dom(M) = B'$ or $Range(M) = G'$. In this case, it is important to note that the finite set $B(\beta_\sigma\omega + \omega) - Dom(M_\sigma)$ is included in $Dom(M)$. To see this, first observe that since G' is infinite and $B(\beta_\sigma\omega+\omega) - Dom(M_\sigma)$ is finite, there must be some pair $(b, g) \in M$ such that $b \in B(\beta_\sigma\omega + 2\omega) - B(\beta_\sigma\omega + \omega)$. Choose such a pair (b, g) and suppose for a contradiction that there is some $b_i \in (\beta_\sigma\omega + \omega) - Dom(M_\sigma)$ which is not in $Dom(M)$. Then we claim that (b_i, g) is a blocking pair for M. That is, by our definition of levels, g prefers b_i to b and hence (b_i, g) is blocking pair because we are assuming that b_i has no partner. We now let $M_{\sigma+1} = M_\sigma \cup M$. We also let $\beta_{\sigma+1} = \beta_\sigma + 2$ if $Dom(M) = B'$ and $\beta_{\sigma+1} = \beta_\sigma + 1$ otherwise. Similarly, we let $\gamma_{\sigma+1} = \gamma_\sigma + 1$ if $Range(M) = G'$ and we let $\gamma_{\sigma+1} = \gamma_\sigma$ if otherwise.

Next we show as in Subcase 1A that $M_{\sigma+1}$ is stable in the sense that there is no blocking pair (b_i, g_j) for $M_{\sigma+1}$ such that either $b_i \in Dom(M_{\sigma+1})$ or $g_j \in Range(M_{\sigma+1})$. The stability of M_σ implies that $b_i \notin Dom(M_\sigma)$ and $g_j \notin Range(M_\sigma)$. If $b_i \in Dom(M)$, then we can argue as before that since $p_{M_{\sigma+1}}(b_i) \in G(\gamma_\sigma\omega + \omega) - Range(M_\sigma)$, it cannot be that $g_j \in G - G(\gamma_\sigma\omega + \omega)$ by the definition of levels, it cannot be that $g_j \in G' = G(\gamma_\sigma\omega + \omega) - Dom(M_\sigma)$ since M is a stable matching in our restricted society, and it cannot be that $g_j \in Range(M_\sigma)$ by the stability of M_σ. Thus there can be no such blocking pair (b_i, g_j) with $b_i \in Dom(M_{\sigma+1})$. Finally, suppose that $g_j \in Range(M)$. Then $p_{M_{\sigma+1}}(g_j) \in B(\beta_\sigma\omega + 2\omega) - Range(M_\sigma)$. Since $b_i \notin Dom(M_\sigma)$, we must have either $b_i \in B'$ or $b_i \in B - B(\beta_\sigma\omega+\omega+\omega)$. The former is not possible since M is a stable matching and the latter is not possible by the definition of levels.

Subcase 1C. $G(\gamma_\sigma\omega + \omega) - Range(M_\sigma)$ is finite.

Then (3) holds where $(G(\gamma_\sigma\omega + \omega) - G(\gamma_\sigma\omega)) - G^*_\sigma$ is finite and $Dom(M_\sigma) =$

$B(\beta_\sigma\omega)$. As above, we let $M_{\sigma+1} = M_\sigma \cup M$ where M is produced by the Gale-Shapley algorithm applied to the restricted preference orders on $B' = B(\beta_\sigma\omega+\omega) - Dom(M_\sigma)$ and $G' = G(\gamma_\sigma\omega+2\omega) - Range(M_\sigma)$. The details are similar to Subcase 1B.

Case 2. Either $B - Dom(M_\sigma)$ is finite or $G - Range(M_\sigma)$ is finite.

Subcase 2A. $B - Dom(M_\sigma)$ is finite and $|B - Dom(M_\sigma)| \leq |G - Range(M_\sigma)|$.

Let $B' = B - Dom(M_\sigma)$ and $n = |B'|$. Our idea is to let $M_{\sigma+1} = M_\sigma \cup M$ where M is the matching that results by running n steps of the Gale-Shapley algorithm for a restricted society whose set of boys is B', whose set of girls is some appropriate subset of $G(\gamma_\sigma\omega+2\omega) - G(\gamma_\sigma\omega)$, and where the ordering are the restrictions of the ordering from our original ordered society $\langle B, G, \{<_i\}_{b_i \in B}, \{<^j\}_{g_j \in G}\rangle$. If we are in cases (1) or (2), then $Range(M_\sigma) = G(\gamma_\sigma\omega)$ and hence we can let $G' = G(\gamma_\sigma\omega+\omega) - Range(M_\sigma)$. That is, in these cases, our assumptions guarantee that $|G'| \geq |B'|$ and that any boy in B' prefers any girl $g' \in G'$ to any girl $g \in G - G(\gamma_\sigma\omega+\omega)$. Moreover the restrictions of the ordering $<_i$ and $<^j$ will either be finite or of order type ω. If we are in case (3), then there are two possible subcases. That is, it could be that the cardinality of $G(\gamma_\sigma\omega+\omega) - Range(M_\sigma)$ is greater than or equal to n in which case we let $G' = G(\gamma_\sigma\omega+\omega) - Range(M_\sigma)$ as we did in cases (1) and (2). If the cardinality of $G(\gamma_\sigma\omega+\omega) - Range(M_\sigma)$ is less than n, then we let $G' = G(\gamma_\sigma\omega+2\omega) - Range(M_\sigma)$. In this situation, again it is easy to see that our assumptions guarantee that $|G'| \geq |B'|$ and that any boy in B' prefers any girl $g' \in G'$ to any girl $g \in G - G(\gamma_\sigma\omega+2\omega)$. Moreover the restrictions of the ordering $<_i$ and $<^j$ will either be finite or of order type ω. In any case, we can run the Gale-Shapley algorithm for the restricted society determined by B' and G' and construct a stable matching M such that $Dom(M) = B'$. We then let $M_{\sigma+1} = M_\sigma \cup M$ and we terminate the algorithm.

Since $Dom(M) = B'$, it is clear that $Dom(M_{\sigma+1}) = B$. We claim that $M_{\sigma+1}$ is a stable matching for $\langle B, G, \{<_i\}_{b_i \in B}, \{<^j\}_{g_j \in G}\rangle$. That is, suppose that (b_i, g_j) is a blocking pair for $M_{\sigma+1}$. Then since the domain of $M_{\sigma+1} = B$, either $b_i \in Dom(M_\sigma)$ or $b_i \in Dom(M)$. By assumption, M_σ has no blocking pair (b, g) with either $b \in Dom(M_\sigma)$ or $g \in Range(M_\sigma)$, so that we cannot have $b_i \in Dom(M_\sigma)$. Thus it must be the case that $b_i \in Dom(M)$. Then it cannot be that $g_j \in G'$, or (b_i, g_j) would be a blocking pair for M. Also, g_j cannot be in $Range(M_\sigma)$ since this would make (b_i, g_j) a blocking pair for M_σ. Thus the only possibility left is that $g_j \in G - (G' \cup Range(M_\sigma))$. But then $p_M(b_i) \in G'$, so that by our observations above, b_i prefers $p_M(b_i)$ to g_j. It follows that there can be no blocking pair for $M_{\sigma+1}$ and hence $M_{\sigma+1}$ is a stable matching of $\langle B, G, \{<_i\}_{b_i \in B}, \{<^j\}_{g_j \in G}\rangle$.

Subcase 2B. $G - Range(M_\sigma)$ is finite and $|G - Range(M_\sigma)| \leq |B - Dom(M_\sigma)|$.

Let $G' = G - Range(M_\sigma)$ and $n = |G'|$. Our idea is to let $M_{\sigma+1} = M_\sigma \cup M$ where M is the matching that results by running n steps of the Gale-Shapley algorithm with the roles of the boys and the girls reversed for a restricted society whose set of boys B' is some appropriate subset of $B(\beta_\sigma\omega + 2\omega) - B(\beta_\sigma\omega)$ and the ordering are the restrictions of the ordering from our original ordered society $\langle B, G, \{<_i\}_{b_i \in B}, \{<^j\}_{g_j \in G}\rangle$. If we are in cases (1) or (3), then $Dom(M_\sigma) = B(\beta_\sigma\omega)$ and hence we can let $B' = B(\beta_\sigma\omega+\omega) - Dom(M_\sigma)$. That is, in these cases, our assumptions guarantee that $|G'| \leq |B'|$ and that any girl in G' prefers any boy $b' \in B'$ to any boy $b \in B - B(\beta_\sigma\omega + \omega)$. Moreover the restrictions of the ordering $<_i$ and $<^j$ will either be finite or of order type ω. If we are in case (2), then there are two possible subcases. That is, it could be that the cardinality of $B(\beta_\sigma\omega + \omega) - Dom(M_\sigma)$ is greater than or equal to n in which case we let $B' = B(\beta_\sigma\omega + \omega) - Dom(M_\sigma)$ as we did in cases (1) and (3). If the cardinality of $B(\beta_\sigma\omega + \omega) - Dom(M_\sigma)$ is less than n, then we let $B' = B(\beta_\sigma\omega + 2\omega) - Dom(M_\sigma)$. In this situation, again it is easy to see that our assumptions guarantee that $|B'| \geq |G'|$ and that any girl in G' prefers any boy $b' \in B'$ to any boy $b \in B - B(\beta_\sigma\omega + 2\omega)$. Moreover the restrictions of the ordering $<_i$ and $<^j$ will either be finite or of order type ω. In all cases, we can run the Gale-Shapley algorithm for the restricted society determined by B' and G' with the roles of the boys and the girls reversed and construct a stable matching M such that $Range(M) = G'$. We then let $M_{\sigma+1} = M_\sigma \cup M$ and we terminate the algorithm.

Since $Range(M) = G'$ it follows that $Range(M_{\sigma+1}) = G$. We claim that $M_{\sigma+1}$ is a stable matching for $\langle B, G, \{<_i\}_{b_i \in B}, \{<^j\}_{g_j \in G}\rangle$. That is, suppose that (b_i, g_j) is a blocking pair for $M_{\sigma+1}$. Then since the range of $M_{\sigma+1} = G$, either $g_j \in Range(M_\sigma)$ or $g_j \in Range(M)$. By assumption, M_σ has no blocking pair (b, g) with either $b \in Dom(M_\sigma)$ or $g \in Range(M_\sigma)$, so that we cannot have $g_j \in Range(M_\sigma)$. Thus it must be the case that $g_j \in Range(M)$. Then it cannot be that $b_i \in B'$, or (b_i, g_j) would be a blocking pair for M. Also, b_i cannot be in $Dom(M_\sigma)$ since this would make (b_i, g_j) a blocking pair for M_σ. Thus the only possibility left is that $b_i \in B - (B' \cup Dom(M_\sigma))$. But then $p_M(g_j) \in B'$, so that by our observations above, g_j prefers $p_M(g_j)$ to b_i. It follows that there can be no blocking pair for $M_{\sigma+1}$ and hence $M_{\sigma+1}$ is a stable matching of $\langle B, G, \{<_i\}_{b_i \in B}, \{<^j\}_{g_j \in G}\rangle$.

Stage λ, λ a limit ordinal

In this case, let $\lambda_0 < \lambda_1 < \ldots$ be a sequence of ordinals such that $lim_i \lambda_i = \lambda$. Then we let $M_\lambda = \bigcup_i M_{\lambda_i}$, $\beta_\lambda = lim_i \beta_{\lambda_i}$ and $\gamma_\lambda = lim_i \gamma_{\lambda_i}$. We can assume by induction that if $\alpha < \eta < \lambda$, then $M_\alpha \subseteq M_\eta$. Hence it follows that if it is the case that for infinitely many i, $Dom(M_{\lambda_i}) = B(\beta_{\lambda_i}\omega)$, then $Dom(M_\lambda) = B(\beta_\lambda\omega)$. Similarly, if for infinitely many i, $Range(M_{\lambda_i}) = G(\gamma_{\lambda_i}\omega)$, then $Range(M_\lambda) = G(\gamma_\lambda\omega)$. Since for each i, either $Dom(M_{\lambda_i}) = B(\beta_{\lambda_i}\omega)$ or $Range(M_{\lambda_i}) = G(\gamma_{\lambda_i}\omega)$, it follows that either $Dom(M_\lambda) = B(\beta_\lambda\omega)$ or $Range(M_\lambda) = G(\gamma_\lambda\omega)$. Now suppose that there is some $n \geq 0$ such that for all $m > n$, $\beta_{\lambda_i} = \beta_\lambda$ and $Dom(M_{\lambda_i}) = B(\beta_\lambda\omega) \cup B^*_{\lambda_i}$ where $B^*_{\lambda_i} \subseteq B(\beta_\lambda\omega + \omega) - B(\beta_\lambda\omega)$. Then we let $\bigcup_i B^*_{\lambda_i} = B^*_\lambda \subset B(\beta_\lambda\omega + \omega) - B(\beta_\lambda\omega)$. Then clearly $Dom(M_\lambda) = B(\beta_\lambda\omega) \cup B^*_\lambda$ and it must be the case that $Range(M_\lambda) = G(\gamma_\lambda\omega)$. Similarly, if there is some

$n \geq 0$ such that for all $m > n$, $\gamma_{\lambda_i} = \gamma_\lambda$, $Range(M_{\lambda_i}) = G(\gamma_\lambda \omega) \cup G^*_{\lambda_i}$ where $G^*_{\lambda_i} \subseteq G(\gamma_\lambda \omega + \omega) - G(\gamma_\lambda \omega)$, we let $\bigcup_i G^*_{\lambda_i} = G^*_\lambda \subset G(\gamma_\lambda \omega + \omega) - G(\gamma_\lambda \omega)$. Then $Range(M_\lambda) = G(\gamma\omega) \cup G^*_\lambda$ and it must be the case that $Dom(M_\lambda) = B(\gamma_\lambda \omega)$.

We claim that there can be no blocking pair (b_i, g_j) such that either $b_i \in Dom(M_\lambda)$ or $g_j \in Range(M_\lambda)$. That is, if there is such a blocking pair (b_i, g_j), then first consider the case where $b_i \in Dom(M_\lambda)$. Then $b_i \in Dom(M_{\lambda_k})$ for some k. But then (b_i, g_j) would be a blocking pair for M_{λ_k}, contrary to our inductive assumptions. Similarly, if $g_j \in Range(M_\lambda)$, then $g_j \in Range(M_{\lambda_k})$ for some k, so that (b_i, g_j) is a blocking pair for M_{λ_k}, contradicting our assumptions.

This completes the construction. It is easy to see by induction that for all $\sigma < \eta$, we either have $\beta_\sigma < \beta_\eta$ or $\gamma_\sigma < \gamma_\eta$ and that for all σ, we must have $\beta_\sigma \leq \beta$ and $\gamma_\sigma \leq \gamma$. Furthermore, if $\sigma < \tau$, then $(\beta_\sigma, \gamma_\sigma) < (\beta_\tau, \gamma_\tau)$ in the usual lexicographic order on $\beta \times \gamma$, which has order type $\beta\gamma$. Thus the construction will stop after at most $\beta\gamma$ steps and produce a stable matching for $\langle B, G, \{<_i\}_{b_i \in B}, \{<^j\}_{g_j \in G}\rangle$. Since we have a well-ordering long enough to accomplish the construction, it follows that arithmetical transfinite recursion (ATR_0) is enough to imply the existence of our stable matching. □

To ensure that an infinite instance, $\langle B, G, \{<_i\}_{b_i \in B}, \{<^j\}_{g_j \in G}\rangle$, of the stable marriage problem has a symmetric stable matching or a stable matching M with domain B, we need some additional hypotheses. We consider the following conditions.

Condition 1(a) For each boy $b \in B$, there exists finite sets $B(b) = \{b = b_{i_1}, b_{i_2}, \ldots, b_{i_n}\} \subseteq B$ and $G(b) = \{g_{j_1}, g_{j_2}, \ldots, g_{j_n}\} \subseteq G$ such that, for each $k \leq n$, $B(b)$ is the set of the first n most preferred boys of g_{j_k}.

Condition 1(b) For each girl $g \in G$, there exists finite sets $B(g) = \{b_{i_1}, b_{i_2}, \ldots, b_{i_n}\} \subseteq B$ and $G(g) = \{g = g_{j_1}, g_{j_2}, \ldots, g_{j_n}\} \subseteq G$ such that, for each $k \leq n$, $G(g)$ is the set of the first n most preferred girls of b_{j_k}.

We will say that a stable marriage problem is *B-bounded* if it satisfies Condition 1(a), *G-bounded* if it satisfies 1(b) and *bounded* if it satisfies both conditions. If there is a function which gives the finite sets described above from the inputs b and/or g, then the problem is said to be *highly B-bounded (highly G-bounded, highly bounded).* Note that for an $\mathbb{N}$-listed problem, we can always find the desired finite sets for $b \in B$ ($g \in G$) if they exist by searching through all possible sets and checking against the list.

Our example in the proof of Theorem 2.6 is in fact highly bounded if f is recursive and has a recursive range. That is, if m is not in the range of f, then $G(b_{2m}) = G(g_{2m}) = \{g_{2m}\}$ and $B(b_{2m}) = B(g_{2m}) = \{b_{2m}\}$ satisfy conditions 1(a) and 1(b). If m is in the range of f, then we can find n such that $f(n) = m$. If $f(n) = m < n$, then the sets $B(b_{2n+1}) = B(b_{2m}) = B(g_{2n+1}) = B(g_{2m}) = \{g_{2n+1}, g_{2m}\}$ and $G(b_{2n+1}) = G(b_{2m}) = G(g_{2n+1}) = G(g_{2m}) = \{b_{2n+1}, b_{2m}\}$ satisfy conditions 1(a) and 1(b). Finally if $f(n) \geq n$, then the sets $B(b_{2n+1}) =$

$B(g_{2n+1}) = \{b_{2n+1}\}$ and $G(b_{2n+1}) = G(g_{2n+1}) = \{g_{2n+1}\}$ satisfy conditions 1(a) and 1(b).

A special case of being B-bounded is when, for each i, there is a j such that g_j prefers b_i first because in that case we can simply let $B(b_i) = \{b_i\}$ and $G(b_i) = \{g_j\}$.

Our next two results concern properties of either B-bounded or bounded instances of the stable marriage problem.

Theorem 2.8 *(RCA_0) The following are equivalent:*

1. ACA_0
2. *Every B-bounded $\mathbb{N}$-listed stable marriage problem has a solution with domain B.*
3. *Every bounded $\mathbb{N}$-listed stable marriage problem has a symmetric solution.*

Proof: Given that the problem is B-bounded, the Gale-Shapley algorithm will produce a stable matching as above. We claim that the domain must be all of B. To see this, we show that the sequence $p_M(b)$ converges for each $b \in B$. Let $B(b) = \{b_{i_1}, \ldots, b_{i_n}\}$ and $G(b) = \{g_{j_1}, \ldots, g_{j_n}\}$ be the sets which witness that condition 1(a) holds for b. Suppose by way of contradiction that some b_{i_k} is not in the domain of M for some $k \leq n$. Then each of the g_{j_l}'s must reject b_{i_k} at some stage s. But this can only happen if g_{j_l} prefers $p_M(g_{j_l})$ over b_{i_k} for each $l \leq n$ since in the Gale-Shapley algorithm, the preferences for g_{j_l} only improve. But the boys that g_{j_l} prefer over b_{i_k} must be among $B(b) - \{b_{i_k}\}$. However this is impossible since the n girls in $G(b)$ cannot all be matched to the $n-1$ boys in $B(b) - \{b_{i_k}\}$.

For the symmetric solution, we add the assumption that the problem is also G-bounded. Then we claim that the matching produced by the Gale-Shapley algorithm will be symmetric. That is, fix any girl $g \in G$ and let $B(g) = \{b_{i_1}, \ldots, b_{i_n}\}$ and $G(g) = \{g_{j_1}, \ldots, g_{j_n}\}$ be the sets which witness that condition 1(b) holds for g. Suppose by way of contradiction that some g_{j_l} is not in the range of M for some $j \leq n$. It follows that, for each $k \leq n$, b_{i_k} never proposed to g_{j_l}, which means that for each $k \leq n$, b_{i_k} prefers $p_M(b_{i_k})$ over g_{j_l}. But this means $p_M(b_{i_k})$ must be in $G(g) - \{g_{i_l}\}$ which is impossible since $|B(g)| = n > |G(g) - \{g_{i_l}\}| = n - 1$

For the reverse direction, note that by our remarks preceding this theorem, the construction of the society $\mathcal{S}_f = \langle B, G, \{<_i\}_{b_i \in B}, \{<^j\}_{g_j \in G}\rangle$ given in Theorem 2.6 which allows us to construct the range of a function $f : \mathbb{N} \to \mathbb{N}$ is a bounded instance of the stable marriage problem. $\square$

Theorem 2.9 *(RCA_0) The following are equivalent:*

1. WKL_0
2. *Every highly B-bounded, effectively $\mathbb{N}$-listed stable marriage problem has a stable matching with domain B.*

3. *Every highly bounded, effectively* $\mathbb{N}$*-listed stable marriage problem has a symmetric stable matching.*

Proof: Fix some highly B-bounded effectively $\mathbb{N}$-listed infinite instance of the stable marriage problem, $\mathcal{S} = \langle B, G, \{<_i\}_{b_i \in B}, \{<^j\}_{g_j \in G}\rangle$. Let $B = \{b_0 < b_1 < \ldots\}$ and $G = \{g_0 < g_1 < \ldots\}$. We first show how to find stable matching with domain B using Bounded Konig's Lemma.

Fix $b_i \in B$ and let $B(b_i)$ and $G(b_i)$ be the finite sets associated with b_i by the B-boundedness condition. Let $L(b_i) = \{g_j : \exists g \in G(b_i)(g_j \leq_i g)\}$. We claim that in any stable matching M, $p_M(b_i) \in L(b_i)$. That is, suppose for a contradiction that $p_M(b_i) \notin L(b_i)$, then for each $g \in G(b_i)$, (b_i, g) is not a blocking pair for M and hence g must marry some boy b which she prefers over b_i. But by assumption, each the boys that g prefers over b_i must be in $B(b_i) - \{b_i\}$. But this is impossible since the number of boys in $B(b_i) - \{b_i\}$ is less than the number of girls in $G(b_i)$. Moreover since $\mathcal{S}$ is highly B-bounded and effectively $\mathbb{N}$-listed, we can effectively find $L(b_i)$ from b_i. Similarly, if $\mathcal{S}$ is highly G-bounded and effectively $\mathbb{N}$-listed, then in any stable matching M of $\mathcal{S}$, it must be the case that each girl g_j must marry some boy b in $L(g_j) = \{b_i \in B : \exists b \in B(g_j)(b_i \leq^j b)\}$ and we can effectively find $L(g_j)$ from g_j.

Thus we can construct an effectively bounded tree T from $\mathcal{S}$ such that for all $\eta \in T$, the set of immediate successors of η in T are precisely $\eta^\frown j$ such that $g_j \in L(b_{|\eta|})$. We can then interpret a path $\pi = (\pi_0, \pi_1, ...)$ through T as specifying a mapping $M_\pi : B \to G$ where $M(b_i) = g_{\pi_i}$. Of course, it is not necessarily the case that M_π is even a 1:1 correspondence much less a stable matching of $\mathcal{S}$. However we can trim T to a tree $T_{\mathcal{S}}$ such that the paths through T correspond exactly to the stable matching of $\mathcal{S}$. That is, we say that a node $\eta = (\eta_0, \ldots, \eta_n)$ in T is an element of $T_{\mathcal{S}}$ if and only if only if

1. the map $M_\eta = \{(b_i, g_{\eta_i}) : i \leq n\}$ is a 1:1 correspondence and
2. there is no $k \leq n$ such that there is a girl g_j with $g_j <_k g_{\eta_k}$ and either (a) $(b_l, g_j) \in M_\eta$ and $b_k <^j b_l$ for some $l \leq n$ or (b) g_j is not in the range of M_η but $t = max\{i : b_i <^j b_k\} \leq n$.

It is easy to check that condition (1) ensures that any path π through $T_{\mathcal{S}}$, M_π is a matching for $\mathcal{S}$ with domain B and that condition (2) ensures that any path π through $T_{\mathcal{S}}$ is a stable matching. Moreover it is easy to check that that if M is a stable matching, then $\pi = (p_M(b_0), p_M(b_1), \ldots)$ is an infinite path through $T_{\mathcal{S}}$. Finally for any n, we can let $t_n = max\{j : \exists i \leq n(g_j \in L(b_i))\}$ $s_n = max\{i : \exists j \leq t_n \exists k \leq n(b_i <^j b_k)\}$, and let $u_n = t_n + s_n$. Then we can use the finite version of the Gale-Shapley algorithm restricted the boys b_i and girls g_j with $i, j \leq u_n$ and their restricted preference orders. It is then easy to see that if N is any stable matching for this restricted society, then $(p_N(b_0), \ldots, p_N(b_n))$ will be a node in $T_{\mathcal{S}}$. Thus $T_{\mathcal{S}}$ will be an infinite bounded tree so that by Bounded Konig's Lemma, $T_{\mathcal{S}}$ has an infinite path π and hence $\mathcal{S}$ has a stable matching M_π with domain B.

In the case that $\mathcal{S} = \langle B, G, \{<_i\}_{b_i \in B}, \{<^j\}_{g_j \in G}\rangle$ is a highly bounded effective $\mathbb{N}$-listed instance of the stable marriage problem, we can modify the construction of T and $T_{\mathcal{S}}$ as follows. First for each $\eta \in T$, the set of immediate successors of η in T are precisely $\eta^\frown j$ such that $g_j \in L(b_{|\eta|/2})$ if length of η is even. If $|\eta| = 2n+1$, then the immediate successors of η are all nodes of the form $\eta^\frown i$ such that $b_i \in L(g_n)$ We can then interpret a path $\pi = (\pi_0, \pi_1, ...)$ through T as specifying a mapping $M \subseteq B \times G$ where $(b_i, g_{\pi_{2i}}) \in M_\pi$ and $(b_{\pi_{2i+1}}, g_i) \in M_\pi$. Of course, again it is the case that M_π may not even even a 1:1 correspondence much less a stable matching of $\mathcal{S}$. However we can trim T to a tree $T_{\mathcal{S}}$ such that the paths through T correspond exactly to the stable matching of $\mathcal{S}$. That is, we say that a node $\eta = (\eta_0, \ldots, \eta_n)$ in T is an element of $T_{\mathcal{S}}$ if and only if

1. the map $M_\eta = \{(b_i, g_{\eta_{2i}}) : 2i \leq n\} \cup \{(b_{\eta_{2i+1}}, g_i) : 2i+1 \leq n\}$ is a 1:1 correspondence and
2. there is no k such that $2k \leq n$ such that there is girl g_j such that $g_j <_k g_{\eta_{2k}}$ and either (a) $(b_l, g_j) \in M_\eta$ and $b_k <^j b_l$ or (b) g_j is not in the range of M_η but $t = 2max\{i : b_i <^j b_k\} \leq n$.
3. there is no k such that $2k+1 \leq n$ such that there is boy b_i such that $b_i <^k b_{\eta_{2k+1}}$ and either (a) $(b_i, g_l) \in M_\eta$ and $g_k <_i g_l$ or (b) b_i is not in the domain of M_η but $t = 2max\{s : g_s <_i g_k\} + 1 \leq n$.

It is easy to check that condition (1) ensures that any path π through $T_{\mathcal{S}}$, M_π is a matching for $\mathcal{S}$ with domain B and range G and that conditions (2) and (3) ensure that any path π through $T_{\mathcal{S}}$ is a stable matching. Moreover it is easy to check that that if M is a stable matching, then $\pi = (p_M(b_0), p_M(g_0), p_M(b_1), p_M(g_1), \ldots)$ is an infinite path through $T_{\mathcal{S}}$. Finally for any n, we can let $t_n = max\{j : \exists i \leq n(g_j \in L(b_i))\}$ $s_n = max\{i : \exists j \leq t_n \exists k \leq n(b_i <^j b_k)\}$, $p_n = max\{i : \exists j \leq n(b_i \in L(g_j))\}$ $q_n = max\{j : \exists i \leq p_n \exists k \leq n(g_j <_i g_k)\}$ and let $u_n = t_n + s_n + p_n + q_n$. Then we can use the finite version of the Gale-Shapley algorithm restricted to the boys b_i and girls g_j with $i, j \leq u_n$ and their restricted preference orders. It is then easy to see that if N is any stable matching for this restrict society, then $(p_N(b_0), p_N(g_0), \ldots, p_N(b_n), p_N(g_n))$ will be a node in $T_{\mathcal{S}}$. Thus $T_{\mathcal{S}}$ will be infinite bounded tree so that by Bounded Konig's Lemma, $T_{\mathcal{S}}$ has an infinite path π and hence $\mathcal{S}$ has a symmetric stable matching M_π.

For the reverse direction, we shall show how to use the existence of symmetric stable matchings in highly bounded effectively $\mathbb{N}$-listed instances of the stable marriage problem can be used to prove Σ^0_1 separation. That is, suppose that we are given two Σ^0_1 formulas $\phi_0(x) = \exists y_1 \ldots \exists y_p \psi_0(y_1, \ldots, y_p, x)$ and $\phi_1(x) = \exists z_1 \ldots \exists z_q) \psi(z_1, \ldots, z_q, x)$ such that X does not occur freely and $\neg \exists n(\phi_0(n) \wedge \phi_1(n))$. We can then construct two increasing sequences of sets $\{A_s\}_{s \in \omega}$ and $\{B_s\}_{s \in \omega}$ where

$$A_s = \{x \leq s : (\exists y_1 \leq s) \ldots (\exists y_p \leq s) \psi_0(y_1, \ldots, y_p, x) \text{ holds}\}$$

and

$$B_s = \{x \leq s : (\exists z_1 \leq s) \ldots (\exists z_q \leq s) \psi_0(z_1, \ldots, z_q, x) \text{ holds}\}.$$

Conceptually, it will be useful in the following argument to consider the sets $A = \bigcup_{s\in\omega} A_s$ and $B = \bigcup_{s\in\omega} B_s$. Of course, we are working RCA_0 so that A and B are merely defined by Σ^0_1 formulas and hence we can not necessarily prove their existence in RCA_0. The existence of the sets is not used in the following argument. We only use the fact that we can effectively compute A_s and B_s uniformly in s. Nevertheless, we shall refer to A and B to help motivate our construction.

Next we construct a highly bounded effectively listed instance of the stable marriage problem. First we let $B = G = \omega$. To avoid confusion, we will let b_i stand for i if we are thinking of i as an element of B and let g_i stand for i if we are thinking of i as an element of G. Let Z denote the integers and let $\langle , \rangle : Z \times \omega \to \omega$ be some fixed recursive pairing function. To determine the preference orderings $<_i$ and $<^j$, we shall specify finite sequences S_i and S^j of pairwise distinct elements. That is, given $S_i = (i_0, \ldots, i_n)$, we define $<_i$ by declaring $g_s <_i g_t$ if and only if

(i) $s = i_p$ and $t = i_q$ where $p < q \leq n$,

(ii) $s = i_p$ for some $p \leq n$ and $t \notin \{i_0, \ldots, i_n\}$, or

(iii) $s, t \notin \{i_0, \ldots, i_n\}$ and $s < t$.

Similarly given $S^j = (j_0, \ldots, j_p)$, we define $<^j$ by declaring that $b_s <^j b_t$ if and only if

(i) $s = j_p$ and $t = j_q$ where $p < q \leq n$,

(ii) $s = j_p$ for some $p \leq n$ and $t \notin \{j_0, \ldots, j_n\}$, or

(iii) $s, t \notin \{j_0, \ldots, j_n\}$ and $s < t$.

Now fix k. We shall define the orderings $<_{\langle n,k\rangle}$ and $<^{\langle n,k\rangle}$. Our idea is the following. Suppose that $k \notin A \cup B$. Then for each n, we will set

$$\begin{aligned} S_{\langle n,k\rangle} &= g_{\langle n,k\rangle}, g_{\langle n-1,k\rangle} && (1)\\ S^{\langle n,k\rangle} &= b_{\langle n+1,k\rangle}, b_{\langle n,k\rangle}. && (2) \end{aligned}$$

Note that for each n, $b_{\langle n,k\rangle}$ is the most preferred boy of some girl, namely $g_{\langle n-1,k\rangle}$. Hence we can find the sets $B(b_{\langle n,k\rangle})$ and $G(b_{\langle n,k\rangle})$ required by the B boundedness condition by setting $B(b_{\langle n,k\rangle}) = \{b_{\langle n,k\rangle}\}$ and setting $G(b_{\langle n,k\rangle}) = \{g_{\langle n-1,k\rangle}\}$. Then by our earlier remarks we know that $b_{\langle n,k\rangle}$ must marry some girl in $L(b_{\langle n,k\rangle}) = \{g_{\langle n-1,k\rangle}, g_{\langle n,k\rangle}\}$. Similarly $g_{\langle n,k\rangle}$ is the most preferred girl of some boy, namely $b_{\langle n,k\rangle}$. Hence we can find the sets $B(g_{\langle n,k\rangle})$ and $G(g_{\langle n,k\rangle})$ required by the G-boundedness condition by setting $B(g_{\langle n,k\rangle}) = \{b_{\langle n,k\rangle}\}$ and setting $G(g_{\langle n,k\rangle}) = \{g_{\langle n,k\rangle}\}$. Then by our earlier remarks we know that $g_{\langle n,k\rangle}$ must marry some girl in $L(g_{\langle n,k\rangle}) = \{b_{\langle n+1,k\rangle}, b_{\langle n,k\rangle}\}$. These choices are pictured as in Figure 1 which we call the basic two-way infinite chain.

It follows that all elements in this basic chain must map to the two elements to which it is connected in the diagram and hence there are precisely two possibilities for any stable matching M on this chain, namely

Figure 1: The basic two-way infinite chain

(A) $(b_{\langle n,k\rangle}, g_{\langle n,k\rangle}) \in M$ for all $n \in Z$ or
(B) $(b_{\langle n,k\rangle}, g_{\langle n-1,k\rangle}) \in M$ for all $n \in Z$.
Our idea is to modify this basic chain so that only (A) is possible if $k \in A$ and only (B) is possible if $k \in B$.

Formally, for each k, we start by defining $S_{\langle n,k\rangle} = g_{\langle n,k\rangle}, g_{\langle n-1,k\rangle}$ and $S^{\langle n,k\rangle} = b_{\langle n+1,k\rangle}, b_{\langle n,k\rangle}$ for all $n < 0$. Then for all $n \geq 0$, we have three cases. Let $A_{-1} = B_{-1} = \emptyset$.

Case 1. Define $S_{\langle n,k\rangle} = g_{\langle n,k\rangle}, g_{\langle n-1,k\rangle}$ and $S^{\langle n,k\rangle} = b_{\langle n+1,k\rangle}, b_{\langle n,k\rangle}$ if $k \notin A_n \cup B_n$.

Case 2. If $k \in A_n$, then define $S_{\langle n,k\rangle} = g_{\langle n,k\rangle}$, and $S^{\langle n,k\rangle} = b_{\langle n,k\rangle}$. (Note that in this case, $p_M(b_{\langle n,k\rangle}) = g_{\langle n,k\rangle}$ for any stable matching M.)

Case 3. If $k \in B_n$, then define $S_{\langle n,k\rangle} = g_{\langle n-1,k\rangle}$, and $S^{\langle n,k\rangle} = b_{\langle n+1,k\rangle}$. (Note that in this case, $p_M(b_{\langle n,k\rangle}) = g_{\langle n-1,k\rangle}$ for any stable matching M.)

Thus we have to consider three cases. Namely if it is never the case that $k \in A_n \cup B_n$, then we will be in the situation of the basic two-way infinite chain described above. That is, if either $n < 0$ or $k \notin A_n \cup B_n$, then the sets $B(b_{\langle n,k\rangle}) = \{b_{\langle n,k\rangle}\}$ and $G(b_{\langle n,k\rangle}) = \{g_{\langle n-1,k\rangle}\}$ will witness that the B-boundedness condition holds for $b_{\langle n,k\rangle}$. Similarly the sets $B(g_{\langle n,k\rangle}) = \{b_{\langle n,k\rangle}\}$ and $G(g_{\langle n,k\rangle}) = \{g_{\langle n,k\rangle}\}$ will witness that the G-boundedness condition holds for $g_{\langle n,k\rangle}$.

If $k \in A_n - A_{n-1}$, then $g_{\langle m,k\rangle}$ is the most preferred girl of $b_{\langle m,k\rangle}$ for all $m \geq n$. It follows that the sets $G(g_{\langle m,k\rangle}) = \{g_{\langle m,k\rangle}\}$ and $B(g_{\langle m,k\rangle}) = \{b_{\langle m,k\rangle}\}$ will witness that G-boundedness condition for $g_{\langle n,k\rangle}$ holds for all $m \geq n$. Similarly for all $i \geq n$, $b_{\langle i,k\rangle}$ is the most preferred boy of $g_{\langle i,k\rangle}$ so that $B(b_{\langle i,k\rangle}) = \{b_{\langle i,k\rangle}\}$ and $G(b_{\langle i,k\rangle}) = \{g_{\langle i,k\rangle}\}$ and will witness that B-boundedness condition for $b_{\langle i,k\rangle}$ holds. Thus the conditions for being highly bounded holds for elements of the basic chain determined by k. Moreover, $b_{\langle i,k\rangle}$ must marry $g_{\langle i,k\rangle}$ for $i \geq n$ since they are the most preferred partner of each other. But this will force $b_{\langle i,k\rangle}$ to marry $g_{\langle i,k\rangle}$ for all $i \in Z$.

Finally consider the case where $k \in B_n - B_{n-1}$. First note that $b_{\langle m,k\rangle}$ is the most preferred boy of $g_{\langle m-1,k\rangle}$ for all $m \geq n$. It follows that the sets $B(b_{\langle m,k\rangle}) =$

$\{b_{\langle m,k\rangle}\}$ and $G(b_{\langle m,k\rangle}) = \{g_{\langle m-1,k\rangle}\}$ will witness that B-boundedness condition for $b_{\langle m,k\rangle}$ holds for all $m \geq n$. Similarly for $i \geq n$, $g_{\langle i,k\rangle}$ is the most preferred girl of $b_{\langle i+1,k\rangle}$ so that $G(g_{\langle i,k\rangle}) = \{g_{\langle i,k\rangle}\}$ and $B(g_{\langle i,k\rangle}) = \{b_{\langle i+1,k\rangle}\}$ will witness that G-boundedness condition for $g_{\langle i,k\rangle}$ holds. Thus the conditions for being highly bounded holds for elements of the basic chain determined by k. Moreover, $b_{\langle i,k\rangle}$ must marry $g_{\langle i-1,k\rangle}$ for $i > n$ since they are the most preferred partner of each other. But this will force $b_{\langle i,k\rangle}$ to marry $g_{\langle i-1,k\rangle}$ for all $i \in Z$.

It follows that any stable matching M has the property that M is symmetric and that if $k \in A$, then $b_{\langle 0,k\rangle}$ must marry $g_{\langle 0,k\rangle}$ and if $k \in B$, then $b_{\langle 0,k\rangle}$ must marry $g_{\langle -1,k\rangle}$. Thus the set X consisting of all $k \in \omega$ such that $(b_{\langle 0,k\rangle}, g_{\langle 0,k\rangle}) \in M$ is separating set for A and B. □

3 Combinatorial Problems equivalent to WKL_0

In this section, we shall consider several combinatorial problems which are equivalent to WKL_0 over RCA_0. We shall start by considering a simple version of Cantor-Schröder-Bernstein Theorem. The Cantor-Schröder-Bernstein Theorem states that if B and G are sets and $f : B \to G$ and $g : G \to B$ are 1:1 functions, then B and G have the same cardinality. Banach strengthened this result by showing that B can be partitioned into two sets B_1 and B_2 such that the function h which is equal to f on B_1 and g^{-1} on B_2 is a bijection from B onto G. Thus we will consider the the following restricted version of this problem.

Problem 1 Suppose that $B, G \subseteq \mathbb{N}$ and $f : B \to G$ and $g : G \to B$ are 1:1 functions such that the sets $f(B)$ and $g(G)$ exist. The problem is to partition B into two sets B_1 and B_2 such that the function $h = f \restriction B_1 \cup g^{-1} \restriction B_2$ is a bijection from A onto B where we write $f \restriction B_1$ for the function f restricted to B_1.

We note that problem 1 is a special case of the standard marriage problem of Philip Hall. That is, suppose that we think of B as set of boys and G as a set of girls. For each boy $b_i \in B$, we say that b_i knows $f(b_i)$ and knows $g^{-1}(b_i)$ if $g^{-1}(b_i)$ is defined. Similarly, for each girl $g_j \in G$, we say that g_j knows $g(g_j)$ and knows $f^{-1}(g_j)$ if $f^{-1}(g_j)$ is defined. Thus in problem 1, if B, $f(B)$, G and $g(G)$ are recursive sets and f and g are partial recursive functions, then we will get a highly recursive society $\mathcal{S} = \langle B, G, K\rangle$ where $K \subseteq B \times G$ is the relation of knowing. That is, $\mathcal{S}$ will have the property that

1. each boy $b_i \in B$ knows at most two girls and we can effectively find the set of girls that b_i knows from b_i,
2. each girl $g_j \in G$ knows at most two boys and we can effectively find the set of boys that g_j knows from g_j,

3. for each finite set of boys $B' \subseteq B$, the cardinality of the sets of all girls G' which are known by at least one boy in B' is greater than or equal to the cardinality of B' and

4. for each finite set of girls $G' \subseteq G$, the cardinality of the sets of all boys B' which are known by at least one girl in G' is greater than or equal to the cardinality of G'.

A society $\mathcal{S}$ is a *recursive society* if B, G and K are recursive sets. We will call a recursive society $\mathcal{S}$ which satisfies (1)-(4) a *degree* ≤ 2 *highly recursive society.* This given, problem 2 is the following.

Problem 2 Given a degree ≤ 2 highly recursive society $\mathcal{S} = \langle B, G, K\rangle$, find a symmetric marriage M, i.e. find a bijection $M : B \to G$ so that for all $b \in B$, $(b, M(b)) \in K$.

We then have the following result about the set of symmetric marriages of a degree ≤ 2 highly recursive society.

Theorem 3.1 *1. If $\mathcal{S} = \langle B, G, K\rangle$ is a degree ≤ 2 highly recursive society, then $\mathcal{S}$ has either finitely many symmetric marriages or has $2^{\aleph_0}$ symmetric marriages.*

2. If A_0 and A_1 are any pair of disjoint r.e. sets, then there is a degree ≤ 2 highly recursive society $\mathcal{S} = \langle B, G, K\rangle$ such that there is an effective 1:1 degree preserving correspondence between the set of all $X \subseteq \mathbb{N}$ such that $A_0 \subseteq X$ and $X \cap A_1 = \emptyset$ and the set of symmetric marriages of $\mathcal{S}$.

Proof:

For the proof of (1), it is easy to see that the knowledge relation can be decomposed into chains of four types as pictured in Figure 2. That is, we can form a graph $\mathcal{K}$ from K whose vertex set is $B \cup G$ and whose edges are sets of the form $\{b, g\}$ where $(b, g) \in K$. Then the connected components of $\mathcal{K}$ will break up into four types, namely, (i) a cycle, (ii) a one way infinite chain starting with a boy b, (iii) a one-way infinite chain staring with a girl g or (iv) a two-way infinite chain.

It is then easy to see that for the one-way infinite chains, there is only one choice for the symmetric marriage M. That is, in Figure 3, M must map b_i to g_i for all i. However for the cycles or two way infinite chains, there are two choices for a symmetric matching M. Thus if $\mathcal{K}$ has only finitely many cycles and two-way infinite chains, then $\mathcal{S}$ will have only finitely many symmetric marriages while if there infinitely many chains which are cycles or two-way infinite chains in $\mathcal{K}$, then $\mathcal{S}$ has $2^{\aleph_0}$ symmetric marriages.

For (2), fix a pair A and B of infinite disjoint *r.e.* sets and recursive enumerations $\{A^s\}_{s\in\omega}$ and $\{B^s\}_{s\in\omega}$ such that, for all s, $A^s, B^s \subseteq \{0, 1, \ldots, s\}$ and there is at most one element of $A \cup B$ which comes into $A \cup B$ at stage s.

Figure 2: Chains for $\mathcal{K}$

We first partition ω into a recursive sequence $(G_0, B_0, G_1, B_1, \ldots)$ of infinite recursive sets . For any fixed i, let $g_i^0 < g_i^1 < \ldots$ and $b_i^0 < b_i^1 < \ldots$ list the elements of G_i and B_i in increasing order. Our symmetrically highly recursive society $S = (B, G, K)$ will be thought of as a bipartite graph with $B = \cup_i B_i$ and $G = \cup_i G_i$. The idea is to construct a connected component of S with vertex set $G_i \cup B_i$ for each i. We construct the i-th component in stages, so that at stage s, we determine the edges out of g_i^k and b_i^k for $k \leq 2s$. We begin as if we are going to construct the two-way infinite chain in which b_i^0 is joined to g_i^0 and g_i^1 and such that, for each $n > 0$, b_i^{2n} is joined to g_i^{2n-2} and g_i^{2n} and b_i^{2n-1} is joined to g_i^{2n-1} and g_i^{2n+1}. See Figure 3.

Observe that there are exactly two possible surjective marriages f for such a component depending on whether $f(b_i^0) = g_i^0$ or $f(b_i^0) = g_i^1$. A marriage $f : B \to G$ for S will code a separating set C_f for A and B by letting $i \in C_f$ if and only if $f(b_i^0) = g_i^1$. Then it is easy to see that all we need to do to ensure that each marriage f of S corresponds to a separating set C_f for A and B is to construct the i-th component so that it is a one-way chain starting in B_i if $i \in A$, a one-way chain starting in G_i if $i \in B$, and the full two-way infinite chain if $i \notin A^s \cup B^s$. Thus we build the chain until we see that $i \in A \cup B$ at some stage s. That is, at each stage t, we add b_i^k and g_i^k for $k \in \{2t, 2t+1\}$ as pictured in Figure 3. Then if $i \in B^s$ omit b_i^{2n} and g_i^{2n} from the chain for all $n \geq s$ so that the chain will be a one-way infinite starting a girl g_i^{2s-2}. If $i \in A^s$, then add b_i^{2s} and we omit g_i^{2s} plus all boys and girls of the form b_i^{2n} and g_i^{2n} for $n > s$ from the chain so that the chain will be a one-way infinite chain starting at b_i^{2s}.

Figure 3: Generic component of the symmetric society

We note that we can consider this example as a recursive version of problem (1) by simply directing the edges of the graph down the left hand side of the graph and up the right hand side of the graph. That is, we can define the function $f : B \to G$ by saying that $f(b^*) = g^*$ is there is a directed edge from b^* to g^* in some component and define the function $g : G \to B$ by saying that $g(g^*) = b^*$ if there is a directed edge from g^* to b^* in some component. $\square$

Next we consider versions of problems 1 and 2 that are equivalent to WKL_0 over RCA_0. We note that Hirst [8] considered versions of problems 1 and 2 that are equivalent to ACA_0 over RCA_0.

Theorem 3.2 *(RCA_0) The following are equivalent.*

1. WKL_0
2. *For any sets $B, G \subseteq \mathbb{N}$ such that there are 1:1 functions $f : B \to G$ and $g : G \to B$ where $f(B)$ and $g(G)$ exists, there exists a partition B_1 and B_2 of B such that $h = f \restriction B_1 \cup g^{-1} \restriction B_2$ is bijection from B onto G.*
3. *For any degree ≤ 2 society $\mathcal{S} = (B, G, K)$ such that $B, G \subseteq \mathbb{N}$ and there are functions K_B and K_G such that for all $b \in B$, $K_B(b)$ is the set of girls that b knows and for all $g \in G$, $K_G(g)$ is the set of all boys that g knows, there is a symmetric marriage.*

Proof: Our proof of part (ii) of the Theorem 3.1 can easily be modified to show that both (2) and (3) imply Σ^0_1 separation which implies WKL_0. Moreover our remarks at the start of this section show that (3) implies (2). Thus we need only show that WKL_0 implies (3) over RCA_0. We know that WKL_0 is equivalent to the Bounded Konig's Lemma over RCA_0 so that we shall show that Bounded Konig's Lemma implies (3).

Thus suppose that $\mathcal{S} = (B, G, K)$ is a degree ≤ 2 society such that $B, G \subseteq \mathbb{N}$ and there are functions K_B and K_G such that for all $b \in B$, $K_B(b)$ is the set of girls that b knows and for all $g \in G$, $K_G(g)$ is the set of all boys that g knows. Let $B = \{i_0 < i_1 < \ldots\}$ and $G = \{j_0 < j_1 < \ldots\}$. To avoid confusion, we shall let b_k stand for i_k and g_k stand for j_k for $k = 0, 1, \ldots$. We define a bounded tree $T \subseteq \omega^{<\omega}$ as follows. First we put $\emptyset$ in T. Then we let $\eta = (\eta_1, \ldots, \eta_k)$ in T if and only if for all i, $g_{\eta_{2i}} \in K_B(b_i)$ and $b_{\eta_{2i+1}} \in K(g_i)$. We can then interpret an infinite path $\pi = (\pi_1, \pi_2, \ldots)$ through T as specifying a relation $M_\pi = \{(b_i, g_{\pi_{2i}}) : i \in \omega\} \cup \{(b_{\pi_{2i+1}}, g_i) : i \in \omega\}$. Of course, M_π is not necessarily a symmetric marriage but we do know that our definition of T ensures that $(b_i, g_{\pi_{2i}}) \in K$ and $(b_{\pi_{2i+1}}, g_i) \in K$ for all i. We can however trim T to get a tree $T_\mathcal{S}$ so that the infinite paths through $T_\mathcal{S}$ correspond exactly to the symmetric marriages of T by saying that a node $(\pi_1, \ldots, \pi_n)$ is in $T_\mathcal{S}$ if and only if $\{(b_i, g_{\pi_{2i}}) : 2i \leq n\} \cup \{(b_{\pi_{2i+1}}, g_i) : 2i+1 \leq n\}$ is a 1:1 correspondence. Finally, we can use the functions K_B and K_G to start to construct $2n$ steps of the chains in the graph $\mathcal{K}$ associated with the knowledge relation K which start at b_i and g_i for all $i \leq n$ as we did in the proof of part (i) of Theorem 3.1. In the worst case, there will be one chain of length $2n$, but in any case, we will be able to use these chains to find finite sets $B' \subseteq B$ and $G' \subseteq G$ such that $\{b_i : i \leq n\} \subseteq B'$, $\{g_i : i \leq n\} \subseteq G'$, and there is a 1:1 correspondence $M : B' \to G'$ such that for all $b \in B'$, $(b, M(b)) \in K$. We can then use M to construct a node $(M(b_0), M^{-1}(g_0), \ldots, M(b_n), M^{-1}(g_n)) \in T_\mathcal{S}$. Thus $T_\mathcal{S}$ is an infinite bounded tree and hence by Bounded Konig's Lemma, $T_\mathcal{S}$ has in infinite path which corresponds to a symmetric marriage of $\mathcal{S}$. $\square$

We pause at this point to make an interesting contrast between problems 1 and 2 and other combinatorial problems such as the problem of showing that any highly recursive graph G for which every finite subgraph of G is k-colorable is k-colorable. Remmel [18] showed that up to a permutation of the colors, for every highly recursive tree $T \subseteq \omega^{<\omega}$, there is a highly recursive graph G such that there is an effective 1:1 correspondence between the set of infinite paths through T and the set of k-colorings of G. Thus the k-colorings of a highly recursive graph G can represent any recursively bounded Π^0_1-class P. Cenzer and Remmel referred to the ability of specific recursively presented instances of a combinatorial problem P to be able to represent an arbitrary recursively bounded Π^0_1 class in the sense above by saying that P *strongly represents* every recursively bounded Π^0_1 class. Note that the symmetric marriages of any degree ≤ 2 society as in (3) above clearly cannot strongly represent an arbitrary recursively bounded Π^0_1 class since there are recursively bounded Π^0_1 classes which have countably infinitely many elements.

Hirst [11] showed that the theorem

(4) Any graph G such that there is a function N such that for any vertex v of G, $N(v)$ equals the set of neighbors of v in G and any finite subgraph of G is k-colorable is itself k-colorable.

is equivalent to WKL_0 over RCA_0. Thus (3) is equivalent to (4) over RCA_0 and yet there is a real mathematical contrast between theorems (3) and (4) in the sense that any instance of (3) has either finitely many or $2^{\aleph_0}$ solutions while there are instances of (4) which have a countably infinite set of solutions. This shows that in some sense, the system WKL_0 is insensitive to the ability of a problem to represent a recursively bounded Π_1^0-class with a countably infinite set of solutions as far as the logical strength of theorem is concerned. It would be interesting to know if there is some weaker version of RCA_0 over which this natural mathematical distinction is reflected in differing logical strengths.

We end this section by considering the proof-theoretic strengths of several results on infinite partially ordered sets (posets). That is, suppose that we start with a poset $\mathcal{A} = (A, \leq^A)$, which consists of a subset A of $\mathbb{N}$ and an ordering relation $\leq^A$. The *width* of $\mathcal{A}$ is the maximum cardinality of an antichain in $\mathcal{A}$ and the *height* of $\mathcal{A}$ is the maximum cardinality of a chain in $\mathcal{A}$. The poset $\mathcal{A} = (A, \leq^A)$ is said to be *n-dimensional* if there are n linear orderings of A, $(A, L_1), \ldots, (A, L_n)$, such that $\leq^A = L_1 \cap \cdots \cap L_n$. The dimension of $\mathcal{A}$ is the least n such $\mathcal{A}$ is n-dimensional.

The first theorem we consider is Dilworth's theorem [3], which states that any poset $\mathcal{A}$ of width n can be covered by n chains. The problem here is to find such a covering of $\mathcal{A}$ by n chains and the set of solutions corresponds to the various coverings of $\mathcal{A}$ by n chains. The effective version of Dilworth's theorem has been analyzed by Kierstead in [13], where he showed that every recursive poset $\mathcal{A}$ of width n can be covered by $(5^n - 1)/4$ recursive chains, while for each $n \geq 2$, there are recursive posets of width n which cannot be covered by $4(n-1)$ chains. See Kierstead's article [15] for details.

There is a natural dual to Dilworth's theorem which says that every poset of height n can be covered by n antichains. The problem again is to find such a covering. The effective version of the latter theorem was analyzed by Schmerl, who showed that every recursive poset of height n can be covered by $(n^2+n)/2$ recursive antichains while for each $n \geq 2$, there is a recursive poset of height n which cannot be covered by $(n^2+n)/2 - 1$ recursive antichains. Furthermore, Szeméredi and Trotter showed that there exist recursive partial orders of height n and recursive dimension 2 which still cannot be covered by $(n^2+n)/2 - 1$ recursive antichains. These results are reported by Kierstead in [13].

The notion of the dimensionality of posets is due to Dushnik and Miller, who showed in [4] that a countable poset (A, R) is n-dimensional if and only if it can be embedded as a subordering in the product ordering $\mathbb{Q}^n$, where $\mathbb{Q}$ is the set of rational numbers under the usual ordering. A (recursive) poset (A, R) has *(*recursive) dimension equal to d, for d finite, if there are d (recursive) linear orderings $(A, L_1), \ldots, (A, L_d)$ such that $R = L_1 \cap \cdots \cap L_d$, but there are not $d-1$ (recursive) linear orderings $(A, L_1'), \ldots, (A, L_{d-1}')$ such that $R = L_1' \cap \cdots \cap L_{d-1}'$. In [14], Kierstead, McNulty and Trotter analyze the recursive dimension of recursive posets and show that in general, the recursive dimension of a poset is not equal to its dimension.

Theorem 3.3 *(RCA_0) The following are equivalent:*

1. *WKL_0*
2. *Dilworth's Theorem: Any poset of width k can be covered by k chains.*

Proof: First we show that the decomposition theorem follows from WKL_0. Let $\mathcal{A} = (A, \leq^A)$ be a poset of width n. Suppose that $A = \{a_0 < a_1 < \ldots\} \subseteq \mathbb{N}$. Let T be the infinite n-ary branching tree. We can think of any infinite path $\pi = (\pi_0, \pi_1, \ldots)$ as representing a partition $(A_1, \ldots, A_n)$ of A where $A_i = \{a_n : \pi_n = i\}$. We can trim the tree T to construct a bounded tree $T_{\mathcal{A}}$ by saying that a node $\eta = (\eta_0, \ldots, \eta_k) \in T$ is in $T_{\mathcal{A}}$ if and only if $(A_1^\eta, \ldots, A_n^\eta)$ is a collection of chains in $\mathcal{A}$ where $A_i^\eta = \{j \leq k : \eta_j = i\}$. It is then easy to see that any infinite path through $T_{\mathcal{A}}$ corresponds to a decomposition into n chains. Finally, it follows from Dilworth's theorem for finite posets that for all k, there is node of length k in $T_{\mathcal{A}}$. We note that it is easy to check that that the proof of the finite version of Dilworth's theorem requires only Σ_1^0 induction and hence can be carried out in RCA_0. Thus Bounded Konig's Lemma suffices to prove Dilworth's theorem over RCA_0.

For the reverse direction, we show that Dilworth's theorem implies Σ_1^0 Separation. That is, we need only show that the set of decompositions of a poset of width n into n chains can represent the class of separating sets for any pair of disjoint r.e. sets. Fix a pair A and B of infinite disjoint r.e. sets and 1:1 enumerations g_A and g_b of A and B respectively.

First consider the case $k = 2$. We begin with the poset $\mathcal{D}_0$ consisting of two one-way chains $\{a_{i,j} : i = 0, 1 \ \wedge \ j \in \mathbb{N}\}$ and $\{b_{i,j} : i = 0, 1 \ \wedge \ j \in \mathbb{N}\}$, where we have $a_{s,j} \leq^{\mathcal{D}_0} a_{t,k}$ and $b_{s,j} \leq^{\mathcal{D}_0} b_{t,k}$ whenever $j < k$ and $s, t \in \{0, 1\}$ and $a_{0,j} \leq^{\mathcal{D}_0} a_{1,j}$ and $b_{0,j} \leq^{\mathcal{D}_0} b_{1,j}$. The two chains are linked by having $a_{0,j} \leq^{\mathcal{D}_0} b_{1,j}$ and similarly $b_{0,j} \leq^{\mathcal{D}_0} a_{1,j}$. We call the elements $\{a_{0,i}, a_{1,i}, b_{0,i}, b_{1,i}\}$, the *$i$-th block* of the poset $\mathcal{D}_0$. The i-th block of $\mathcal{D}_0$ is pictured in Figure 4(A).

Our final poset $\mathcal{D} = (D, \leq^D)$ will consist of the poset $\mathcal{D}_0$ together with an infinite set E whose relations to the elements of $\mathcal{D}_0$ and among themselves is to be specified in stages. Now it is clear that a decomposition of this poset, up to renaming the chains, is completely determined by the choice, for each i, of either

(a) putting $a_{0,i}$ and $a_{1,i}$ in one chain and $b_{0,i}$ and $b_{1,i}$ in the other, or

(b) putting $a_{0,i}$ and $b_{1,i}$ in one chain and $a_{1,i}$ and $b_{0,i}$ in the other.

Thus we can think of a chain decomposition $h : D \to \{1, 2\}$ as coding up a set C_h where $i \in C_h$ if and only if we use choice (b) for the i-th component, that is, if and only if $h(a_{0,i}) = h(b_{1,i})$. Now the idea is to define the relations between the elements $\mathcal{D}_0$ and remaining set E so that we introduce an element $f \in E$, at stage s, in the i-th component between $a_{0,i}$ and $a_{1,i}$ if i comes into

Figure 4: Blocks for width 2 poset

B at stage s, i.e., if $g_B(s) = i$. See Figure 4(B). This will force f, $a_{0,i}$ and $a_{1,i}$ to be in the same chain. We introduce an element $e \in E$ in the i-th component between $b_{0,i}$ and $a_{1,i}$ if i comes into A at stage s. See Figure 4(C). This will force f, $b_{0,i}$ and $a_{1,i}$ to be in the same chain. Finally we have no new element in the i-th component if $i \notin A \cup B$. Now suppose that h is a decomposition of $\mathcal{D}$ into two chains. We define a separating set S for A and B by putting i into S if and only if $h(a_{0,i}) = h(a_{1,i})$. Furthermore, there is a one-to-one correspondence $h \to C_h$ between the decompositions of $\mathcal{D}$ into two chains and the separating sets of A and B.

For the case where $k > 2$, one simply adds to the poset described a set of $k-2$ infinite one-way chains so that any two elements from different chains are incomparable and any element in these $k-2$ chains are incomparable with any element in $\mathcal{D}$. $\square$

Theorem 3.4 *(RCA_0) The following are equivalent:*

1. *WKL_0*

2. *Any poset of height k can be covered by k antichains.*

Proof: First we show that the decomposition theorem follows from WKL_0. Let $\mathcal{A} = (A, \leq^A)$ be a poset of height n. Suppose that $A = \{a_0 < a_1 < \ldots\} \subseteq \mathbb{N}$. Let T be the infinite n-ary branching tree. We can think of any infinite path $\pi = (\pi_0, \pi_1, \ldots)$ as representing a partition $(A_1, \ldots, A_n)$ of A where $A_i = \{a_n : \pi_n = i\}$. We can trim the tree T to construct a bounded tree $T_{\mathcal{A}}$ by saying

Figure 5: Height 2 poset

that a node $\eta = (\eta_0, \ldots, \eta_k) \in T$ is in $T_{\mathcal{A}}$ if and only if $(A_1^\eta, \ldots, A_n^\eta)$ is a collection of antichains in $\mathcal{A}$ where $A_i^\eta = \{j \leq k : \eta_j = i\}$. It is then easy to see that any infinite path through $T_{\mathcal{A}}$ corresponds to a decomposition into n antichains. Finally, it follows from the fact that any finite poset of height n can be decomposed in the n antichains that for all k, there is node of length k in $T_{\mathcal{A}}$. Again, one can easily check that the proof that every finite set of height n can be covered by n antichains requires only Σ_1^0 induction and hence can be carried out in RCA_0. Thus Bounded Konig's Lemma suffices to prove that any poset of height n can be decomposed into n antichains over RCA_0.

For the reverse direction, we show that our dual to Dilworth's theorem implies Σ_1^0 Separation. That is, we show that the set of decompositions of a poset of height n into n antichains can represent an the class of separating sets for any pair of disjoint r.e. sets. Fix a pair A and B of infinite disjoint r.e. sets and 1:1 enumerations g_A and g_B of A and B.

Again we shall initially consider the case $n = 2$. The poset $\mathcal{D} = (D, \leq_D)$ will consist of two parts. The first part of the poset will consist of an antichain $c_0, c_1, \ldots$, and the second part will consist of two antichains $a_0, a_1, \ldots$ and $b_0, b_1, \ldots$ where $a_0 \leq_D b_0$ and, for each i, $a_i \leq_D b_i$ and $a_i \leq_D b_{i+1}$. See Figure 5.

We will complete the partial ordering on $\mathcal{D}$ by specifying the relations between the two parts in stages. Clearly, up to renaming the antichains, there is a unique decomposition of the second part of the poset into two antichains. We can then think of a decomposition $f : D \to \{0, 1\}$ of $\mathcal{D}$ into two antichains as coding up a set C_f by specifying $i \in C_f$ if and only if f assigns c_i to the same antichain as the a's. The construction is simple in this case. For each i, we define c_i to be greater than a_s if $g_A(s+1) = i$ and incomparable to a_s otherwise. Similarly we define c_i to be less than b_s if $g_B(s+1) = i$ and incomparable to b_s otherwise. It is easy to check that the resulting poset is of height 2. Now suppose that f is a decomposition of $\mathcal{D}$ into two antichains. We define a separating set S for A and B by putting i into S if and only if $f(c_i) = f(a_0)$. Furthermore, up to renaming the antichains, there is a one-to-one correspondence $f \to C_f$ between decompositions of $\mathcal{D}$ into two antichains and separating sets of A and B.

For the case where $k > 2$, one simply adds to the poset described a set of $k-2$ recursive infinite antichains, all of whose elements are comparable with every element of $\mathcal{D}$ and so that elements from different antichains are also comparable $\square$

Finally we consider a results on dimension of posets that is equivalent to WKL_0 over RCA_0. We say that a poset $\mathcal{A} = (\mathbb{N}, \leq^A)$ has *local dimension* $\leq d$ if for each n, the restriction $\mathcal{A} \restriction n$ of $\mathcal{A}$ to $\{0, 1, \ldots, n\}$ has dimension $\leq d$.

Theorem 3.5 *(RCA_0) The following are equivalent:*

1. *WKL_0*
2. *Every poset $\mathcal{A} = (\mathbb{N}, \leq^A)$ of local dimension $\leq d$ has dimension $\leq d$.*

Proof: First we show that (2) follows from WKL_0. Let $\mathcal{A} = (\mathbb{N}, \leq^A)$ be a poset of local dimension $\leq d$. We can code a set of d linear orderings of A, $(A, L_1), \ldots, (A, L_d)$ as a path through a bounded tree as follows. Given d linear orderings of $\{0, 1, \ldots, n-1\}$, there clearly are $(n+1)^d$ ways to extend these d linear orderings to d linear orderings on $\{0, 1, \ldots, n\}$. One can fix some effective enumeration of these extensions for each n, so that it then becomes possible to code each d-tuple of linear orderings by a function $f : A \to \mathbb{N}$ where $f(n) \leq (n+1)^d - 1$ for all n. Thus the set of solutions for the d-dimensionality problem of a poset $\mathcal{A}$ can be represented as the set of all $f : A \to \mathbb{N}$ such that $f(n) \leq (n+1)^d - 1$ for all n and for all k, $f(1), \ldots, f(k)$ codes an n-tuple of linear orderings , $(\{0, \ldots, k\}, L_1), \ldots, (\{0, \ldots, k\}, L_n)$ whose intersection is $(\{0, \ldots, k\}, \leq^A)$. This set can clearly be represented as the set of infinite paths through a bounded tree $T_{\mathcal{A}}$. Moreover, since $\mathcal{A}$ has local dimension $\leq d$, the tree will be $T_{\mathcal{A}}$ infinite. Thus Bounded Konig's Lemma suffices to show that $\mathcal{A}$ has dimension $\leq d$.

For the reverse direction, we show that the dual theorem implies Σ^0_1 Separation. That is, we show that the set of d linear orderings whose intersection is a poset of local dimension $\leq d$ can represent the class of separating sets for any pair of r.e. sets A and B. So fix a pair A and B of disjoint *r.e.* sets and recursive enumerations $\{A^s\}_{s \in \omega}$ and $\{B^s\}_{s \in \omega}$ such that, for all s, $A^s, B^s \subseteq \{0, 1, \ldots, s\}$ and there is at most one element of $A \cup B$ which comes into $A \cup B$ at stage s.

We consider the case of two dimensional partial orderings. First we partition $\mathbb{N}$ into two infinite recursive sets $C = \{c_0 < c_1 < \cdots\}$ and $D = \{d_0 < d_1 < \cdots\}$. For each i, we let $C_i = \{c_{5i}, c_{5i+1}, c_{5i+2}, c_{5i+3}, c_{5i+4}\}$. We shall define a recursive partial ordering $<_P$ on $\mathbb{N}$ in stages. Given any two sets E and F, $E <_P F$ will denote that, for any $e \in E$ and $f \in F$, $e <_P f$. We start by defining $<_P$ so that $C_0 <_P C_1 <_P C_2 <_P \cdots$. This means that if $<_1$ and $<_2$ are two linear orderings such that $<_1 \cap <_2 = <_P$, then the only difference between $<_1$ and $<_2$ on C is how $<_1$ and $<_2$ order the elements within the blocks C_i. For each block C_i, $<_P$ is defined so that we have the Hasse diagram in Figure 6(A).

Figure 6: The i-th block of the dimension 2 poset

It is then easy to check that, up to a permutation of the indices of the linear orderings $<_1$ and $<_2$, there are precisely two ways to define $<_1$ and $<_2$ on C_i so that $<_1 \cap <_2$ equals $<_P$ restricted to A_i, namely,

(I) $c_{5i} <_1 c_{5i+1} <_1 c_{5i+2} <_1 c_{5i+3} <_1 c_{5i+4}$ and
$c_{5i+2} <_2 c_{5i+4} <_2 c_{5i+3} <_2 c_{5i} <_2 c_{5i+1}$, or

(II) $c_{5i} <_1 c_{5i+1} <_1 c_{5i+2} <_1 c_{5i+4} <_1 c_{5i+3}$ and
$c_{5i+2} <_2 c_{5i+3} <_2 c_{5i+4} <_2 c_{5i} <_2 c_{5i+1}$.

Note that the difference between (I) and (II) is that in the ordering where the elements c_{5i}, c_{5i+1} precede the elements $c_{5i+2}, c_{5i+3}, c_{5i+4}$, we have c_{5i+3} preceding c_{5i+4} in (I), while in (II) c_{5i+4} precedes c_{5i+3}.

We can thus use a pair of linear orderings $<_1$ and $<_2$ such that $<_1 \cap <_2 = <_P$ is defined within the blocks C_i to code a set $S(<_1, <_2) \subseteq \mathbb{N}$ by declaring $i \in S$ if and only if $<_1$ and $<_2$ are of type (I) on C_i.

The key to our ability to code up a tree of separating sets for a pair of disjoint *r.e.* sets A and B is the following. If we add an element d to the Hasse diagram as pictured in Figure 6(B), then only linear orderings $<_1$ and $<_2$ of type (I) can be extended to $C_i \cup \{d\}$ so that $<_1 \cap <_2 = <_P$ and if we add an element d to the Hasse diagram as pictured in Figure 6(C), then only linear orderings $<_1$ and $<_2$ of type (II) can be extended to $C_i \cup \{d\}$ so that $<_1 \cap <_2 = <_P$.

That is, it is easy to check that, up to a permutation of indices there is only one way to define linear orderings $<_1$ and $<_2$ on $C_i \cup \{d\}$ so that $<_1 \cap <_2 = <_P$ if $<_P$ has the Hasse diagram as pictured in Figure 6(B), namely

(I'): $c_{5i} <_1 d <_1 c_{5i+1} <_1 c_{5i+2} <_1 c_{5i+3} <_1 c_{5i+4}$ and
$c_{5i+2} <_2 c_{5i+4} <_2 d <_2 c_{5i+3} <_2 c_{5i} <_2 c_{5i+1}$.

Similarly, up to a permutation of indices, there is only one way to define linear orderings $<_1$ and $<_2$ on $C_i \cup \{d\}$ so that $<_1 \cap <_2 = <_P$ if $<_P$ has the Hasse diagram as pictured in Figure 6(C), namely

(II'): $c_{5i} <_1 d <_1 c_{5i+1} <_1 c_{5i+2} <_1 c_{5i+4} <_1 c_{5i+3}$ and
$c_{5i+2} <_2 c_{5i+3} <_2 d <_2 c_{5i+4} <_2 c_{5i} <_2 c_{5i+1}$.

Now to complete our definition of $<_P$ on $\mathbb{N}$, we proceed in stages as follows.

Stage 0 If $i \in A^0$, let $C_{i-1} <_P \{d_0\} <_P C_{i+1}$, and define $<_P$ on $C_i \cup \{d_0\}$ so that we have a Hasse diagram as in Figure 6(B). If $i \in B^0$, let $C_{i-1} <_P \{d_0\} <_P C_{i+1}$ and define $<_P$ on $C_i \cup \{d_0\}$ so that we have a Hasse diagram as in Figure 6(C). If $A^0 \cup B^0 = \emptyset$, define $\{d_0\} <_P C$. Note this defines $<_P$ on all of $C \cup \{d_0\}$ by transitivity.

Stage $s > 0$. Assume we have defined $<_P$ on $C \cup \{d_0, \ldots, d_{s-1}\}$ so that for all $j < s$, $C_{i-1} <_P \{d_j\} <_P C_{i+1}$ if $i \in (A^j \cup B^j) \setminus (A^{j-1} \cup B^{j-1})$ and $\{d_j\} <_P C \cup \{d_0, \ldots, d_{j-1}\}$ otherwise. Then if $i \in A^s \setminus A^{s-1}$, let $C_{i-1} <_P \{d_s\} <_P C_{i+1}$ and define $<_P$ on $C_i \cup \{d_s\}$ so that we have a Hasse diagram as pictured in Figure 6(B). If $i \in B^s \setminus B^{s-1}$, let $C_{i-1} <_P \{d_s\} <_P C_{i+1}$ and define $<_P$ on $C_i \cup \{b_s\}$ so that we have a Hasse diagram as pictured in Figure 6(C). If $(A^s \cup B^s) \setminus (A^{s-1} \cup B^{s-1}) = \emptyset$, define $\{d_s\} <_P C \cup \{d_0, \ldots, d_{s-1}\}$. Again this defines $<_P$ on all of $C \cup \{d_0, \ldots, d_s\}$ by transitivity.

This completes our definition of $<_P$ on $\mathbb{N}$. It is easy to see that the definition of $<_P$ is completely effective. Given our remarks prior to our definition the stages, it is routine to check that up to a permutation of indices, if $<_1$ and $<_2$ are two linear orderings of $\mathbb{N}$, then $<_1 \cap <_2 = <_P$ if and only if $A \subseteq S(<_1, <_2)$ and $B \cap S(<_1, <_2) = \emptyset$. $\square$

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