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Index sets for computable differential equations

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Index sets are used to measure the complexity of properties associated with the differentiability of real functions and the existence of solutions to certain classic differential equations. The new notion of a locally computable real function is introduced and provides several examples of Σ^0_4 complete sets.

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1 Introduction

Computable analysis studies the effective content of theorems and constructions in analysis. In this paper, we study the complexity of the derivative of a real function of several variables and the complexity of the solutions of differential equations of the form $\frac{dy}{dx} = F(x,y)$ and the wave equation from the point of view of index sets.

Index sets play an important role in the study of computable functions and computably enumerable sets (see, for example, Soare [21]) . Index sets for computable combinatorics have been studied by Gasarch and others [8, 9]; the latter paper provides a survey of such results. Index sets for Π^0_1 classes were developed by the authors in [3] and applied to several areas of computable mathematics including computable algebra and logic, computable orderings, computable combinatorics, and computable analysis.

In this paper, we use index sets to develop a complexity measure for the class of computably continuous functions. This follows the path of four recent papers [3, 5, 6, 7] where we studied index sets for Π^0_1 classes and computably continuous functions. The results of those papers assign a precise level of complexity in the arithmetic hierarchy to various properties of classes and functions. For example, the complexity of a set having measure one is Π^0_1 complete, the complexity of a set having cardinality ≥ 2 is Σ^0_2 complete and the complexity of a function having a computable fixed point is Σ^0_3 complete.

The key to the development of a successful theory of index sets for various properties associated with the derivatives of computably continuous functions is to choose an appropriate definition of an index of a computably continuous function. We define the notion of an index for a computable real function of n variables by defining a Π^0_2 set I^n of indices a such that the computable function φ_a defines a computable real function $F_a : \mathbb{R}^n \longrightarrow \mathbb{R}$, where $\mathbb{R}$ denotes the reals. In fact, I^n is Π^0_2 complete. This means that the most meaningful index set results that we obtain involve conditions whose complexity is greater than Π^0_2. Nevertheless, there are a number of results that we can obtain for less complex conditions. For example, we show that $\{\langle a,b\rangle \in I^n \times I^n : \frac{\partial F_a}{\partial x_i} = F_b\}$ is a relative Π^0_1 set in I^n, that is, it is the intersection of Π^0_1 set with I^n.

We shall consider index sets of computably continuous functions whose derivatives have various properties and index sets of computably continuous functions which are the solutions of differential equations of the form $\frac{dy}{dx} = F(x,y)$ and the wave equation.

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The complexity of differentiation is one of the first problems studied in descriptive analysis and set theory. In particular, Mazurkiewicz [12] showed in 1936 that the set of everywhere differentiable functions is complete coanalytic (Π^1_1) in the space of continuous functions on the interval. The differentiability of computable functions was studied by Kushner in [11]. We show that $\{a \in I^n : F_a \text{ has a computable derivative}\}$ is Σ^0_3 complete. On the other hand, we show that both $\{a \in I^n : F_a \text{ has a continuous derivative}\}$ and $\{a \in I^n : F_a \text{ has a continuous, but not computable, derivative}\}$ are Π^0_3 complete. These theorems improve the result of Myhill [13] that a computable function can have a non-computable derivative. For a fixed computable point c, we show that $\{a \in I^n : \dfrac{dF_a}{dx}(c) \text{ exists}\}$ is Π^0_3 complete. Our version of Mazurkiewicz' theorem shows that $\{a \in I^n : F_a \text{ is everywhere differentiable}\}$ is Π^1_1 complete. While it may seem surprising that the complexity of such fundamental sets have not been previously established, a good notion of an index for a computable real function is not obvious, and hence all these results seem to be new.

We also consider the notion, due to Pour-El and Zhong [18] of a *nowhere computably differentiable* function and a new notion of a *locally computably differentiable* function. Informally, a function F is nowhere computable if for any computable F_e and any open set U, $F(x) \neq F_e(x)$ for some $x \in U$. F is locally computable if, for every bounded open set U, there is a computable F_e such that $F = F_e$ on U. We show that

$$\{a \in I^n : F'_a \text{ is nowhere computably differentiable}\}$$

is Π^0_3 complete and that $\{a \in I^n : F'_a \text{ is locally computable}\}$ is Σ^0_4 complete. It should be noted that natural examples of complete Σ^0_4 sets are relatively rare.

Next we consider the complexity of Peano's classical existence theorem for differential equations of the form $y' = F(x,y)$. Peano's existence theorem states that if $F(x,y)$ is continuous on a closed rectangle, then $y' = F(x,y)$ has a continuously differentiable solution in some closed interval. Pour-El and Richards 16] first studied the computable version of Peano's existence theorem and constructed a computable F on the unit square such that $y' = F(x,y)$ has no computable solution on any interval. We shall show that

$$\{a \in I^2 : \frac{dy}{dt} = F_a(t,y),\ y(0) = 0 \text{ has a computable solution}\}$$

is Σ^0_3 complete.

Finally we consider the wave equation in three dimensions,

$$u_{xx} + u_{yy} + u_{zz} = u_{tt}, \tag{1}$$

with initial conditions $u_t(x,y,z,0) = 0$ and $u(x,y,z) = F(x,y,z)$. The wave equation can be solved by Kirchoff's formula. Pour-El and Richards [17] constructed a computable function F such that the corresponding wave equation has a unique solution which is not computable. We show that the set of indices a such that the equation corresponding to F_a has a computable solution is Σ^0_3 complete.

2 Index sets for continuous functions

In this section, we present enumerations of the computably continuous functions on the space $\Re^n$ and then define and classify the basic index sets needed for the analysis of differential equations.

The space $\Re$ has a computable basis of dyadic rational open intervals. For each n, the space $\Re^n$ has a computable basis of finite products $G_1 \times G_2 \times \cdots \times G_n$ of dyadic rational open intervals. Let $U^n_0, U^n_1, \ldots$ be an effective enumeration of these basic open intervals for the space $\Re^n$. This means that we can uniformly compute from n and e the sequence $\langle p_1, q_1, p_2, q_2, \ldots, p_n, q_n\rangle$ such that $U^n_e = (p_1,q_1) \times (p_2,q_2) \times \cdots \times (p_n,q_n)$. Moreover, it is easy to show that there exists such an enumeration which has the property that whenever $U^n_a \subset U^n_b$, then $b \leq a$. This means that the larger intervals occur closer to the beginning of our enumeration. Furthermore, it is not difficult to show that there exists such an enumeration such that we can uniformly compute from k, m and n, a bound $e = e(k,m,n)$ such that any basic set U_a of diameter $\geq 2^{-m}$ which is contained in $[-k,k]^n$ satisfies that $a \leq e$. We shall also study the compact subspace $[0,1]^n$. It is easy to see that there exist enumerations of the basic open sets of $[0,1]^n$ that have similar properties.

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A continuous function $F : \Re^n \longrightarrow \Re^n$ may be represented by a map $f : \omega \longrightarrow \omega$, where we interpret $f(a) = b$ to mean that the image of the interval U_a is included in the interval U_b. For any element x, $F(x)$ is then the unique y such that $y \in U_{f(m)}$ for every m such that $x \in U_m$. For the real line, this is essentially the representation described in Weihrauch [22]. To ensure the continuity of F, we must assume that $U_m \subset U_p$ implies that $U_{f(m)} \subset U_{f(p)}$. To ensure that the map f actually represents a function, we need a (local) modulus of convergence function d such that whenever U_m has diameter $< d(k)$, $U_{f(m)}$ has diameter $< 2^{-k}$. For a compact subspace such as $[0, 1]^n$, a global modulus of convergence function can then be obtained. For the real line, we must have a family of modulus functions d_n on the interval $[-n, n]$ for each $n \geq 1$.

We will say that F is *computably continuous* (or just *computable*) if F may be represented by a computable function f with computable modulus function d when $X = \{0, 1\}^\omega$ or if $X = [0, 1]$. When $X = \Re$, we will say that F is computably continuous if F may be represented by a computable function f with a uniformly computable family of modulus functions $\{d_n\}_{n\geq 1}$, where d_n is a modulus function on $[-n, n]$.

Here is a formal definition.

Definition 2.1 Let the space X have computable basis $\{U_i^X\}_{i\in\mathbb{N}}$, let Y have basis $\{U_i^Y\}_{i\in\mathbb{N}}$ and let F be a continuous function from X into Y.

1. A function $f : \mathbb{N} \longrightarrow \mathbb{N}$ *represents* F if, for all a and b, $F[U_a^X] \subset U_{f(a)}^Y$.
2. F *has index* e, written $F = F_e$, if the (total) recursive function φ_e represents F.
3. F is *computable* if $F = F_e$ for some e.

We note that $F : \Re \longrightarrow \Re$ is computable if and only if, for every n, there exists m such that we can specify $F(x)$ within 2^{-n} given an approximation of x which is within 2^{-m}. Of course, not every φ_e will represent a function. Let $I(X, Y)$ be the set of indices e such that φ_e represents a computable function from X to Y and let $I^n = I(\Re^n, \Re)$. It follows from [5, Theorem 5.1] that, for each n, I^n is Π^0_2 complete. We will sketch the proof here.

Proposition 2.2 *For each* n, I^n *is* Π^0_2.

P r o o f . The conditions on e that ensures that φ_e represents a computably continuous function F_e on $\Re^n$ are the following. Let $\mathrm{Tot} = \{e : \varphi_e \text{ is total}\}$.

(1) φ_e is a total function (i. e. $e \in \mathrm{Tot}$).

(2) $(\forall m, n)\,(U_m \subset U_n \rightarrow U_{\varphi_e(m)} \subset U_{\varphi_e(n)})$.

(3) $(\forall k, m)(\exists r)(\forall t)\,[U_t \subseteq [-k, k]^n \,\&\, \mathrm{diam}(U_t) < 2^{-r} \rightarrow \mathrm{diam}(U_{\varphi_e(t)}) < 2^{-m}]$.

While condition (3) has a Π^0_3 form, it can be restated as a Π^0_2 condition. That is, by condition (2), it follows that we can restrict ourselves to basic sets U_t which are the products of rational intervals where the end points have the form $j/2^r$. This condition implies that there is a uniformly computable family of modulus functions. That is, fix k and m and suppose that r satisfies

$$U_t \subseteq [-k-1, k+1]^n \,\&\, \mathrm{diam}(U_t) < 2^{-r} \rightarrow \mathrm{diam}(U_{\varphi_e(t)}) < 2^{-m}$$

for all subintervals of $[-k-1, k+1]^n$. Now suppose that $\mathrm{diam}(U_t) < 2^{-r-1}$ and let $U_t = G_1 \times \cdots \times G_n \subseteq [-k, k]^n$ where each G_i is a dyadic interval. Since each G_i is a dyadic interval, it follows that there exist $H_i = [j_i/2^r, (j_i+1)/2^r]$ such that $G_i \subset H_i$. Let $U_s = H_1 \times \cdots \times H_n$. Then clearly $U_s \subset [-k-1, k+1]^n$ and $\mathrm{diam}(U_s) < 2^{-r}$ so that by assumption $\mathrm{diam}(U_{\varphi_e(s)}) < 2^{-m}$. But then $U_t \subset U_s$ so that $\mathrm{diam}(U_{\varphi_e(t)}) < 2^{-m}$ as well. Thus we can compute the necessary modulus r from k and m by computing $\varphi_e(t)$ for all t such that U_t is a product of rational intervals of the form $[j_i/2^r, (j_i+1)/2^r] \subseteq [-k, k]$ until we find a large enough r (which must exist by condition (3)) such for all such $\mathrm{diam}(U_{\varphi_e(s)}) < 2^{-m}$. It follows that condition (3) can be replaced by the following condition which is clearly Π^0_2 since the quantifier on t ranges over a finite set.

(3*) $(\forall k, m)(\exists r)(\forall t)[U_t = [j_1/2^r, (j_1+1)/2^r] \times \cdots \times [j_n/2^r, (j_n+1)/2^r] \subseteq [-k, k]^n$
$\rightarrow \mathrm{diam}(U_{\varphi_e(t)}) < 2^{-m}]$. □

For any property $\mathcal{R}$ of a function, let $I^n(\mathcal{R})$ be the set of indices e such that F_e has property $\mathcal{R}$. The remainder of this section is devoted to calculating the complexity of a few simple properties. For rational numbers, we have the uniform result.

Proposition 2.3 $\{(e, r, p_1, \ldots, p_n, q) \in \mathbb{N}^2 \times \mathbb{Q}^{n+1} : e \in I^n \,\&\, F_e(\vec{p}) - q < 2^{-r}\}$ *is* Π^0_2. *In fact, it is the intersection of a* Σ^0_1 *set with* $I^n \times N \times \mathbb{Q}^{n+1}$.

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Proof. Clearly

$$F_e(\vec{p}) - q < 2^{-r} \iff (\exists a)\,[\vec{p} \in U_a \,\&\, U_{\varphi_e(a)} \subset (q - 2^{-r}, q + 2^{-r})]$$

is a Σ^0_1 condition. □

Next fix a pair of computable reals x and y in $\Re$. That is, fix a pair of total computable functions $(\varphi_{e_1}, \varphi_{e_2})$ and $(\varphi_{f_1}, \varphi_{f_2})$ such that

(i) for all i, $\varphi_{e_1}(i) = x_i$ and $\varphi_{f_1}(i) = y_i$ are rational numbers, and $\lim_{n\to\infty} x_i = x$ and $\lim_{n\to\infty} y_i = y$ and

(ii) for all $m > 0$, if $i > \varphi_{e_2}(m)$, then $|x_i - x| < 2^{-m}$, and if $j > \varphi_{f_2}(m)$, then $|y_j - y| < 2^{-m}$.

Then it is easy to see that $\{s : x \in U_s\}$ is a Σ^0_1 set. That is,

$$x \in U_s \iff (\exists i \geq \varphi_{e_2}(m+1))\,((x_i - \frac{1}{2^m}, x_i + \frac{1}{2^m}) \subseteq U_s).$$

Now the predicate $x \in \overline{U_s}$ is Π^0_1 since $x \in \overline{U_s}$ if and only if $(\forall t)\,[U_s \cap U_t = \emptyset \to x \notin U_t]$. It certainly follows that $\{e \in I^1 : F_e(x) = y\}$ is Π^0_2. That is,

$$F_e(x) = y \iff e \in I^1 \,\&\, \forall s\,(x \in U_s \to y \in U_{\varphi_e(s)}).$$

In fact, the property that $F_e(x) = y$ is Π^0_1 relative to I^1 since we can substitute the Π^0_1 condition $y \in \overline{U_{\varphi_e(s)}}$ for the Σ^0_1 condition $y \in U_{\varphi_e(s)}$ in the above equation.

Thus we have proved the following

Proposition 2.4 *For any computable reals x and y, the set $\{a \in I^1 : F_a(x) = y\}$ is Π^0_1 relative to I^1, i. e., it is the intersection of a Π^0_1 set with I^1, and hence it is a Π^0_2 set.*

If $a \in I^n$, then we shall write $F_a \equiv c$ if $F_a(x_1, \dots, x_n) = c$ for all $(x_1, \dots, x_n) \in \mathcal{R}^n$.

Proposition 2.5

1. *For any fixed computable real c, $\{a \in I^1 : F_a \equiv c\}$ is Π^0_1 relative to I^1, i. e., it is the intersection of a Π^0_1 set with I^1, and hence it is a Π^0_2 set.*
2. *$\{a \in I^1 : F_a$ is a constant function$\}$ is Π^0_1 relative to I^n and hence it is a Π^0_2 set.*

Proof. We have

$$F_a \equiv c \iff a \in I^1 \,\&\, (\forall t)\,\left(c \in \overline{U_{\varphi_a(t)}}\right),$$

and also F_e is a constant function if and only if $e \in I^n$ and $(\forall s)(\forall t)\,\left(U_{\varphi_e(s)} \cap U_{\varphi_e(t)} \neq \emptyset\right)$. □

By the same type of arguments, we can show that if $\vec{x}$ is an n-tuple of computable reals in $\Re^n$ and y is a computable real, then $\{e \in I^n : F_e(\vec{x}) = y\}$ is Π^0_1 relative to I^n and hence is a Π^0_2 set. Similarly, if c is a computable real, then $\{a \in I^n : F_a \equiv c\}$ and $\{a \in I^n : F_a$ is a constant function$\}$ are Π^0_1 relative to I^n.

It is easy to see that I^n is closed under scalar multiplication, sum, product, and composition. That is, we have the following

Proposition 2.6

1. *There is a computable function $\alpha : \mathbb{N} \times \mathbb{Q} \longrightarrow \mathbb{N}$ such that for all a, if $U_a = (p, q)$, then $U_{\alpha(a,c)} = (cp, cq)$.*
2. *There is a computable function β such that, for all a and b, if $U_a = (p_1, p_2)$ and $U_b = (q_1, q_2)$, then $U_{\beta(a,b)} = (p_1 + q_1, p_2 + q_2)$.*
3. *There is a computable function γ such that, for all a and b, if $U_a = (p_1, p_2)$ and $U_b = (q_1, q_2)$, then $U_{\gamma(a,b)} = (u, v)$, where $u = \min\{p_1q_1, p_1q_2, p_2q_1, p_2q_2\}$ and $v = \max\{p_1q_1, p_1q_2, p_2q_1, p_2q_2\}$.*

It then easily follows from Proposition 2.6 that the following holds.

Proposition 2.7

1. *There is a computable function $\lambda : \mathbb{N} \times \mathbb{Q} \longrightarrow \mathbb{N}$ such that $F_{\lambda(a,c)} = cF_a$.*
2. *There is a computable function $\mu : \mathbb{N}^2 \longrightarrow \mathbb{N}$ such that $F_{\mu(a,b)} = F_a + F_b$.*

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3. *There is a computable function* $\nu : \mathbb{N}^2 \longrightarrow \mathbb{N}$ *such that* $F_{\nu(a,b)} = F_a \cdot F_b$.
4. *There exist computable functions* $\delta^n : \mathbb{N}^{n+1} \longrightarrow \mathbb{N}$ *such that*

$$F_{\delta^n(b,a_1,\dots,a_n)}(\vec{x}\,) = F_b(F_{a_1}(\vec{x}\,), F_{a_2}(\vec{x}\,), \dots, F_{a_m}(\vec{x}\,)).$$

As an application of parts 1 and 2 of Theorem 2.7 we have the following

Proposition 2.8 $\{(a,b) \in I^n \times I^n : F_a = F_b\}$ *is* Π^0_1 *relative to* I^n.

Proof. $F_a = F_b \iff F_a - F_b = 0.$ □

3 Complexity of differentiation

In this section, we consider the complexity of various classes of computable functions that are characterized by properties of the derivatives.

Theorem 3.1 $\{\langle a,b\rangle \in I^1 \times I^1 : \dfrac{dF_a}{dx} = F_b\}$ *is* Π^0_1 *relative to* $I^1 \times I^1$*, i. e., it is the intersection of* $I^1 \times I^1$ *with a* Π^0_1 *set, and hence is a* Π^0_2 *set.*

Proof. First consider the case $n = 1$. It is well known that (see [22, p. 184]) that integration is computable. Then by the fundamental theorem, we can say that

$$\frac{dF_a}{dx} = F_b \iff (\forall p < q) \int_p^q F_b(x)dx = F_a(q) - F_a(p).$$

By continuity, we can restrict p, q to rationals and for p and q rational numbers, the equality

$$(\forall p < q) \int_p^q F_b(x)dx = F_a(q) - F_a(p)$$

is Π^0_1 by our arguments above. For higher dimension, we use the fundamental theorem for line integrals to say that

$$\frac{dF_a}{dx} = \langle F_{b_1}, F_{b_2}, \dots, F_{b_n}\rangle \iff (\forall p,q) \int_p^q \langle F_{b_1}, F_{b_2}, \dots, F_{b_n}\rangle \cdot dr = F_a(q) - F_a(p).$$

Here again by continuity, we can restrict ourselves to the case where p and q are rational vectors and the vector function r is the straight line from p to q. □

J. R. Myhill [13] first constructed a computable function f with continuous derivative f' such that f' is not computable. In fact, Myhill constructed f so that $f'(1)$ is not computable. Note that if f'' is computable, then f' will in fact be computable by the computability of the antiderivative.

Theorem 3.2 $\{e : F_e$ *is computably differentiable*$\}$ *is* Σ^0_3 *complete.*

Proof. By Theorem 3.1, we have

$$a \in I^n(\text{computably differentiable}) \iff (\exists b_1, \dots, b_n \in I^n)\big(\frac{dF_a}{dx} = \langle F_{b_1}, \dots, F_{b_n}\rangle\big).$$

Thus $\{e : F_e \text{ is computably differentiable}\}$ is Σ^0_3.

For the other direction, it suffices to consider the case where $n = 1$. We will define a reduction of the Σ^0_3 complete set $\mathrm{Rec} = \{e : W_e \text{ is computable}\}$ to the index set for differentiable functions. Here W_e is the e-th computably enumerable (c. e) set, that is, W_e equals the domain of φ_e.

Following Myhill's example, we define the real number $\sigma_a = \sum_{n\in W_a\oplus\mathbb{N}} 10^{-n}$, where $W_a \oplus \mathbb{N}$ is defined as $\{2n : n \in W_a\} \cup \{2n+1 : n \in \mathbb{N}\}$. It is easy to see that σ_a is computable if and only if W_a is computable. Let α_a be a computable function that gives a computable enumeration of the set $W_a \oplus \mathbb{N}$.

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We will use the canonical "pulse function" $\Phi(x)$ as described in [17]. That is, Φ is the C^∞ function with support $[-1/2, 1/2]$ defined by

$$\Phi(x) = \begin{cases} (1+x)\mathrm{e}^{-x^2/(1-4x^2)} & \text{for } 1/2 < x < 1/2, \\ 0 & \text{elsewhere.} \end{cases}$$

Note that $2 \geq \Phi(x) \geq 0$ for all x and $\Phi'(0) = 1$.

Now define the computable real function $F_{\varphi(a)}$ by $F_{\varphi(a)}(x) = \sum_{k=0}^{\infty} 10^{-(k+\alpha_a(k))}\Phi(10^k(x))$. Then $F_{\varphi(a)}$ is computable since $\sum_{k=0}^{n} 10^{-(k+\alpha_a(k))}\Phi(10^k(x))$ is within 10^{-n} of $F_{\varphi(a)}(x)$ for all x. This series is uniformly convergent and we can differentiate to get

(2) $$F'_{\varphi(a)}(x) = \textstyle\sum_{k=0}^{\infty} 10^{-\alpha_a(k)}\Phi'(10^k(x)).$$

Note that $F'_{\varphi(a)}(0) = \sigma_a$ and is computable if and only if W_a is computable. Thus if W_a is not computable, then $F'_{\varphi(a)}(1)$ is not computable. and hence $F_{\varphi(a)}$ is not computably differentiable. On the other hand, if W_a is computable, then it easily follows from (2) that $F'_{\varphi(a)}$ is computable. □

This result can be extended to the notion of "nowhere computable" functions. In [18], Pour-El and Zhong construct a function F which is computable and differentiable on the unit ball such that $F'(x)$ is not computable for any x in a dense set of rational points. Let us say that G is *nowhere computable* on D if, for any basic open set $U \subset D$ and any computable function F, there is a point $x \in U$ such that $F(x) \neq G(x)$. Then we say that F is *nowhere computably differentiable* on D if F' is nowhere computable on D. Clearly the function of Pour-El and Zhong is nowhere differentially computable on the unit ball.

Theorem 3.3 *$\{e \in I^n : F_e$ is nowhere computably differentiable$\}$ is Π^0_3 complete.*

P r o o f. A modification of Theorem 3.1 which restricts p, q to a basic open set U_c shows that

$$\{\langle a, b, c\rangle : \frac{dF_a}{dx} = F_b \text{ on } U_c\}$$

is Π^0_2. Then F_a is nowhere differentiably computable if and only if

$$(\forall b)(\forall c)\,[b \in I^n \rightarrow \frac{dF_a}{dx} \neq F_b \text{ on } U_c].$$

It follows that $\{e \in I^n : F_e$ is nowhere computably differentiable$\}$ is Π^0_3.

For the completeness, we adapt the proof of Theorem 3.2 along the lines suggested by Pour-El and Zhong [18]. Let $G_a(x) = \sum_{k=0}^{\infty} 10^{-(k+\alpha_a(k))}\Phi(10^k x)$. Hence G_a is (uniformly) computable, $|G_a(x)| \leq 2$, G_a has support $[-\frac{1}{2}, \frac{1}{2}]$ and $G'_a(0) \equiv_{\mathrm{T}} W_a$. That is, as in Theorem 3.2, we can show that $G'_{\varphi(a)}(0) = \sum_{n \in W_a \oplus \mathbb{N}} 10^{-n} = \sigma_a$, which is Turing equivalent to W_a and hence is computable if and only if W_a is computable. Note also that for all $x \neq 0$, $G'_{\varphi(a)}(x) = \sum_{k=0}^{\infty} 10^{-\alpha_a(k)}\Phi'(10^k x)$ is computable. That is, there are only finitely many k such that $-\frac{1}{2} \leq 10^k x \leq \frac{1}{2}$ and hence $\Phi'(10^k x) = 0$ for all but finitely many k. Since $\Phi'(x)$ is computable for all x, it follows that $G'_{\varphi(a)}(x)$ is computable for all $x \neq 0$.

Now for integers k, l such that $-3^k < l < 3^k$, let $H_{a,k,l}(x) = G_a(2 \cdot 3^k(x - l/3^k))$, and let

$$H_{a,k}(x) = 3^{-k} \textstyle\sum_{-3^k < l < 3^k \;\&\; l \not\equiv 0 \bmod 3} H_{a,k,l}(x).$$

Then $H_{a,k}$ has similar properties to G_a. That is, $|H_{a,k}(x)| \leq 2/3^k$, $H_{a,k}$ has support

$$\textstyle\bigcup_{-3^k < l < 3^k \;\&\; l \not\equiv 0 \bmod 3} [\frac{l}{3^k} - \frac{1}{4 \cdot 3^k}, \frac{l}{3^k} + \frac{1}{4 \cdot 3^k}],$$

$H'_{a,k}(l/3^k) \equiv_{\mathrm{T}} W_a$ for each l with $-3^k < l < 3^k$ such that $l \not\equiv 0 \bmod 3$, and $H'_{a,k}(y)$ is computable for all y not of the form $l/3^k$ with $-3^k < l < 3^k$ such that $l \not\equiv 0 \bmod 3$.

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Now let $F_{\psi(a)}(x) = \sum_{k=1}^{\infty} 4^{-k} H_{a,k}(x)$. By the uniformity of our definitions, $F_{\psi(a)}$ is computable. We claim that $F_{\psi(a)}(l/3^k) \equiv_{\mathrm{T}} W_a$ for all k and l such that $-3^k < l < 3^k$ and $l \not\equiv 0 \bmod 3$. To see this, fix $x = l_0/3^{k_0}$ with $-3^{k_0} < l_0 < 3^{k_0}$ such that $l \not\equiv 0 \bmod 3$. Next observe that if $k > k_0$, there is a neighborhood about x on which $H_{a,k}$ is identically zero. That is, if $H_{a,k}$ is not identically 0 in a neighborhood of x, then it must be the case that there is some integer s such that $-3^k \le s \le 3^k$, 3 does not divide s, and

$$\frac{s}{3^k} - \frac{1}{4 \cdot 3^k} \le \frac{l_0}{3^{k_0}} \le \frac{s}{3^k} + \frac{1}{4 \cdot 3^k}.$$

But that we would mean $s - \frac{1}{4} \le l_0 3^{k-k_0} \le s + \frac{1}{4}$ and, hence, $s = l_0 3^{k-k_0}$ which would violate the fact the 3 does not divide s. Thus $F'_{\psi(a)}(x) = 4^{-k_0} H'_{a,k_0}(x) + \sum_{1 \le k < k_0} 4^{-k} H'_{(a,k)}(x)$. By our observation above, $H'_{(a,k)}(x)$ is computable for each $k > k_0$ and hence $\sum_{1 \le k < k_0} 4^k H'_{(a,k)}(x)$ is computable. Thus $F'_{\psi(a)}(l_0/3^{k_0}) \equiv_{\mathrm{T}} H'_{a,k_0}(l_0/3^{k_0}) \equiv_{\mathrm{T}} W_a$.

Since the set of fractions of the form $l/3^k$ such that 3 does not divide l is dense, it is immediate that $F_{\psi(a)}$ is nowhere computably differentiable if W_a is not computable. On the other hand, if W_a is computable, then $\dfrac{dF_{\psi(a)}}{dx}$ may be computed as in the proof of Theorem 3.2. □

Now a function F is *not* nowhere computable if there exists *some* neighborhood U such that F is computable on U. This is a rather weak positive condition. A stronger, more natural, positive condition would be that F is computable on *every* neighborhood U where it may be the case that there are different computable functions on different neighborhoods U. Thus we will say that a computable function F on $\Re$ is *locally computably differentiable* if, for every basic open set U, there exists a computable function F_e such that $\dfrac{dF}{dx} = F_e$ on U.

Theorem 3.4

1. *$\{e \in I^n : F_e$ is locally computably differentiable$\}$ is Π^0_4 complete.*
2. *The set of $e \in I^n$ such that F_e is locally computably differentiable but not computably differentiable is Π^0_4 complete.*

Proof.

1. It follows from the definition that this property is Π^0_4. That is, F_a is locally computably differentiable if and only if $(\forall c)(\exists b)\,(\dfrac{dF_a}{dx} = F_b$ on $U_c)$.

For the completeness, let A be an arbitrary Π^0_4 set and let B be a Σ^0_3 relation so that $a \in A$ if and only if $(\forall n)\,(\langle n, a\rangle \in B)$. Since B is Σ^0_3 and Rec is Σ^0_3-complete, there is a one-to-one total recursive function g such that $\langle n, a\rangle \in B$ if and only if $g(\langle n, a\rangle) \in \mathrm{Rec}$. Thus, if $F_{\varphi(a)}$ is the computable real function defined in Theorem 3.2, then for all n and a, $F_{\varphi(g(\langle n,a\rangle))}$ is a computable function with support $[-1/2, 1/2]$ which is computably differentiable if and only if $\langle n, a\rangle \in B$. In particular, $F'_{\varphi(g(\langle n,a\rangle))}(0) = \sum_{n \in W_{g(\langle n,a\rangle)} \oplus N} 10^{-n} = \sigma_{g(\langle n,a\rangle)}$ is computable if and only if $\langle n, a\rangle \in B$, $F'_{\varphi(g(\langle n,a\rangle))}(x)$ is computable for all $x \ne 0$, and $F'_{\varphi(g(\langle n,a\rangle))}(x) = 0$ if $x \notin (-1/2, 1/2)$.

Then we can define the function $F_{\psi(a)}$ by patching together these functions $F_{\varphi(g(\langle n,a\rangle))}$ as follows:

$$F_{\psi(a)}(x) = \begin{cases} 0 & \text{if } x \le -\frac{1}{2}, \\ F_{\varphi(g(\langle n,a\rangle))}(x - n) & \text{if } n - \frac{1}{2} \le x \le n + \frac{1}{2} \text{ and } n \ge 0. \end{cases}$$

If $a \in A$, then $F_{\varphi(g(\langle n,a\rangle))}$ is computably differentiable for each n and it follows that $F_{\psi(a)}$ is locally computably differentiable. If $a \notin A$, then there is some n such that $F_{\varphi(g(\langle n,a\rangle))}$ is not computably differentiable and it follows that $F_{\psi(a)}$ is not computably differentiable on $(n - \frac{1}{2}, n + \frac{1}{2})$ and is thus not locally computably differentiable.

2. We modify the proof of Theorem 3.2 further as follows. Let $K = \{e : e \in W_e\}$ be the usual complete Σ^0_1 set. In the argument above we have $F'_{\varphi(g(\langle n,a\rangle))}(0) = \sigma_{g(\langle n,a\rangle)} = \sum_{n \in W_{g(\langle n,a\rangle)} \oplus \mathbb{N}} 10^{-n}$, where $W_{g(\langle n,a\rangle)} \oplus \mathbb{N} = \{2m : m \in W_{g(\langle n,a\rangle)}\} \cup \{2m+1 : m \in \mathbb{N}\}$. Define

$$W_{f(n,a)} = \begin{cases} \{2m+2 : m \in W_{g(\langle n,a\rangle)}\} \cup \{2m+3 : m \in \mathbb{N}\} & \text{if } n \notin K, \\ \{2m+2 : m \in W_{g(\langle n,a\rangle)}\} \cup \{2m+3 : m \in \mathbb{N}\} \cup \{0\} & \text{if } n \in K. \end{cases}$$

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Let $\sigma_{n,a} = \sum_{n \in W_{f(n,a)}} 10^{-n}$. We can uniformly define a computable function $\alpha_{f(n,a)}$ whose range is $W_{f(n,a))}$ and define a computable function ψ such that

$$F_{\psi(n,a)}(x) = \sum_{k=0}^{\infty} 10^{-(k+\alpha_{f(n,a)}(k))} \Phi(10^k(x)).$$

Thus as in part 1., $F'_{\psi(n,a)}(0)$ will be computable if and only if $\langle n, a \rangle \in B$. Now modify the definition from part 1. so that

$$F_{\psi(a)}(x) = \begin{cases} 0 & \text{if } x \leq -\frac{1}{2}, \\ F_{\psi(n,a)}(x-n) & \text{if } n - \frac{1}{2} \leq x \leq n + \frac{1}{2} \text{ and } n \geq 0. \end{cases}$$

It follows as above that $F_{\psi(a)}$ is locally computable if and only if $a \in A$. However, we also have the following. If $n \in K$, then $\sigma_{n,a} \geq 1$. On the other hand, if $n \notin K$, then $\sigma_{n,a} \leq 1/100$. Thus we have $n \notin K$ if and only if $F'_{\psi(a)}(n) \leq 1/10$. This clearly implies that $F'_{\psi(a)}$ is not a computable function so that $F_{\psi(a)}$ can never be computably differentiable. □

Next we consider the property of being differentiable at a particular point. We just give the result for $n = 1$.

Lemma 3.5 *If F is continuous, then the following are equivalent for any real number c:*

1. *F is differentiable at c.*

2. *For every rational ε, there exist rationals $m < M$ and δ such that $M - m < \varepsilon$ and, for all rationals $q \neq c$ in $(c - \delta, c + \delta)$, $m \leq (F(q) - F(c))/(q - c) \leq M$.*

Proof. If F is differentiable at c, then $F'(c)$ is the limit of $(F(q) - F(c))/(q - c)$, so for any rationals m and M such that $\varepsilon/3 < F'(c) - m < \varepsilon/2$ and $\varepsilon/3 < M - F'(c) < \varepsilon/2$, such a rational δ must exist.

Now suppose that 2. holds. For each $\varepsilon = 2^{-n}$, choose m_n and M_n such that $M_n - m_n < \varepsilon$ and δ_n such that $m_n \leq (F(q) - F(c))/(q - c) \leq M_n$ for all rationals $q \neq c$ in $(c - \delta_n, c + \delta_n)$. We claim that for each n, k, $m_k \leq M_n$. To see this, let δ be the minimum of δ_n and δ_k and let $q \neq c$ be any rational in $(c - \delta, c + \delta)$. Then $m_k \leq (F(q) - F(c))/(q - c) \leq M_n$. It follows that $\{m_k\}_{k \geq 0}$ has a supremum and that $\{M_n\}_{n \geq 0}$ has an infimum. Since $M_n - m_n < 2^{-n}$, these must be equal. Denote this common value by L. We claim that $F'(c) = L$. For any given n, let m_n, M_n and δ_n be given for $\varepsilon = 2^{-n}$ as above. Now for any rational $q \neq c$ in $(c - \delta, c + \delta)$, we have $m_n \leq (F(q) - F(c))/(q - c) \leq M_n$ and we also have $m_n \leq L \leq M_n$ and $M_n - m_n < \varepsilon$. It follows that $|L - \dfrac{F(q) - F(c)}{q - c}| < \varepsilon$. For any irrational $x \neq c$ in $(c - \delta, c + \delta)$, the continuity of F implies that $|L - \dfrac{F(x) - F(c)}{x - c}| \leq \varepsilon$ as well. Thus $\lim_{x \to c} \dfrac{F(x) - F(c)}{x - c} = L$, as desired. □

Theorem 3.6 *For any computable real c, $\{e : F'_e(c) \text{ exists}\}$ is Π^0_3 complete.*

Proof. The upper bound on the complexity easily follows from Lemma 3.5. That is, if one considers condition 2. of Lemma 3.2, then is easy to see that this is Π^0_3 condition when $F = F_e$ is computable and c is computable. That is, the condition $q \neq c$ is Π^0_1 and, for $q \neq c$, $(F(q) - F(c))/(q - c)$ is uniformly computable from q. It follows from Proposition 2.3 that the conditions that $(F(q) - F(c))/(q - c) < m$ and $(F(q) - F(c))/(q - c) > M$ are Σ^0_1 conditions. Thus the conditions that $(F(q) - F(c))/(q - c) \geq m$ and $(F(q) - F(c))/(q - c) \leq M$ are Π^0_1 conditions. It follows that if we write out condition 2. from Lemma 3.5 when $F = F_e$ is computable and c is computable, then it will be a Π^0_3 condition.

For the completeness, let A be a Π^0_3 complete set. Since the set $\text{Fin} = \{e : W_e \text{ is finite}\}$ is a complete Σ^0_2 set, we may assume that there is a function φ such that, for each a, $a \in A$ if and only if for all m, $W_{\varphi(a,m)}$ is finite. We may assume without loss of generality that, for each s, there is at most one m and one n such that $n \in W_{\varphi(a,m),s+1} - W_{\varphi(a,m),s}$, and furthermore $n < s$. In addition, we may assume that $W_{\varphi(a,m)} \cap W_{\varphi(a,k)} = \emptyset$ for any $m \neq k$.

We will define a reduction ψ of A such that $a \in A$ ifand only if $F_{\psi(a)}$ is differentiable at $x = 0$. The function $F_{\psi(a)}$ is defined uniformly as a limit of a sequence $G_{a,s}$ as follows:

Initially set $G_{a,0} \equiv 0$.

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At stage $s+1$, there are two cases. If no element comes into any $W_{\varphi(a,m)}$ at stage s, then $G_{a,s+1} = G_{a,s}$. Otherwise, let m and n be given so that $n \in W_{\varphi(a,m),s+1} - W_{\varphi(a,m),s}$ and let $G_{a,s+1} = G_{a,s} + f_{m,s}$, where $f_{m,s}(x)$ is defined as follows:

$$f_{m,s}(x) = \begin{cases} 2^{3s+8-m}(x-2^{-s}-2^{-s-2})^2(x-2^{-s}+2^{-s-2})^2 & \text{if } 2^{-s}-2^{-s-2} \le x \le 2^{-s}+2^{-s-2}, \\ 0 & \text{otherwise.} \end{cases}$$

It is easy to check that $f_{m,s}$ has the following properties:

(a) $f_{m,s}(2^{-s}) = 2^{-m-s}$, and this is the maximum of $f_{m,s}$, and

(b) $0 = f_{m,s}(2^{-s}-2^{-s-2}) = f'_{m,s}(2^{-s}-2^{-s-2}) = f_{m,s}(2^{-s}+2^{-s-2}) = f'_{m,s}(2^{-s}+2^{-s-2})$.

Since $|G_{a,s+1} - G_{a,s}| \le 2^{-s}$, it follows that the limit $F_{\psi(a)}$ exists and is computable. Moreover, it is easy to see that $F_{\psi(a)}$ is differentiable at all points other than 0. That is, the intervals $\{[2^{-s}-2^{-s-2}, 2^{-s}+2^{-s-2}]\}_{s\ge 0}$ are pairwise disjoint so that if $x \ne 0$, then either $F_{\psi(a)}$ is zero in a neighborhood of x or x belongs to $[2^{-s}-2^{-s-2}, 2^{-s}+2^{-s-2}]$ and $F_{\psi(a)}(x) = f_{m,s}(x)$. Next observe that for each m, n and s such that $n \in W_{\varphi(a,m),s+1} - W_{\varphi(a,m),s}$, and each $x \in [2^{-s}-2^{-s-2}, 2^{-s}+2^{-s-2}]$, we have that $x = 2^{-s} \pm a$, where $0 \le a \le 2^{-s-2}$. Thus

$$\begin{aligned} \frac{F_{\psi(a)}(x)}{x} &= \frac{2^{3s+8-m}(\pm a - 2^{-s-2})^2(\pm a + 2^{-s-2})^2}{2^{-s} \pm a} = \frac{2^{3s+8-m}(a^2 - 2^{-2s-4})^2}{2^{-s} \pm a} \\ &\le \frac{2^{3s+8-m}(2^{-2s-4})^2}{2^{-s}-2^{-s-2}} = \frac{2^{-s-m}}{2^{-s}-2^{-s-2}} = \frac{2^{-m}}{\frac{3}{4}} = \frac{4}{3}\cdot 2^{-m}. \end{aligned}$$

Thus for $x \in [2^{-s}-2^{-s-2}, 2^{-s}+2^{-s-2}]$, we have

(i) $F_{\psi(a)}(2^{-s})/2^{-s} = 2^{-m}$, and

(ii) $0 \le F_{\psi(a)}(x)/x \le \frac{4}{3}\cdot 2^{-m}$.

Suppose now that $a \in A$. Then for each m, there are only finitely many s such that $W_{\varphi(a,m),s+1} - W_{\varphi(a,m),s}$ is nonempty. Choose t large enough so that, for all $k \le m$, $W_{\varphi(a,k)} = W_{\varphi(a,k),t-1}$. This implies that for all $x \le 2^{-t}$, we have $0 \le F_{\psi(a)}(x)/x \le \frac{4}{3}\cdot 2^{-m}$. Since this is true for each fixed m, it follows that $F_{\psi(a)}$ has derivative 0 at $x = 0$.

Next suppose that $a \notin A$. Then, for some m, there are infinitely many s such that $F_{\psi(a)}(2^{-s})/2^{-s} = 2^{-m}$, whereas $F_{\psi(a)}(2^{-s}-2^{-s-2}) = 0$. It is immediate that $F_{\psi(a)}$ is not differentiable at $x = 0$. □

Now a computable function may have a derivative which is not continuous as well as not computable.

Lemma 3.7 *A continuous function $F : [0,1] \longrightarrow [0,1]$ is continuously differentiable if and only if, for all rational $\varepsilon > 0$, there exists a rational $\delta > 0$ such that, for all rationals $p < q$ and $r < s$ where all of $\{q-p, s-r, |s-p|, |s-q|, |r-p|, |r-q|\}$ are less than δ, we have $\left\| \frac{F(q)-F(p)}{q-p} - \frac{F(s)-F(r)}{s-r} \right\| < \varepsilon$.*

P r o o f. If F is continuously differentiable, then the condition easily follows for all *real* p, q, r, s. Now suppose that the condition is satisfied. Then the function $G(x,y) = (F(y)-F(x))/(y-x)$ is uniformly continuous on the dense set consisting of all rational pairs $\langle p, q\rangle$ such that $p \ne q$. It follows from basic analysis that $G(x,y)$ has a unique extension to a continuous function (still denoted by G) on the square. But for any x, $G(x,x) = \lim_{y\to x} G(x,y) = \lim_{y\to x} \dfrac{F(y)-F(x)}{y-x} = G'(x)$. □

Theorem 3.8

1. *The set U of all $e \in I^1$ such that F_e is continuously differentiable is Π^0_3 complete.*

2. *The set V of all $e \in I^1$ such that F_e is continuously differentiable but not computably differentiable is Π^0_3 complete.*

3. *The set W of $e \in I^1$ such that F_e is continuously differentiable and not locally computably differentiable is Σ^0_4 complete.*

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P r o o f. The upper bound on the complexity of U, V, and W for computable continuous functions on [0,1] as well as any other rational interval follows from Lemmas 3.5 and 3.7. The more general condition for $\Re$ is that the condition of Lemma 3.7 holds for all intervals $[-n, n]$ with p, q, r, s restricted to $[-n, n]$.

The completeness of property 1. follows from the proof of Theorem 3.6. That is, the function $F_{\psi(a)}$ given there will not even be differentiable at $x = 0$ if $a \notin A$ and will be continuously differentiable everywhere if $a \in A$.

The completeness of property 2. follows from the proof of Theorem 3.2. That is, the function $F_{\varphi(a)}$ given there will always be continuously differentiable and will fail to be computably differentiable if and only if W_a is *not* a computable set.

3. follows from the proof of Theorem 3.4, since the function $F_{\psi(a)}$ defined there is always continuously differentiable. □

The set of continuous functions which are differentiable was shown to be a complete coanalytic set in the space of continuous functions by Mazurkiewicz [12]. We use a modified version of Mazurkiewicz's proof in the following.

Theorem 3.9 $\{e \in I^1 : F_e$ *is everywhere differentiable*$\}$ *is* Π^1_1 *complete.*

P r o o f. It follows from Lemma 3.5 that the property of being differentiable at a real point c is uniformly Π^0_3 relative to c. Thus the property of being everywhere differentiable is Π^1_1.

For the completeness, we will make use of the Σ^1_1 complete set $\{a : P_a \neq \emptyset\}$, where P_a is the a-th Π^0_1 class in ω^ω. For simplicity of the construction below, we will replace ω^ω by $\{1, 2, \dots\}^\omega$. Given a string $\sigma = (\sigma(0), \dots, \sigma(n))$, we shall write $\sigma^\frown k$ for the string $(\sigma(0), \dots, \sigma(n), k)$. We shall write $\tau \sqsubseteq \sigma$ if τ is an initial segment of σ, i. e., if $\tau = (\sigma(0), \dots, \sigma(m))$ for some $m \leq n$. We note that one can uniformly construct from a, a primitive recursive tree $T_a \subseteq \{1, 2, \dots\}^{<\omega}$ such that $x \in P_a$ if and only if $x \restriction n \in T_a$ for all n. See [3, 2, 5] for details.

We will define a computable function φ such that P_a is empty if and only if $F_{\varphi(a)}$ is everywhere differentiable. For any finite sequence $\sigma \in \{1, 2, \dots\}^n$, define dyadic rationals

$$q_\sigma = 2^{-\sigma(0)} + 2^{-\sigma(1)-3\sigma(0)} + \cdots + 2^{-\sigma(n)-3(\sum_{s=0}^{n-1}\sigma(s))} \text{ and } r_\sigma = q_\sigma + 2^{-3\sigma(n)-3(\sum_{s=0}^{n-1}\sigma(s))},$$

and let $J(\sigma) = [q_\sigma, r_\sigma]$. Thus, if $S(\sigma, n) = \sum_{k=0}^{n} \sigma(k)$, then $r_\sigma = q_\sigma + 2^{-3S(\sigma,n)}$. Moreover, if $\tau = \sigma^\frown k$, then

$$\begin{aligned} q_\tau &= q_\sigma + 2^{-k-3S(\sigma,n)} > q_\sigma, \\ r_\tau &= q_\sigma + 2^{-k-3S(\sigma,n)} + 2^{-3k-3S(\sigma,n)} = q_\sigma + 2^{-3S(\sigma,n)}(2^{-k} + 2^{-3k}) < q_\sigma + 2^{-3S(\sigma,n)} = r_\sigma. \end{aligned}$$

Thus $q_\sigma < q_{\sigma^\frown k} < r_{\sigma^\frown k} < r_\sigma$. Moreover, if $k < l$, then

$$\begin{aligned} r_{\sigma^\frown l} &= q_\sigma + 2^{-3S(\sigma,n)}(2^{-l} + 2^{-3l}) \;=\; q_\sigma + 2^{-k-3S(\sigma,n)}(2^{-l-k} + 2^{-2l-(l-k)}) \\ &< q_\sigma + 2^{-k-3S(\sigma,n)} = q_{\sigma^\frown k}. \end{aligned}$$

Thus $J_{\sigma^\frown l} \cap J_{\sigma^\frown k} = \emptyset$. It follows that if σ and τ are incompatible, then $J(\sigma)$ and $J(\tau)$ are disjoint. Also, if $|\sigma| = k$, then $\operatorname{diam}(J(\sigma)) \leq 2^{-3k}$.

For an infinite sequence $x \in \{1, 2, \dots\}^\omega$, let $r_x = \lim_n r_{x\restriction n} = 2^{-\sigma(0)} + \sum_{n\geq 0} 2^{-\sigma(n+1)-3(\sum_{s=0}^{n}\sigma(s))}$. Then r_x is the unique element of the intersection $\bigcap_n J(x \restriction n)$. Thus $P_a \neq \emptyset$ if and only if there exist $t \in [0, 1]$ and $x \in P_a$ such that, for all n, $t \in J(x \restriction n)$. Let $J_a = \{t : (\exists x \in P_a)(\forall n)\,(t \in J(x \restriction n))\}$. Then $J_a \subset [0, 1]$ and is nonempty if and only if P_a is nonempty.

Our goal is to define $F_{\varphi(a)}$ such that $F_{\varphi(a)}$ is differentiable at t if and only if $t \notin J_a$. To define $F_{\varphi(a)}$, we first need to define a family of functions $F_\sigma(t)$ for each $\sigma \in \{1, 2, \dots\}^{k+1}$. Let $J(\sigma) = [q, r]$ as defined above and let

$$F_\sigma(t) = \begin{cases} \dfrac{(t-q)^2(r-t)^2}{(r-q)^{7/2}} & \text{if } q \leq t \leq r, \\ 0 & \text{otherwise.} \end{cases}$$

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Then $F_\sigma(q) = F_\sigma(r) = 0$, $F'_\sigma(q) = F'_\sigma(r) = 0$, and F_σ has a maximum at $t = (q+r)/2$ of $F((q+r)/2) = (r-q)^{1/2}/16$. The key fact here is that if $t \in [q,r]$, then there is a point $s \in [q,r]$ such that

$$\frac{|F(t)-F(s)|}{|t-s|} \geq \frac{1}{16\sqrt{r-q}}.$$

In fact, we can choose s to be one of q, r, or $(q+r)/2$. That is, suppose without loss of generality that $q \leq t \leq (q+r)/2$. Then clearly $|t-q| \leq (r-q)/2$ and $|(q+r)/2 - t| \leq (r-q)/2$. Moreover either $F(t) - F(q) \geq \sqrt{r-q}/32$ or $F((q+r)/2) - F(t) \leq \sqrt{r-q}/32$. It then follows that

$$\text{either } \frac{|F(t)-F(q)|}{|t-q|} \geq \frac{1}{16\sqrt{r-q}} \text{ or } \frac{|F(t)-F((q+r)/2)|}{|t-\frac{q+r}{2}|} \geq \frac{1}{16\sqrt{r-q}}.$$

Now define the computable function $F_{\varphi(a)}$ by $F_{\varphi(a)}(t) = \sum_{t\in J(\sigma),\sigma\in T_a} F_\sigma(t)$. Note that for $|\sigma| = k$, we have $F_\sigma(t) \leq 2^{-1.5k}$ so that the sum of $F_\sigma(t)$ for all σ with $|\sigma| > k$ is $\leq 2^{-1.5k}(1/(1-2^{-1.5})) \leq 3(2^{-1.5k})$. Hence we may compute $F_{\varphi(a)}(t)$ within $3(2^{-1.5k})$ by finding those unique σ with $|\sigma| = 0,1,\dots,k$ such that $t \in J(\sigma)$. This shows that $F_{\varphi(a)}$ is computable.

Suppose that $x \in P_a$, so that, for all n, $r_x \in J(x \upharpoonright n)$. For each such n, there is a point $s_n \in J(x \upharpoonright n)$ with $|s_n - r_x| \leq 2^{-3n}$ and $|F(r_x) - F(s_n)|/|r_x - s_n| \geq 2^{1.5n-4}$. This clearly implies that F_a is not differentiable at any $t \in J_a$. Thus, if $P_a \neq \emptyset$, then $F_{\varphi(a)}$ is not everywhere differentiable.

Next suppose that $P_a = \emptyset$. This implies that any $t \in [0,1]$ belongs to only finitely many intervals $J(\sigma)$ such that $\sigma \in T_a$. That is, if $t \in J_\sigma \cap J_\tau$, then by construction either $\tau \sqsubseteq \sigma$ or $\sigma \sqsubseteq \tau$.

We claim that $F_{\varphi(a)}$ is differentiable everywhere. There are three cases. First, if t is not one of the dyadic rationals of the form q_σ, then there is an open interval about t which meets only finitely many intervals $J(\sigma)$ such that $\sigma \in T_a$. Thus $F_{\varphi(a)}$ is a finite sum of differentiable functions on that interval and hence $F_{\varphi(a)}$ is differentiable at t. If $t = q_\sigma$ and the set of nodes above σ in T_a is finite, then again there is an open interval about r_σ which meets only finitely many intervals $J(\tau)$ such that $\tau \in T_a$. Thus $F_{\varphi(a)}$ is a finite sum of differentiable functions on that interval and hence $F_{\varphi(a)}$ is differentiable at q_σ. Finally we consider the case where $t = q_\sigma$ and set of nodes above σ in T_a is infinite. Now $q_\sigma \in (q_\tau, r_\tau)$ for all initial segments τ of σ. Then consider $G_\sigma = F_{\varphi(a)} - \sum_{\tau\sqsubseteq\sigma} F_\tau$. We need only check that G_σ is differentiable at t. It follows from our construction that there is an interval $(t-\varepsilon, t]$ where G_σ is zero. Now consider a u such that $q_\sigma = t < u < r_\sigma$. Recall that $r_\sigma - q_\sigma = 2^{-3(\sum_{s=0}^{|\sigma|}\sigma(s))} = \delta$. Now there are two subcases. First it could be that u is in the open set $(q_\sigma, r_\sigma) - \bigcup_{k\geq 1} J_{\sigma^\frown k}$. In that, case, $G_\sigma(u) = 0$ and hence $|G_\sigma(u) - G_\sigma(t)|/|u-t| = 0$. Otherwise, there is some k such that $u \in J(\sigma^\frown k) = [q_\sigma + 2^{-k}\delta, q_\sigma + 2^{-k}\delta + 2^{-3k}\delta]$. Now u can be in only finitely many intervals of the form J_τ. Thus for some finite set R of nodes including and possibly extending $\sigma^\frown k$, we have

$$G_\sigma(u) - G_\sigma(t) = \textstyle\sum_{\tau\in R} F_\tau(u) \leq \sum_{\tau\in R} \dfrac{(r_\tau - q_\tau)^{1/2}}{16}.$$

But for each $\tau \in R$, $r_\tau - q_\tau$ is of the form $2^{-3k-3p}\delta$ so that there is some $w \geq 1$ such that

$$\textstyle\sum_{\tau\in R} \dfrac{(r_\tau - q_\tau)^{1/2}}{16} \leq \sum_{p=0}^{w} \dfrac{(2^{-3k-3p}\delta)^{1/2}}{16} = \sqrt{\delta}\, 2^{-1.5k-4}(\sum_{p=0}^{w} 2^{-1.5p}) \leq \sqrt{\delta}\, 2^{-1.5k-3}.$$

Since $u - t \geq 2^{-k}\delta$, we can conclude that $|F_a(u) - F_a(t)|/|u-t| \leq 2^{-.5k-3}\delta^{-.5}$. It follows that $F'_{\varphi(a)}(t) = 0$ so that $F_{\varphi(a)}$ is differentiable at t. □

4 Differential equations

In this section, we determine the complexity of the index set corresponding to the property that there exists a computable solution to the ordinary differential equation

$$\frac{d\varphi}{dt} = F(t, \varphi(t)), \quad \varphi(0) = 0$$

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for a continuous function $F(x,y)$. Peano's existence theorem states that, if $F(x,y)$ is continuous on the rectangle $-a \le x \le a$, $-b \le y \le b$, where $a,\ b > 0$, then this differential equation has a continuously differentiable solution on $[-\alpha,\alpha]$, where $\alpha = \min\{a, b/M\}$, and $M = \max\{|F(x,y)| : -a \le x \le a,\ -b \le y \le b\}$.

Pour-El and Richards [16] first showed that the computable version of Peano's theorem fails by constructing a function $F(x,y)$ computable on $\{0 \le x \le 1, -1 \le y \le 1\}$ such that no solution of $\frac{d\varphi}{dt} = F(t,\varphi(t))$ is computable on any interval $[0,\delta]$, $\delta > 0$.

Simpson [19] gave a simpler construction and showed the equivalence, over the system RCA_0, of Peano's Existence Theorem with WKL_0 (Weak König's Lemma). We will employ Simpson's version from [20] to derive an index sets result which improves the theorem of Pour-El and Richards.

Theorem 4.1 *The set A of all $a \in I^2$ such that there exist a $\delta > 0$ such that the differential equation $\frac{d\varphi}{dt} = F_a(t,\varphi(t))$ has a computable solution φ on $[-\delta,\delta]$ with $\varphi(0) = 0$ is Σ^0_3 complete.*

Proof. It follows from Theorem 3.2 and the remarks from Section 2 concerning composition that A is Σ^0_3. For the completeness, we will reduce the set

$$S = \{\langle a,b\rangle : W_a \cap W_b = \emptyset \text{ and } W_a \text{ and } W_b \text{ have a computable separating set}\}$$

to A. Note that is shown in [3] that $S_{a,b}$ is Σ^0_3 complete. That is, we will define a primitive recursive function ψ such that $\varphi_{\psi(a,b)}$ represents a computably continuous function $F_{\psi(a,b)}$ defined on the rectangle $[-1,1]\times[-1,1]$ such that $y' = F_{\psi(a,b)}(x,y)$ has a computable solution with $y(0) = 0$ on some interval $(-\delta,\delta)$ with $\delta > 0$ if and only if W_a and W_b have a computable separating set.

Simpson [20] constructed a computably continuous function $f_{a,b}(x,y)$ on the rectangle $|x| \le 1$, $|y| \le 1$ such that $|f_{a,b}(x,y)| \le 1$, $f_{a,b}(-x,y) = -f_{a,b}(x,y)$, and for each $n \ge 1$, if $y = \varphi(x)$ is any solution of $y' = f(x,y)$ on the interval $-2^{-n+1} \le x \le -2^{-n}$, then

(1) $\varphi(-2^{-n+1}) = \varphi(-2^{-n})$,

(2) $n \in W_a$ and $\varphi(-2^{-n+1}) = 0$ imply $\varphi(-2^{n} - 2^{-n+1}) > 2^{-3(n+2)}$, and

(3) $n \in W_b$ and $\varphi(-2^{-n+1}) = 0$ imply $\varphi(-2^{n} - 2^{-n+1}) < 2^{-3(n+2)}$.

We shall give some of the details of this construction since we need to verify that the construction is uniform in a and b and hence we can define a computable function φ such that for each a and B, $\varphi(a,b)$ defines a computable function $F_{\varphi(a,b)} = f_{a,b}$.

For any a and s, we let $W_{a,s}$ denote the set of elements enumerated into W_a by the end of stage s. We assume that if $x \in W_{a,s}$, then $x \le s$. Let $q(x) = \max\{1-|x|,0\}$ and for $n \in \mathbb{N}$, let

$$h_{n,a,b}(x) = \begin{cases} 2^{-k}q(2^k(x-\frac{1}{2})) & \text{if } n \in W_{a,k+1} - W_{a,k},\\ -2^{-k}q(2^k(x-\frac{1}{2})) & \text{if } n \in W_{b,k+1} - W_{a,k},\\ 0 & \text{otherwise.}\end{cases}$$

Here we will make the convention that $W_{c,0} = W_{c,1} = \emptyset$ for all c. This implies that $h_{n,a,b}(0) = h_{n,a,b}(1) = 0$ for all a and b. Note that if $n \in W_a$ ($n \in W_b$), then $h_{n,a,b}$ is positive (negative) on an interval on length 2^{-k+1} centered at $x = 1/2$ for some $k \ge 2$. If $y' = h_{n,a,b}(x)$ for $0 \le x \le 1$, then there are three possibilities.

1. If $n \notin W_a \cup W_b$, then $y = 0$ for $0 \le x \le 1$.

2. If $n \in W_{a,k+1} - W_{a,k}$, then y is non-decreasing on $[0,1]$ and $y(1) = 2^{-2k} + y(0)$.

3. If $n \in W_{b,k+1} - W_{b,k}$, then y is non-increasing on $[0,1]$ and $y(1) = -2^{-2k} + y(0)$.

Let $s(x,y) = 9x(1-x)y^{\frac{1}{3}}$. Then $y' = s(x,y)$ with $y(0) = y_0 \neq 0$ has unique solution

$$y = (\operatorname{sgn} y_0)[x^2(3-2x) + |y_0|^{\frac{2}{3}}]^{\frac{3}{2}}.$$

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Here $\operatorname{sgn} t = 1$ if $t > 0$ and $\operatorname{sgn} t = -1$ if $t < 0$. For $y_0 = 0$, there is a family of solutions, for $0 \leq c \leq 1$:

$$y = \begin{cases} 0 & \text{for } 0 \leq x \leq c, \\ \pm[x^2(3-2x) - c^2(3-2c)]^{\frac{3}{2}} & \text{for } c \leq x \leq 1. \end{cases}$$

Now let y be a solution of $y' = s(x,y)$ on $[0,1]$. Then for $y_0 \neq 0$, $y(1) = (\operatorname{sgn} y_0)[1 + |y_0|^{\frac{2}{3}}]^{\frac{3}{2}}$. In particular, if $|y_0| = 2^{-2k}$, then $|y(1)| = (1 + 2^{-2k/3})^{\frac{3}{2}}$ and if $|y_0| = -2^{-2k}$, then $|y(1)| = -(1 + 2^{-2k/3})^{\frac{3}{2}}$. It follows that $|y(1) - 1| \leq 2^{-2k+2}$ if $y_0 > 0$ and $|y(1) - 1| < 2^{-2k+2}$ if $y_0 < 0$. Furthermore, y can be approximated by $(\operatorname{sgn} y_0)[x^2(3-x)]^{\frac{3}{2}}$ with error $< 2^{-2k+2}$ on $[0,1]$. Finally if $y_0 = 0$, then $(\operatorname{sgn} y_0)[x^2(3-x)]^{\frac{3}{2}}$ is a solution.

Now define $j_{n,a,b}$ as follows:

$$j_{n,a,b}(x,y) = \begin{cases} h_{n,a,b}(x) & \text{for } 0 \leq x \leq 1, \\ s(x-1,y) & \text{for } 1 \leq x \leq 2, \\ -s(x-2,y) & \text{for } 2 \leq x \leq 3, \\ -h_{n,a,b}(x-3) & \text{for } 3 \leq x \leq 4. \end{cases}$$

Simpson proves that $j_{n,a,b}(x,y)$ has the following properties. If $y' = j_{n,a,b}(x,y)$ over $0 \leq x \leq 4$, then $y(4-x) = y(x)$ and $y(2) > 1$ if $n \in W_a$, $y(2) < -1$, if $n \in W_b$, and $-1 \leq y(2) \leq 1$ otherwise. Note that since $h_{n,a,b}(0) = h_{n,a,b}(1) = 0$ for all a and b, it follows that $j_{n,a,b}(0,y) = j_{n,a,b}(4,y) = 0$ for all a and b. Thus we can extend $j_{n,a,b}$ to the whole $\mathbb{R}^2$ if we define $j_{n,a,b}(x,y) = 0$ if $x \notin [0,4]$.

If $y(x)$ is a solution of $y' = j_{n,a,b}(x,y)$ over $0 \leq x \leq 4$, then $y(2)$ determines the solution of $y(x)$ throughout $1 \leq x \leq 2$ and hence also for $0 \leq x \leq 1$. Since $h_{n,a,b}(x) = h_{n,a,b}(1-x)$ and $s(x,y) = s(1-x,y)$, it follows that $j_{n,a,b}(x,y) = -j_{n,a,b}(4-x,y)$. This implies that $y_1(x) = y(4-x)$ is also a solution on $[0,4]$. But then since $y_1(2) = y(2)$, then it must be the case that $y_1(x) = y(x)$ on $[0,4]$ and hence $y(x) = y(4-x)$ on $[0,4]$ so that $y(0) = y(4)$. If in addition, $y(0) = 0$, then $y(2) > 1$ if $n \in W_a$, and $y(2) < -1$ if $n \in W_b$. Finally if $n \notin W_a \cup W_b$, then $-1 \leq y(2) \leq 1$.

Note that under the transformation

$$\hat{x} = 2^{n+2}(x + 2^{-n+1}), \quad \hat{y} = 2^{3(n+2)} \cdot y$$

a solution of $y' = j_{n,a,b}(x,y)$ on the interval $0 \leq x \leq 4$ becomes a solution to

$$y' = 2^{-2(n+2)} j_n(2^{n+2}(x + 2^{-n+1}), 2^{3(n+2)}y)$$

on the interval $-2^{-n+1} \leq x \leq -2^{-n}$. This given, Simpson defines $f_{a,b}(x,y)$ for $x \leq 0$ by

$$f_{a,b}(x,y) = \textstyle\sum_{n=1}^{\infty} 2^{-2(n+2)} j_n(2^{n+2}(x + 2^{-n+1}), 2^{3(n+2)}y),$$

and for $x \geq 0$ by $f_{a,b}(x,y) = -f_{a,b}(-x,y)$. It is easy to see that the construction is completely uniform and that for any a and b, $f_{a,b}$ is computably continuous on $[-1,1] \times [-1,1]$. Thus the desired function φ exists.

Next suppose that y is any computable continuous solution of $y' = f_{a,b}(x,y) = F_{\varphi(a,b)}(x,y)$ with $y(0) = 0$ which is defined on some interval $[-\delta, 0]$. Thus, there is some N such that y is defined on $[-2^{-N}, 0]$. Then for each $n \geq N$, it follows from the properties of $j_{n,a,b}$ that $y(-2^{-n+1}) = y(-2^{-n})$. Since $\lim_{n\to\infty} -2^{-n} = 0$, it follows by continuity that $y(-2^{-n}) = 0$ for all $n \geq N$. But the properties of the $j_{n,a,b}$'s then ensure that for each $n \geq N$, $n \in W_a$ if and only if $y(-2^{-n} - 2^{-n-1}) > 2^{-3(n+2)}$ and $n \in W_b$ if and only if $y(-2^{-n} - 2^{-n-1}) < -2^{-3(n+2)}$. But then we can compute a separating class C for W_a and W_b as follows. For $n < N$, let $n \in C$ if and only if $n \in W_a$. Since we know that either $y(-2^{-n} - 2^{-n-1}) < 2^{-4(n+2)}$ or $y(-2^{-n} - 2^{-n-1}) > -2^{-4(n+2)}$ (possibly both), we approximate $y(-2^{-n} - 2^{-n-1})$ until one of the two conditions holds. If the former, then we know that $n \notin W_a$, so we make $n \notin C$ and if the latter, then we know that $n \notin B$, so we put $n \in C$. This shows that if $F_{\psi(a,b)}$ has a computable solution, then W_a and W_b have a computable separating set.

Now suppose that W_a and W_b have a computable separating set C. We will show how to compute a solution $g(x)$ to the differential equation $y' = F_{\varphi(a,b)}(x,y)$ where $y(0) = 0$ on $-1 \leq x \leq 1$. Since by definition

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$F_{\varphi(a,b)}(x,y) = -F_{\varphi(a,b)}(-x,y)$ and $g(0) = 0$, it is enough to compute g in $[-1,0]$. Since $g(0) = 0$ and under the transformation

$$\hat{x} = 2^{n+2}(x + 2^{-n+1}), \quad \hat{y} = 2^{3(n+2)}y,$$

a solution of $y' = j_{n,a,b}(x,y)$ on the interval $0 \le x \le 4$ becomes a solution of

$$y' = 2^{-2(n+2)} \cdot j_{n,a,b}(2^{n+2}(x + 2^{-n+1}), 2^{3(n+2)}y)$$

on $[-2^{-n+1}, -2^{-n}]$, we must define $g(-2^{-n}) = 0$ for all $n \ge 0$ and $g(x) = 2^{-3(n+2)}G(2^{n+2}(x + 2^{-n+1}))$ for $-2^{-n+1} \le x \le -2^{-n}$, where $G(x)$ is a solution of $y' = j_{n,a,b}(x,y)$ on $0 \le x \le 4$. Thus we need only show that we can compute the function $G(x)$ where $G(x)$ is a solution of $y' = j_{n,a,b}(x,y)$ on $0 \le x \le 4$.

First suppose that $n \in C$ so that $n \notin W_b$. Let k be given. There are two cases. If $n \in W_{a,k+1}$, then we can compute the exact solution of $y' = h_{n,a,b}(x)$ for $0 \le x \le 1$ and we will have $y(1) > 0$. This in turn allows us to compute the unique solution y for $1 \le x \le 2$. By symmetry, we can also compute y on [2,4]. In the second case, suppose that $n \notin W_{a,k+1}$. It follows from the discussion above that $0 \le y \le 2^{-2k}$ on [0,1] and $y - [x^2(3-2x)]^{\frac{3}{2}}| < 2^{-2k+2}$ on [1,2]. This means that we can approximate a solution $y = G(x)$ within 2^{-2k+2} on [0,2] (and also on [2,4] by symmetry). But this is enough to tell us that the solution G is computable on [0,4] as desired.

Similarly suppose that $n \notin C$ so that $n \notin W_a$. Let k be given. Again there are two cases. If $n \in W_{b,k+1}$, then we can compute the exact solution of $y' = h_{n,a,b}(x)$ for $0 \le x \le 1$ and we will have $y(1) < 0$. This in turn allows us to compute the unique solution y for $1 \le x \le 2$. By symmetry, we can also compute y on [2,4]. In the second case, suppose that $n \notin W_{a,k+1}$. It follows from the discussion above that $0 \le y \le 2^{-2k}$ on [0,1] and $y - [x^2(3-2x)]^{\frac{3}{2}}| < 2^{-2k+2}$ on [1,2]. This means that we can approximate a solution $y = G(x)$ within 2^{-2k+2} on [0,2] (and also on [2,4] by symmetry). But again this is enough to tell us that the solution G is computable on [0,4].

Thus we have shown that $y' = F_{\varphi(a,b)}(x,y)$ has computable solution with $y(0) = 0$ if and only if W_a and W_b can be separated by a computable set. □

We observe that the continuous solution φ of the differential equation $\frac{d\varphi}{dt} = f(t, \varphi(t))$ with $\varphi(0) = 0$ is always continuously differentiable on its domain if f continuous. Thus the differential equation $\frac{d\varphi}{dt} = F_a(t, \varphi(t))$ in Theorem 4.1 always has a computably differential solution φ with $\varphi(0) = 0$ on some interval $[-\delta, \delta]$ with $\delta \ge 0$ if it has a computably continuous solution φ with $\varphi(0) = 0$ on some interval $[-\delta, \delta]$. We say that a solution φ to the differential equation $\frac{d\varphi}{dt} = F_a(t, \varphi(t))$ with $\varphi(0) = 0$ is *locally computable on* $[-1, 1]$ if for every open set U such that the closure of U is contained in $(0,1) \times (0,1)$, there is a computable function F_e such that $\varphi = F_e$ on U.

Theorem 4.2 *The set* LC *of all* $a \in I^2$ *such that the differential equation* $\frac{d\varphi}{dt} = F_a(t, \varphi(t))$ *has a solution* φ *with* $\varphi(0) = 0$ *which is locally computable on* $[-1, 1]$ *is* Π^0_4 *complete.*

Proof. It is easy to see that LC is Π^0_4 by writing out the definition.

To see that LC is Π^0_4-complete, let A be a Π^0_4 complete set and let B be a Σ^0_3 set such that $a \in A$ if and only if $\langle n, a \rangle \in B$ for all n. Since B is 1:1 reducible to

$$S = \{\langle a, b \rangle : W_a \cap W_b = \emptyset \text{ and } W_a \text{ and } W_b \text{ have a computable separating set}\},$$

it follows from our proof of Theorem 4.1 that there is a primitive recursive function ψ such that $\langle n, a \rangle \in B$ if and only if the ordinary differential equation $\frac{d\varphi}{dt} = F_{\psi(n,a)}(t, \varphi(t))$, $\varphi(0) = 0$ has a computable solution on $[-1, 1]$. Furthermore, one can check that our definitions ensure that $F_{\psi(n,a)}(x,y) = 0$ for $|x| \ge 1$ and $F_{\psi(n,a)})(x,y) = -F_{\psi(n,a)}(-x,y)$. It follows from the argument above that the solution φ of $y' = F_{\psi(n,a)}(x,y)$ with $\varphi(0) = 0$ will always have $\varphi(-1) = \varphi'(-1) = 0 = \varphi(1) = \varphi'(1)$.

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Now define $F_{\theta(a)}(x,y)$ for $-1 \le x \le 1$ setting for $1-2^n \le x \le 1-2^{-n-1}$

$$F_{\theta(a)}(x,y) = F_{\psi(n,a)}(2^{n+1}(x-1+2^{-n}), 2^n y).$$

For $-1 \le x \le 0$, we set $F_{\theta(a)}(x,y) = -F_{\theta(a)}(-x,y)$. Finally we set $F_{\theta(a)}(x,y) = 0$ if $|x| \ge 1$. Note that our proof of Theorem 4.1 ensures that for any c and d, $F_{c,d}(x,y) = 0$ for $|x| \ge 1$, it easily follows that $F_{\theta(a)}$ is a computably continuous function.

Now suppose that the solution φ of $y' = F_{\theta(a)}(x,y)$ with $\varphi(0) = 0$ is locally computable on $[-1,1]$. Then for each n, $f_{n,a}(x) = 2^n\varphi(1-2^{-n}+2^{-n-1}x)$ restricted to $[1-2^{-n}, \le 1-2^{-n-1}]$ is a solution of $y' = F_{\psi(n,a)}(x,y)$ with $f_{n,a}(0) = 0$. Thus if $\langle n,a\rangle \notin B$ for some n, i.e. $a \notin A$, then $f_{n,a}(x)$ can not be computable on $[-1,1]$ and hence φ is not computable on $[1-2^{-n}, \le 1-2^{-n-1}]$. Thus if there exists an n such that $\langle n,a\rangle \notin B$, then φ is not locally computably on $[-1,1]$. On the other hand, if for all n, $\langle n,a\rangle \notin B$, i.e. $a \in A$, then for all n, $f_{n,a}(x)$ is computable on $[-1,1]$ and hence φ is computable on $[1-2^{-n}, \le 1-2^{-n-1}]$. But this ensures that φ is locally computable on $[0,1]$ and hence by symmetry it is locally computable on $[-1,1]$. Thus we have proved that $a \in A$ if and only if $\theta(a) \in LC$ and hence LC is a Π^0_4 complete set. □

We end this section, by considering the problem of whether a given *wave equation*

$$u_{xx} + u_{yy} + u_{zz} - u_{tt} = 0$$

with initial conditions $u_t(x,y,z,0) = 0$ and $u(x,y,z,0) = F(x,y,z)$ has a computable solution. Myhill [13] constructed a real computable functions f such that $f'(1)$ is not computable. Pour-El and Richards [17] adapted Myhill's example from [13] to construct a computable function $F(x,y,z) = f(\varrho)$ such that the corresponding wave equation has no computable solution and in fact, for the unique solution u, $u(0,0,0,1) = f(1) + f'(1)$ and is thus not computable. We can now give an index set version of this result. We note that Pour-El and Zhong [18] recently strengthened this result to make the unique solution *nowhere computable*, but we do not have a corresponding index set result.

Theorem 4.3 *Let $Wave$ equal the set of all $a \in I^3$ such that the wave equation $u_{xx} + u_{yy} + u_{zz} - u_{tt} = 0$ with initial conditions $u_t(x,y,z,0) = 0$ and $u(x,y,z,0) = F_a(x,y,z)$ has a computable solution. Then $Wave$ is Σ^0_3 complete.*

Proof. One can verify the Σ^0_3 upper bound on the complexity of our index set by observing that $a \in Wave$ if and only if there exists an e such that $u(x,y,z) = F_e$ satisfies the defining conditions. It is then easy to check that the defining conditions are Π^0_2.

For the completeness, we will give a reduction of the well-known Σ^0_3 complete set $\{e : W_e \text{ is computable}\}$ to $Wave$. Following Myhill's example, we define the real number $\sigma_a = \sum_{n \in W_a \oplus \mathbb{N}} 10^{-n}$ so that σ_a is computable if and only if W_a is computable. Assume that we have a uniformly computable one-to-one enumeration $\alpha_a(k)$ of the set $W_{\psi(a)} = W_a \oplus \mathbb{N}$.

We will use the canonical "pulse function" $\Phi(x)$ as defined in the proof of Theorem 3.2 which is a C^∞ function with support $[-1/2, 1/2]$ such that $\varphi(x) \ge 0$ for all x and $\varphi'(0) = 1$. Next let ϱ denote the spherical coordinate $\sqrt{x^2+y^2+z^2}$ and define the computable real function $F_{\varphi(a)}$ as in the proof of Theorem 3.2 by

$$F_{\varphi(a)}(\varrho) = \textstyle\sum_{k=0}^\infty 10^{-(k+\alpha_a(k))}\Phi(10^k(\varrho-1)).$$

Then $F_{\varphi(a)}(x,y,z) = F_{\varphi(a)}(\varrho)$ is computable and we have

$$F'_{\varphi(a)}(\varrho) = \textstyle\sum_{k=0}^\infty 10^{-a(k)}\Phi'(10^k(\varrho-1)). \tag{3}$$

Thus $F'_{\varphi(a)}(1) = \sigma_a$ and hence $F'_{\varphi(a)}(1)$ is computable if and only if W_a is computable.

Kirchhoff's formula [17] gives the solution of (1) as

$$u(\vec{x},t) = \textstyle\iint_S [F(\vec{x}+t\vec{n}) + t(\operatorname{grad} F)(\vec{x}+t\vec{n})\cdot\vec{n}]d\sigma(\vec{n}). \tag{4}$$

Here we will have $(\operatorname{grad} F)(\varrho\cdot\vec{n}) = F'(\varrho)\vec{n}$. It then follows from Kirchhoff's formula that the unique solution u to (1) satisfies $u(0,0,0,1) = F_{\varphi(a)}(1) + F'_{\varphi(a)}(1)$.

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Suppose now that W_a is not computable. Then as we have seen $F'_{\varphi(a)}(1)$ is not computable, so that, since $F_{\varphi(a)}$ is computable, $u(0,0,0,1)$ is not computable and hence the solution u is not computable.

On the other hand, suppose that W_a is computable and let $F = F_{\varphi(a)}$. It follows from (3) that F' is computable and hence the solution may be computed by Kirchhoff's formula as

$$u(\vec{x}\,,t) = \iint_S [F(\vec{x} + t\vec{n}\,) + tF'(\vec{x} + t\vec{n}\,) \cdot \vec{n}\,]d\sigma(\vec{n}\,).$$

□

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