Effective Categoricity of Abelian p-Groups

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Abstract

We investigate effective categoricity of computable Abelian p-groups $\mathcal{A}$. We prove that all computably categorical Abelian p-groups are relatively computably categorical, that is, have computably enumerable Scott families of existential formulas. We investigate which computable Abelian p-groups are Δ^0_2 categorical and relatively Δ^0_2 categorical.

1 Introduction

In computable model theory we are interested in effective versions of model theoretic notions and constructions. We consider in particular computability theoretic bounds on the complexity of isomorphisms of structures within the same isomorphism type. This paper is a sequel to [6] where we looked at equivalence structures. Here we will examine computable Abelian groups. We consider only

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countable structures for computable languages, and for infinite structures we may assume that their universe is ω. We identify sentences with their Gödel codes. The *atomic diagram* of a structure $\mathcal{A}$ for L is the set of all quantifier-free sentences in L_A, L expanded by constants for the elements in A, which are true in $\mathcal{A}$. A structure is *computable* if its atomic diagram is computable. In other words, a structure $\mathcal{A}$ is computable if there is an algorithm that determines for every quantifier-free formula $\theta(x_0, \ldots, x_{n-1})$ and every sequence $(a_0, \ldots, a_{n-1}) \in A^n$, whether $\mathcal{A} \vDash \theta(a_0, \ldots, a_{n-1})$. The *elementary diagram* of $\mathcal{A}$ is the set of all sentences of L_A that are true in $\mathcal{A}$. A structure $\mathcal{A}$ is *decidable* if its elementary diagram is computable. For $n > 0$, the *n-diagram* of $\mathcal{A}$ is the set of all Σ_n sentences of L_A that are true in $\mathcal{A}$. A structure is *n-decidable* if its n-diagram is computable.

A computable structure $\mathcal{A}$ is *computably categorical* if for every computable isomorphic copy $\mathcal{B}$ of $\mathcal{A}$, there is a *computable* isomorphism from $\mathcal{A}$ onto $\mathcal{B}$. For example, the ordered set of rational numbers is computably categorical, while the ordered set of natural numbers is not. Moreover, Goncharov and Dzgoev [15], and Remmel [32] independently proved that a computable linear ordering is computably categorical if and only if it has only finitely many successors. Furthermore, Goncharov and Dzgoev [15], and Remmel [33] established that a computable Boolean algebra is computably categorical if and only if it has finitely many atoms (see also LaRoche [25]). Miller [30] proved that no computable tree of height ω is computably categorical. Lempp, McCoy, Miller, and Solomon [26] characterized computable trees of finite height that are computably categorical. Nurtazin [31], and Metakides and Nerode [28] established that a computable algebraically closed field of finite transcendence degree over its prime field is computably categorical.

The present paper will be concerned with the categoricity of Abelian p-groups Goncharov [12] and Smith [35] characterized computably categorical Abelian p-groups as those that can be written in one of the following forms: $(\mathbb{Z}(p^\infty))^l \oplus \mathcal{G}$ for $l \in \omega \cup \{\infty\}$ and $\mathcal{G}$ is finite, or $(\mathbb{Z}(p^\infty))^n \oplus \mathcal{G} \oplus (\mathbb{Z}(p^k))^\infty$, where $n, k \in \omega$ and $\mathcal{G}$ is finite. Goncharov, Lempp, and Solomon [18] proved that a computable, ordered, Abelian group is computably categorical if and only if it has finite rank. Similarly, they showed that a computable, ordered, Archimedean group is computably categorical if and only if it has finite rank.

In the recent paper [6], the authors showed that a computable equivalence structure $\mathcal{A}$ is computably categorical if and only if $\mathcal{A}$ has at most finitely many finite equivalence classes, or $\mathcal{A}$ has only finitely many infinite classes and there is a finite bound on the size of the finite classes and there is at most one finite k such that $\mathcal{A}$ has infinitely many classes of size k. We characterized the relatively Δ^0_2 categorical equivalence structures as those with either finitely many infinite equivalence classes, or with an upper bound on the size of the finite equivalence classes. We also consider the complexity of isomorphisms for structures $\mathcal{A}$ and $\mathcal{B}$ such that both $Fin^{\mathcal{A}}$ and $Fin^{\mathcal{B}}$ are computable, or Δ^0_2. Finally, we show that every computable equivalence structure is relatively Δ^0_3 categorical.

For any computable ordinal α, we say that a computable structure $\mathcal{A}$ is *Δ^0_α categorical* if for every computable structure $\mathcal{B}$ isomorphic to $\mathcal{A}$, there is a Δ^0_α

isomorphism form $\mathcal{A}$ onto $\mathcal{B}$. Lempp, McCoy, Miller, and Solomon [26] proved that for every $n \geq 1$, there is a computable tree of finite height, which is Δ^0_{n+1} categorical but not Δ^0_n categorical. We say that $\mathcal{A}$ is *relatively computably categorical* if for every structure $\mathcal{B}$ isomorphic to $\mathcal{A}$, there is an isomorphism that is computable relative to the atomic diagram of $\mathcal{B}$. Similarly, a computable $\mathcal{A}$ is *relatively Δ^0_α categorical* if for every $\mathcal{B}$ isomorphic to $\mathcal{A}$, there is an isomorphism that is Δ^0_α relative to the atomic diagram of $\mathcal{B}$. Clearly, a relatively Δ^0_α categorical structure is Δ^0_α categorical. We are especially interested in the case when $\alpha = 2$. McCoy [27] characterized, under certain restrictions, Δ^0_2 categorical and relatively Δ^0_2 categorical linear orderings and Boolean algebras. For example, a computable Boolean algebra is relatively Δ^0_2 categorical if and only if it can be expressed as a finite direct sum $c_1 \vee \ldots \vee c_n$, where each c_i is either atomless, an atom, or a 1-atom. Using an enumeration result of Selivanov [34], Goncharov [13] showed that there is a computable structure, which is computably categorical but not relatively computably categorical. Using a relativized version of Selivanov's enumeration result, Goncharov, Harizanov, Knight, McCoy, Miller, and Solomon [16] showed that for each computable successor ordinal α, there is a computable structure, which is Δ^0_α categorical but not relatively Δ^0_α categorical.

It is not known whether for a computable limit ordinal α, there is a computable structure that is Δ^0_α categorical but not relatively Δ^0_α categorical (see [16]). It is also not known whether for any computable successor ordinal α, there is a rigid computable structure that is Δ^0_α categorical but not relatively Δ^0_α categorical. Another open question is whether every Δ^1_1 categorical computable structure must be relatively Δ^1_1 categorical (see [17]).

There are syntactic conditions that are equivalent to relative Δ^0_α categoricity. These conditions involve the existence of certain families of formulas, that is, certain Scott families. Scott families come from Scott's Isomorphism Theorem, which says that for a countable structure $\mathcal{A}$, there is an $L_{\omega_1\omega}$ sentence whose countable models are exactly the isomorphic copies of $\mathcal{A}$. A *Scott family* for a structure $\mathcal{A}$ is a countable family Φ of $L_{\omega_1\omega}$ formulas, possibly with finitely many fixed parameters from A, such that:

(i) Each finite tuple in $\mathcal{A}$ satisfies some $\psi \in \Phi$;

(ii) If $\overrightarrow{a}$, $\overrightarrow{b}$ are tuples in $\mathcal{A}$, of the same length, satisfying the same formula in Φ, then there is an automorphism of $\mathcal{A}$ that maps $\overrightarrow{a}$ to $\overrightarrow{b}$.

A *formally c.e. Scott family* is a c.e. Scott family consisting of finitary existential formulas. A *formally Σ^0_α Scott family* is a Σ^0_α Scott family consisting of computable Σ_α formulas. Roughly speaking, computable infinitary formulas are $L_{\omega_1\omega}$ formulas in which the infinite disjunctions and conjunctions are taken over computably enumerable (c.e.) sets. We can classify computable formulas according to their complexity as follows. A computable Σ_0 or Π_0 formula is a finitary quantifier-free formula. Let $\alpha > 0$ be a computable ordinal. A computable Σ_α formula is a c.e. disjunction of formulas $(\exists \overrightarrow{u})\theta(\overrightarrow{x}, \overrightarrow{u})$, where θ is computable Π_β for some $\beta < \alpha$. A computable Π_α formula is a c.e. conjunction of formulas $(\forall \overrightarrow{u})\theta(\overrightarrow{x}, \overrightarrow{u})$, where θ is computable Σ_β for some $\beta < \alpha$. Precise definition of computable infinitary formulas involves assigning indices to the formulas, based on Kleene's system of ordinal notations (see [2]). The important property of these formulas is given in the following theorem due to Ash.

Theorem 1.1 *For a structure $\mathcal{A}$, if $\theta(\overrightarrow{x})$ is a computable Σ_α formula, then the set $\{\overrightarrow{a} : \mathcal{A} \models \theta(\overrightarrow{a})\}$ is Σ^0_α relative to the atomic diagram of $\mathcal{A}$.*

An analogous result holds for computable Π_α formulas.

It is easy to see that if $\mathcal{A}$ has a formally c.e. Scott family, then $\mathcal{A}$ is relatively computably categorical. In general, if $\mathcal{A}$ has a formally Σ^0_α Scott family, then $\mathcal{A}$ is relatively Δ^0_α categorical. Goncharov [13] showed that if $\mathcal{A}$ is 2-decidable and computably categorical, then it has a formally c.e. Scott family. Ash [1] showed that, under certain decidability conditions on $\mathcal{A}$, if $\mathcal{A}$ is Δ^0_α categorical, then it has a formally Σ^0_α Scott family. For the relative notions, the decidability conditions are not needed. Moreover, Ash, Knight, Manasse, and Slaman [3], and independently Chisholm [8] established the following result.

Theorem 1.2 *Let $\mathcal{A}$ be a computable structure. Then the following are equivalent:*

(a) A *is relatively Δ^0_α categorical;*

(b) A *has a formally Σ^0_α Scott family;*

(c) A *has a c.e. Scott family consisting of computable Σ_α formulas.*

Cholak, Goncharov, Khoussainov, and Shore [9] gave an example of a computable structure that is computably categorical, but ceases to be after naming any element of the structure. Such a structure is not relatively computably categorical. On the other hand, Millar [29] previously established that if a structure $\mathcal{A}$ is 1-decidable, then any expansion of $\mathcal{A}$ by finitely many constants remains computably categorical. Khoussainov and Shore [22] proved that there is a computably categorical structure without a formally c.e. Scott family whose expansion by any finite number of constants is computably categorical. A similar result was established by Kudinov by a different method. Using a modified family of enumerations constructed by Selivanov [34], Kudinov produced a computably categorical, 1-decidable structure without a formally c.e. Scott family.

A structure is *rigid* if it does not have nontrivial automorphisms. A computable structure is Δ^0_α *stable* if every isomorphism from $\mathcal{A}$ onto a computable structure is Δ^0_α. If a computable structure is rigid and Δ^0_α categorical, then it is Δ^0_α stable. A *defining family* for a structure $\mathcal{A}$ is a set Φ of formulas with one free variable and a fixed finite tuple of parameters from A such that:

(i) Every element of A satisfies some formula $\psi \in \Phi$;

(ii) No formula of Φ is satisfied by more than one element of A.

A defining family Φ is *formally* Σ^0_α if it is a Σ^0_α set of computable Σ_α formulas. In particular, a defining family Φ is *formally c.e.* if it is a c.e. set of finitary existential formulas. For a rigid computable structure $\mathcal{A}$, there is a formally Σ^0_α

Scott family iff there is a formally Σ^0_α defining family. We say that two tuples $\langle g_1, \ldots, g_n \rangle$ and $\langle g'_1, \ldots, g'_n \rangle$ in $\mathcal{G}$ are *automorphic* if there is an automorphism of $\mathcal{G}$ taking each g_i to g'_i.

In Section 2, we investigate algorithmic properties of Abelian groups and their characters, as well as the provide a connection between equivalence structures and Abelian p-groups. In Section 3, we examine effective categoricity of Abelian p-groups. We show that every computably categorical Abelian p-group is also relatively computably categorical.

The notions and notation of computability theory are standard and as in Soare [36]. We fix $\langle \cdot, \cdot \rangle$ to be a computable bijection from ω^2 onto ω. Let $(W_e)_{e \in \omega}$ be an effective enumeration of all c.e. sets.

2 Computable Abelian p-groups and Equivalence Structures

A computable Abelian p-group $\mathcal{A} = (A, +^{\mathcal{A}}, 0)$ where we assume that $A = \omega$ and that 0 is the identity for the operation $+^{\mathcal{A}}$. It is immediate that the subtraction function $-^{\mathcal{A}}$ as well as the inverse function are also computable. In this section, we will focus on direct sums of cyclic groups and quasicyclic groups and their connection with equivalence structures.

Some definitions are needed. Let p be a prime number. The group $\mathcal{G}$ is said to be a p-group if, for all $g \in G$, $|g|$ is a power of p. $\mathbb{Z}(p^n)$ is the cyclic group of order p^n. $\mathbb{Z}(p^\infty)$ denotes the quasicyclic p-group, the direct limit of the sequence $\mathbb{Z}(p^n)$ and also the set of rationals in $[0, 1)$ of the form $\frac{i}{p^n}$ with addition modulo 1.

Definition 2.1 *The* period *of G is $max\{|g| : g \in G\}$ if this quantity is finite, and ∞ otherwise.*

The subgroups $p^\alpha G$, where α is an ordinal, are defined recursively as follows.

$p^0 G = G$, $pG = \{px : x \in G\}$

$p^{\alpha+1} G = p(p^\alpha G)$ and

$p^\lambda G = \bigcap_{\alpha < \lambda} p^\alpha G$ for limit λ.

The *length* of $\mathcal{G}$, $lh(\mathcal{G})$, is the least ordinal α such that $p^{\alpha+1}(G) = p^\alpha(G)$. The *divisible part* of $\mathcal{G}$ is $D(\mathcal{G}) = p^{lh(G)}$ and is a subgroup of $\mathcal{G}$. $\mathcal{G}$ is said to be *reduced* if $D(\mathcal{G}) = \{0\}$.

For an element $g \in G$, the height $ht(g)$ is ∞ if $g \in p^{lh(G)}$ and is otherwise the least α such that $g \notin p^{\alpha+1} G$. For a computable group G, $ht(g)$ can be an arbitrary computable ordinal. The height of G is the supremum of $\{ht(g) : g \in G\}$.

Here are some classic results about Abelian p-groups which we will need. The reader is referred to Fuchs [11] for a full development of the theory of infinite Abelian groups.

Theorem 2.2 1. ***(Baer)*** *For any p-group $\mathcal{G}$, there exists a subgroup $\mathcal{A}$ such that $G = \mathcal{A} \oplus D(\mathcal{G})$.*

2. ***(Prüfer)*** *If G is a countable Abelian p-group, then G is a direct sum of cyclic groups if and only if all nonzero elements have finite height.*

Definition 2.3 *Let $\mathcal{A}$ be a subgroup of $\mathcal{G}$.*

1. *$\mathcal{A}$ is a* direct summand *of $\mathcal{G}$ if there exists a subgroup $\mathcal{B}$ of $\mathcal{G}$ such that $\mathcal{G} = \mathcal{A} \oplus \mathcal{B}$.*

2. *$\mathcal{A}$ is a* pure *subgroup of $\mathcal{G}$ if $A \cap p^n G = p^n A$ for all n, that is the height of an element $a \in A$ is the same in $\mathcal{A}$ as it is in $\mathcal{G}$.*

We need some results from group theory on direct summands. See [11] for details.

Theorem 2.4 1. *(Kulikov) If $\mathcal{A}$ has finite period and is a pure subgroup of $\mathcal{G}$, then $\mathcal{A}$ is a direct summand of $\mathcal{G}$.*

2. *(Baer) Any divisible subgroup $\mathcal{D}$ of a group $\mathcal{A}$ is a direct summand of $\mathcal{A}$.* $\square$

The *Ulm subgroups* G^α are defined by $G^\alpha = p^{\omega\alpha}G$. The αth *Ulm factor* G_α of G is $G^\alpha/G^{\alpha+1}$ and the *Ulm length* $\lambda(A)$ of G is the least α such that $G^\alpha = G^{\alpha+1}$.

It follows from Theorem 2.2 that each Ulm factor is a direct sum of cyclic groups. Thus each Ulm factor G_α is a direct sum of cyclic groups. Now consider the sequence of

$$P_\alpha(G) = G_\alpha \cap \{x \in G : px = 0\}.$$

Let $u_\alpha(G) = \dim_{\mathbb{Z}_p} P_\alpha(G)/P_{\alpha+1}(G)$.

Theorem 2.5 (Ulm) *Two Abelian p-groups G and H are isomorphic if and only if they have the same Ulm sequence, that is, if and only if $\lambda(G) = \lambda(H)$ and $u_\alpha(G) = u_\alpha(H)$ for all α.*

Definition 2.6 1. *$\oplus_\alpha \mathcal{H}$ denotes the direct sum of α copies of $\mathcal{H}$ where $\alpha \leq \omega$.*

2. *If $\mathcal{A} = \oplus_{i<\omega} Z(p^{n_i})$, then the* character *of $\mathcal{A}$ is*

$$\chi(\mathcal{A}) = \{(n, k) : card(\{i : n_i = n\}) \geq k\}.$$

3. *If $\mathcal{G} = \mathcal{A} \oplus \oplus_\alpha Z(p^\infty)$ for some $\alpha \leq \omega$ and some $\mathcal{A}$ as above, then $\chi(\mathcal{G}) = \chi(\mathcal{A})$.*

4. *$\mathcal{G}$ is said to have* bounded character *if for some finite b and all $(n, k) \in \chi(\mathcal{G})$, $n \leq b$, and is said to have* unbounded character *otherwise.*

In the previous paper [6], a similar notion of character for equivalence structures was studied and structures of various characters were constructed. We will show that for a general class of such structures, a corresponding p-group *with the same character* may be constructed from a given equivalence structure. Here are the basic definitions of computable equivalence structures.

An equivalence structure $\mathcal{A} = (A, E^{\mathcal{A}})$ consists of a set with a binary relation that is reflexive, symmetric, and transitive. An equivalence structure $\mathcal{A}$ is *computable* if A is a computable subset of ω and E is a computable relation. If A is an infinite set (which is usual), we may assume, without loss of generality, that $A = \omega$. The $\mathcal{A}$-equivalence class of $a \in A$ is

$$[a]^{\mathcal{A}} = \{x \in A : xE^{\mathcal{A}}a\}.$$

We generally omit the superscript $^{\mathcal{A}}$ when it can be inferred from the context.

Definition 2.7 (*i*) *Let* $\mathcal{A}$ *be an equivalence relation. The* character *of* $\mathcal{A}$ *is the set*

$$\chi(\mathcal{A}) = \{\langle k, n\rangle :\ n, k > 0 \text{ and } \mathcal{A} \text{ has at least } n \text{ equivalence classes of size } k\}.$$

(*ii*) *We say that* $\mathcal{A}$ *has* bounded character *if there is some finite* k *such that all finite equivalence classes of* $\mathcal{A}$ *have size at most* k*.*

For both groups and equivalence relations, we may define the notion of a *character* as a subset K of $(\omega - \{0\}) \times \omega$ such that for all n and k, $(k, n+1) \in K \Rightarrow (k, n) \in K$. Let $o_{\mathcal{G}}(g)$ be the order of g in $\mathcal{G}$. The $\mathcal{G}$ may be omitted when it is clear.

Theorem 2.8 (Khisamiev [20]) *Suppose that* $\mathcal{G}$ *is a computable Abelian* p*-group and* $\mathcal{G}$ *is isomorphic to* $\oplus_\alpha Z(p^\infty \oplus \oplus_{i<\omega} Z(p^{n_i})$ *Then*

1. $\{\langle a, n\rangle : o(g) = p^n\}$ *is computable*
2. $\{\langle a, n\rangle : ht(g) \geq n\}$ *is* Σ^0_1.
3. $D(\mathcal{G})$ *is a* Π^0_2 *set (recall that* $D(\mathcal{G})$ *is the divisible part of* $\mathcal{G}$.
4. *The character* $\chi(\mathcal{G})$ *is a* Σ^0_2 *set.*

Proof: (1) $o(g) = p^n \iff p^n \cdot g = 0 \;\&\; p^{n-1} \cdot g \neq 0$.

(2) $ht(g) \geq p^n \iff (\exists h) p^n \cdot h = g$.

(3) Under the hypothesis, $ht(g) \geq \omega$ implies that $g \in D(\mathcal{G})$, so that $g \in D(\mathcal{G}) \iff (\forall n) ht(g) \geq n$.

(4) We have the following characterization of $\chi(\mathcal{G})$ by Theorem 2.4: $\langle n, k\rangle \in \chi(\mathcal{G})$ if and only if there exist $g_0, \dots, g_{k-1}$ such that, for all $i < k$, $o(g) = p^n$ and $ht(g) = 0$ and

$$(*)(\forall c_0, c_1, \dots, c_{k-1} < p^n)(c_0 \cdot g_0 + \cdots + c_{k-1} \cdot g_{k-1} = 0 \rightarrow (\forall i < k) c_i = 0)].$$

That is, if $\langle n, k\rangle \in \chi(\mathcal{G})$, then $\mathcal{G}$ has at least n summands $\langle g_1\rangle, \ldots, \langle g_k\rangle$ isomorphic to $Z(p^k)$ and the sequence $g_1, \ldots, g_n$ satisfies (*). On the other hand, if $g_0, \ldots, g_{k-1}$ satisfy (*), then they generate a bounded, pure subgroup $\mathcal{A}$ of $\mathcal{G}$ which must be a summand by Theorem 2.4. Hence $\mathcal{G}$ has at least k summands of the form $Z(p^n)$. $\square$

Here is a connection between equivalence structures and p-groups.

Theorem 2.9 *Let p be a prime number and let $\mathcal{A}$ be a computable equivalence structure with character K and with α infinite equivalence classes. We write $Inf(\mathcal{A})$ to denote the set of elements in $\mathcal{A}$ whose equivalence classes are infinite. Then there exists a computable p-group $\mathcal{G}$ isomorphic to*

$$\mathcal{H} \oplus \bigoplus_{\alpha} Z(p^\infty)$$

where $\mathcal{H}$ is a direct sum of cyclic p-groups with character K. Furthermore, if $Inf(\mathcal{A})$ is Σ^0_1, then $D(\mathcal{G})$ is also Σ^0_1 and if $Inf(\mathcal{A})$ is computable, then $D(\mathcal{G})$ is also computable.

Proof: Let $\mathcal{A}$, α and K be given as stated and let $\equiv$ denote $\equiv^{\mathcal{A}}$. First define the computable set B of basic elements of $\mathcal{G}$ to consist of $\{a \in A : (\forall m < a)\neg(m \equiv a)$. Then enumerate B in numerical order as $b_0, b_1, \ldots$. The group $\mathcal{G}$ will be the direct sum of $\mathcal{G}_i$, where the groups $\mathcal{G}_i$ and $\mathcal{G}$ are constructed in stages $\mathcal{G}^s_i$, Initially, $\mathcal{G}^0_0 = \{0, 1, \ldots, p-1\}$ is a copy of Z_p and in general $\mathcal{G}^s_i$ is a copy of $Z(p^k)$, where $k = card(\{j < s : j \equiv b_i\})$ and $\mathcal{G}^s$ is isomorphic to the direct sum $\mathcal{G}^s_0 \oplus \mathcal{G}^s_1 \oplus \cdots \oplus G^s_s$.

At stage $s+1$, we initiate the component group G^{s+1}_{s+1} and, for each $i \leq s$, we check to see whether $s+1 \neq b_i$ and $s+1 \equiv b_i$. If not, just let $\mathcal{G}^{s+1}_i = \mathcal{G}^s_i$. If so, then extend $\mathcal{G}^s_i$ to a copy of $Z(p^{k+1})$, where G^s_i is a copy of $Z(p^k)$. That is, given that G^s_i is a cyclic group of order p^k with generator a, we put a new element b into G^{s+1}_i so that $p \cdot b = a$ and also add elements to represent $i \cdot b + g$ for $i = 1, 2, \ldots, p-1$ and $g \in G^s_i$. Then we also add elements to $\mathcal{G}$ to represent the new elements of $G^{s+1}_0 \oplus \cdots \oplus G^{s+1}_{s+1}$. This is done so that the elements of $\mathcal{G}^{s+1}$ are an initial segment of ω.

$\mathcal{G}$ will be a computable group, since for each $a \in \omega$, $a \in \mathcal{G}^a$ and for any two elements $a \leq b$, $a+^{\mathcal{G}}$ is defined by stage b.

It is clear that if $card([b_i]) = n$ in $\mathcal{A}$, then for some s and all $t \geq s$, $\mathcal{G}^t_i$ is isomorphic to $Z(p^n)$ and if $card([b_i]) = \omega$ in $\mathcal{A}$, then the inverse limit $\mathcal{G}_i$ of $\langle G^s_i : s < \omega\rangle$ will be a copy of $Z(p^\infty)$. Thus $\mathcal{G}$ has character K and has α components of $Z(p^\infty)$, as desired.

For the furthermore clause, suppose that $Inf(\mathcal{A})$ is a Σ^0_1 (respectively, computable) set. Then $\{i : b_i \in Inf(\mathcal{A})\}$ is also Σ^0_1 (respectively computable). Now a sequence $\langle c_0, \ldots, c_{k-1}\rangle \in \mathcal{G}$ is in $D(\mathcal{G})$ if and only if, for all $i < k$, if $c_i \neq 0$, then $i \in Inf(\mathcal{A})$. $\square$

There are several corollaries to results from [6].

Corollary 2.10 *For any Σ_2^0 character K, there is a computable Abelian p-group $\mathcal{G}$ with character K and with $D(\mathcal{G})$ isomorphic to $\oplus_\omega Z(p^\infty)$. Furthermore, $D(\mathcal{G})$ is a Σ_1^0 set.*

Proof: By Lemma 2.3 of [6], there will be a computable equivalence structure $\mathcal{A}$ with character K such that $Inf(\mathcal{A})$ is Σ_1^0 and the result is now immediate from Theorem 2.9. □

Corollary 2.11 *For any $r \leq \omega$ and any bounded character K, there is a computable p-group $\mathcal{G}$ with character K and with $D(\mathcal{G})$ isomorphic to $\oplus_r Z(p^\infty)$. Furthermore, $D(\mathcal{G})$ is a computable set.*

Proof: By Lemma 2.4 of [6], there exists a computable equivalence structure $\mathcal{A}$ with character K, with exactly r equivalence classes, and with $Inf(\mathcal{A})$ computable. The result now follows from Theorem 2.9. □

If the character is not bounded, then the notions of an s-function and an s_1-function are important. These functions were introduced by Khisamiev in [19]. The s-functions are called *limitwise monotonic* in [21].

Definition 2.12 *Let $f : \omega^2 \to \omega$. The function f is an s-*function *if the following hold:*

1. *For every i and s, $f(i,s) \leq f(i,s+1)$;*
2. *For every i, the limit $m_i = lim_s f(i,s)$ exists.*

   *We say that f is an s_1-*function *if, in addition:*
3. *For every i, $m_i < m_{i+1}$.*

The following result about characters, s-functions and s_1-functionsis immediate from Lemma 2.6 of [6].

Lemma 2.13 *Let $\mathcal{G}$ be a computable p-group with infinite character and with $D(\mathcal{G})$ isomorphic to $\oplus_r Z(p^\infty)$ where r is finite. Then*

1. *There exists a computable s-function f with corresponding limits $m_i = lim_s f(i,s)$ such that $\langle k, n\rangle \in \chi(\mathcal{G}))$ if and only if $card(\{i : k = m_i\}) \geq n$.*
2. *If the character is unbounded, then there is a computable s_1-function f such that $\langle m_i, 1\rangle \in \chi(\mathcal{G})$ for all i.* □

Corollary 2.14 *Let K be a Σ_2^0 character, and let r be finite.*

1. *Let f be a computable s-function with the corresponding limits $m_i = lim_s f(i,s)$ such that $\langle n, k\rangle \in K \iff card(\{i : k = m_i\}) \geq n$. Then there is a computable Abelian p-group $\mathcal{G}$ with $\chi(\mathcal{G})) = K$ and with $D(\mathcal{G})$ isomorphic to $\oplus_r Z(p^\infty)$.*

2. *Let f be a computable s_1-function with corresponding limits $m_i = lim_s f(i,s)$ such that $\langle m_i, 1\rangle \in K$ for all i. Then there is a computable Abelian p-group $\mathcal{G}$ with $\chi(\mathcal{G}) = K$ and with $D(\mathcal{G})$ isomorphic to $\oplus_r Z(p^\infty)$.*

Proof: This follows from Theorem 2.9 and from Lemma 2.8 of [6] where corresponding equivalence structures are constructed. $\square$

3 Categoricity of Abelian p-groups

The computably categorical Abelian p-groups were characterized by Smith [35] as follows.

Theorem 3.1 (Smith) *A computable Abelian p-group $\mathcal{G}$ is computably categorical if and only if either*

1. *$\mathcal{G} \approx \oplus_\omega Z(p^\infty) \oplus F$, or*
2. *$\mathcal{G} \approx \oplus_r Z(p^\infty) \oplus \oplus_\omega Z(p^m) \oplus F$, where F is a finite Abelian p-group and $r, m \in \omega$.*

Lemma 3.2 1. *If the computable groups $\mathcal{G}$ and $\mathcal{H}$ are relatively Δ^0_α categorical and $\mathcal{G}$ and $\mathcal{H}$ are Σ^0_1 definable in $\mathcal{G} \oplus \mathcal{H}$, then $\mathcal{G} \oplus \mathcal{H}$ is relatively Δ^0_α categorical.*

2. *If the computable groups $\mathcal{G}_1, \mathcal{G}_2, \ldots, \mathcal{G}_k$ are relatively computably categorical each $\mathcal{G}_i$ is Σ^0_1 definable in $\mathcal{G} = \mathcal{G}_1 \oplus \cdots \oplus G_n$, then $\mathcal{G}$ is relatively Δ^0_2 categorical.*

Proof: (1) The Scott formulas for $\mathcal{G}$ and $\mathcal{H}$ may be modified for $\mathcal{G} \oplus \mathcal{H}$ to quantify only over $\mathcal{G}$ and $\mathcal{H}$. Then for an element $a = g + h \in \mathcal{G} \oplus \mathcal{H}$, the Scott formula is

$$(\exists y \in G)(\exists z \in H)[x = y + z \;\&\; \phi^G(y) \;\&\; \psi^H(z)],$$

where ϕ^G is the Scott formula for g, relativized to $\mathcal{G}$, and ψ^H is the Scott formula for h, relativized to $\mathcal{H}$. It can be checked that these formulas will be Σ^0_α. For tuples $\langle a_1, \ldots, a_n\rangle$, the method is the same. If $\langle a_1, \ldots, a_n\rangle$ and $\langle a'_1, \ldots, a'_n\rangle$ satisfy the same Scott formula, then we have $a_i = g_i + h_i$ and $b_i = g'_i + h'_i$ where g_i and g'_i satisfy the same Scott formula in $\mathcal{G}$ so that there is an automorphism Φ of $\mathcal{G}$ taking g_i to g'_i and similarly there is automorphism Ψ of $\mathcal{H}$ taking h_i to h'_i. Then the mapping taking $x + y$ to $\Phi(x) + \Psi(y)$ will be an automorphism of $\mathcal{G} \oplus \mathcal{H}$ taking each a_i to a'_i. Therefore these formulas make up a Scott family as desired, so that $\mathcal{G} \oplus \mathcal{H}$ is relatively Δ^0_α categorical.

(2) The relativized Scott formulas will now be Σ^0_2 and the proof follows as in part (1). $\square$

We can now investigate the relative computable categoricity of computable Abelian p-groups.

Theorem 3.3 *If $\mathcal{G}$ is a computably categorical Abelian p-group, then $\mathcal{G}$ is relatively computably categorical.*

Proof: By Theorem 3.1, we have an expression for the form of $\mathcal{G}$. Any finite structure is certainly relatively computably categorical, so we may ignore the $\mathcal{F}$ by Lemma 3.2.

(1) If all summands are of the form $Z(p^\infty)$, then the Scott sentence for a tuple $\langle g_1, \dots, g_n \rangle$ simply tells the order o_i of each g_i and tells whether $c_1 \cdot g_1 + \cdots c_n \cdot g_n = 0$ for all $c_1 < o_1, \dots, c_n < o_n$. Suppose that single elements g and g' have the same order p^k. Then there are divisible subgroups $\mathcal{D}$ and $\mathcal{D}'$ of $\mathcal{G}$ containing g and g' (respectively), each isomorphic to $Z(p^\infty)$. Since g and g' have the same order, there is an isomorphism taking $\mathcal{D}$ to $\mathcal{D}'$ which maps g to g' and since $\mathcal{D}$ and $\mathcal{D}'$ are direct summands of $\mathcal{G}$ by Theorem 2.4, this may be extended to an isomorphism of $\mathcal{G}$ taking g to g'. This shows that groups of type (1) are relatively computably categorical.

(2) We may assume that $\mathcal{F} = 0$ and $\mathcal{G} = \mathcal{D} \oplus \mathcal{H}$, where $\mathcal{D} = \oplus_r Z(p^\infty)$ and let $\mathcal{H} = \oplus_\omega Z(p^m)$. We claim that $\mathcal{D}$ is Δ^0_1 definable, by the following. First, note that $g \in \mathcal{D}$ if and only if g is divisible by p^m, so that $\mathcal{D}$ is Σ^0_1. Second, note that there are exactly p^{kr} elements in $\mathcal{D}$ of order $\le p^k$. Now given $g \in G$, it follows that $g \in \mathcal{D}$ if and only if, for any p^{kr} distinct elements of $\mathcal{D}$ with order at most p^{kr}, our g equals one of those elements. This gives a Π^0_1 formula for $\mathcal{D}$. The Scott formula for a single element $g \in \mathcal{G}$ tells whether $c \cdot g \in \mathcal{D}$ for $c = 1, p, \dots, p^n = o(g)$. Now suppose that g_1 and g_2 have the same Scott formula. If both are divisible, they are automorphic as in part (1). Now suppose $g_1 \notin \mathcal{D}$ and that $p^k \cdot g_1$ is not divisible for any $p^k < o(g_1)$. Then $g_i = d_i + h_i$ where $d_i \in \mathcal{D}$ and $h_i \in \mathcal{H}$ with $o(h_i) = p^m$. By the previous argument, we may assume that $d_1 = d_2$. Now for each i, h_i generates a pure subgroup $\mathcal{H}_i$ of $\mathcal{H}$ of order p^m, so by Theorem 2.4, we have $\mathcal{H} = \mathcal{H}_i \oplus \mathcal{C}_i$ for some (isomorphic) subgroups C_i of $\mathcal{H}$. There is certainly an isomorphism of $\mathcal{H}_1$ onto $\mathcal{H}_2$ taking h_1 to h_2 and this may be extended to an automorphism of $\mathcal{G}$ taking g_1 to g_2. Now suppose that $p^k \cdot g_1$ is divisible for some k with $p^k < o(g_1)$. Then $g_1 = d_1 + h_1$, where $p^k \cdot h_1 = 0$ and hence h_1 is divisible by p^{m-k}. Thus we can find d'_1 and h'_1 with $g'_1 = d'_1 + h'_1$ such that $g_1 = p^{m-k}(d'_1 + h'_1)$ and similarly for g_2 and g'_2. It follows from the previous argument that g'_1 and g'_2 are automorphic and the same automorphism takes g_1 to g_2.

For a sequence $\langle g_1, \dots, g_n \rangle$ from $\mathcal{G}$, the Scott formula includes the Scott formulas for each element and also tells which linear combinations $c_1 \cdot g_1 + \cdots + c_n \cdot g_n = 0$ and which are divisible, where each $c_i < o(g_i)$. We prove by induction on n that if $\langle g_1, g_2, \dots g_n \rangle$ and $\langle g'_1, g'_2, \dots g'_n \rangle$ satisfy the same Scott formula, then they are automorphic. The case $n = 1$ is given above. For $n > 1$, suppose that $\langle g_1, g_2, \dots g_n \rangle$ and $\langle g'_1, g'_2, \dots g'_n \rangle$ satisfy the same Scott formula; it follows that $\langle g_1, g_2, \dots g_{n-1} \rangle$ and $\langle g'_1, g'_2, \dots g'_{n-1} \rangle$ also satisfy the same Scott formula, and are therefore automorphic by induction. There are two cases. If some constant c_i in a true equation is not divisible by p, then without loss of generality we can solve the equation for g_n and use the observation above that $\langle g_1, g_2, \dots g_{n-1} \rangle$ and $\langle g'_1, g'_2, \dots g'_{n-1} \rangle$ are automorphic. If all constants of all true equations are divisible by p, then we may find a_i and a'_i with $g_i = pa_i$ and $g'_i = pa'_i$ and it suffices to show that $\langle a_1, \dots, a_n \rangle$ and $a'_1, \dots, a'_n \rangle$ are automorphic. After some finite number of divisions, we will eventually get coefficients not divisible by p.

□

Note that this argument depends on the fact that in $\oplus_\infty Z(p^m)$ an element has order $\leq p^k$ if and only if it is divisible by p^{m-k}. This is not true in the group $\oplus_\infty Z(p^m) \oplus \oplus_\infty Z(p^n)$ where $m \neq n$. Of course any group which is not computably categorical cannot be relatively computably categorical, so Theorem 3.3 characterizes the relatively computably categorical Abelian p-groups.

Next we consider Δ^0_2 categoricity. The first case is when the reduced part of $\mathcal{G}$ has finite period.

Theorem 3.4 *Suppose that $\mathcal{G}$ is isomorphic to $\oplus_\alpha Z(p^\infty) \oplus \mathcal{H}$, where $\mathcal{H}$ has finite period and $\alpha \leq \omega$. Then $\mathcal{G}$ is relatively Δ^0_2 categorical.*

Proof: Since the period p^r is finite, $\mathcal{G}$ is a direct sum of computably categorical groups of the form $\oplus_\alpha Z(p^\infty)$ and $\oplus_\omega Z(p^m)$, together with some finite F. The Scott formulas are similar to those given above for the computably categorical groups, except that when g is not divisible, we need to ask whether it is divisible by p^k for each $k < r$, or is not divisible by p^k. The latter question is Π^0_1, so the Scott formulas are a conjunction of Σ^0_1 and Π^0_1. Likewise for a sequence of elements, we need to ask whether each linear combination is divisible by p^k. After that, the argument is essentially the same as in Theorem 3.3. □

There is a special case when $\mathcal{G}$ has no divisible part.

Theorem 3.5 *Suppose that $\mathcal{G}$ is a computable Abelian p-group with all elements of finite height. Then $\mathcal{G}$ is relatively Δ^0_2 categorical. [Note: These are exactly the reduced Abelian p-groups of length at most ω.]*

Proof: For any finite subgroup $\mathcal{F}$ of $\mathcal{G}$ and any finite sequence $\overrightarrow{g}$ of elements of $\mathcal{F}$, the formula $\phi_{\overrightarrow{g},\mathcal{F}}(\overrightarrow{x})$ gives the atomic diagram of $\mathcal{F}[\overrightarrow{g}]$ and also states that $\mathcal{F}$ is a pure subgroup. The latter question is a Π_1 condition,

$$(\forall g \in F)(\forall n)(\forall x)[p^n \cdot x = g \;\Rightarrow\; (\exists y \in F) p^n \cdot y = g].$$

The Scott formula for $\overrightarrow{g}$ states that there exists a finite set $F = \{a_1, \ldots, a_t\}$ so that $\phi_{\overrightarrow{g},\mathcal{F}}(\overrightarrow{g})$ and furthermore, no subgroup of $\mathcal{F}$ is pure. If $\overrightarrow{g}$ and $\overrightarrow{g}'$ satisfy the same Scott formula, then there are isomorphic pure subgroups $\mathcal{F}$ containing $\overrightarrow{g}$ and $\mathcal{F}'$ containing $\overrightarrow{g}'$. Since $\mathcal{F}$ and $\mathcal{F}'$ are pure, there exist isomorphic summands $\mathcal{H}$ and $\mathcal{H}'$ so that $\mathcal{G} = \mathcal{F} \oplus \mathcal{H} = \mathcal{F}' \oplus \mathcal{H}'$ so that the isomorphism between $\mathcal{F}$ and $\mathcal{F}'$ may be extended to an automorphism of $\mathcal{G}$.

Now $\mathcal{G} = \oplus_{n<\omega} Z(p^{i_n})$ where each i_n is finite, so that for each k, $\oplus_{n<k} Z(p^{i_n})$ is a pure subgroup of $\mathcal{G}$ and any finite sequence $\overrightarrow{g}$ will be included in one of these pure subgroups. Thus every $\overrightarrow{g}$ satisfies some Scott formula. □

We claim that no other Abelian p-groups are relatively Δ^0_2 categorical. We first show this for groups which are products of cyclic and quasicyclic groups. It turns out that even for a group $\mathcal{G}$ of infinite period with only finitely many $Z(p^\infty)$ components, $\mathcal{G}$ is not relatively Δ^0_2 categorical. This differs from equivalence structures, where any structure with only a finite number of infinite equivalence classes is relatively Δ^0_2 categorical. For equivalence structures, each class is necessarily computable but $D(\mathcal{G})$ need not be computable even when there is just one copy of $Z(p^\infty)$.

Theorem 3.6 *Suppose that the computable group $\mathcal{G}$ is isomorphic to $\oplus_\alpha Z(p^\infty) \oplus \mathcal{H}$ for some group $\mathcal{H}$ with infinite period and all elements of finite height, where $\alpha \neq 0$. Then $\mathcal{G}$ is not relatively Δ^0_2 categorical.*

Proof: Let $\mathcal{G} = \mathcal{D} \oplus \mathcal{H}$, where $\mathcal{D}$ is divisible and $\mathcal{H}$ is a product of cyclic groups of unbounded order. Suppose $\mathcal{G}$ had a Σ^0_2 family of Scott sentences. We will show that there is an element of the divisible part $\mathcal{D}$ whose Scott formula is satisfied by some element of $\mathcal{H}$. But there can be no automorphism of G mapping a divisible element to a non-divisible element. This contradiction will show that there is no such Scott sentence.

We first assume that $\alpha = \omega$.

Let a be an element of $\mathcal{D}$ of order p and satisfies a Σ^0_2 Scott formula $\Psi(x, \overrightarrow{d})$. We observe first that the parameters $\overrightarrow{d}$ may be assumed to be independent and in fact to belong to different components in the product decomposition of $\mathcal{G}$. For the finite summands, the parameter can be assumed to be a generator, and for the quasicyclic summands, the parameter can be taken to have maximal order (and therefore generate any other possible parameters.)

Of course any other divisible element of order p must satisfy the same formula, so we may assume that a belongs to a subgroup $\mathcal{A}$ isomorphic to $Z(p^\infty)$ which does not contain any of the parameters. Then, by choosing witnesses $\overrightarrow{c}$ to instantiate the existentially quantified variables in Ψ, we have a computable Π^0_1 formula $\theta(x, \overrightarrow{d}, \overrightarrow{c})$ satisfied by a.

We can now use the fact that this Π^0_1 sentence is true in $\mathcal{G}$ if and only if it is true in all finite subgroups of $\mathcal{G}$ containing $a, \overrightarrow{c}, \overrightarrow{d}$.

Let $a, \overrightarrow{c}, \overrightarrow{d}$ generate a finite subgroup $\mathcal{F}_1$ of $\mathcal{G}$ and let $A_1 = \mathcal{A} \cap \mathcal{F}_1$ a finite subgroup of $\mathcal{A}$ of order p^m and $\mathcal{F}_1 = \mathcal{A}_1 \oplus B_1$ for some group $\mathcal{B}$. Now find a factor group $\mathcal{H}_1 \subset \mathcal{H}$ of $\mathcal{G}$ of order type $\geq p^m$ and independent of $\mathcal{F}_1$; this exists since $\mathcal{H}$ has infinite period. We may assume without loss of generality that $|\mathcal{H}_1| = p^m$ and that each of $\overrightarrow{d}$ is in $\mathcal{H}_1$. Let ϕ be an isomorphism from $\mathcal{A}_1 \oplus F_1$ to $\mathcal{H}_1 \oplus F_1$ which is the identity on $\mathcal{H}_1$ and let $a' = \phi(a)$ and let $\overrightarrow{b}$ be the image of $\overrightarrow{c}$ under this mapping.

We claim that $\theta(a', \overrightarrow{d}, \overrightarrow{b})$. Now let $\mathcal{H}'$ be any finite subgroup of $\mathcal{G}$ containing $a', \overrightarrow{d}, \overrightarrow{b}$; we may assume that $\mathcal{H}' = \mathcal{H}_1 \oplus \mathcal{F}_2$ where $\mathcal{F}_1 \subseteq \mathcal{F}_2$. Furthermore, we may assume (by taking an automorphism of $\mathcal{G}$ if necessary) that $\mathcal{F}_2 \cap \mathcal{A}_1 = \emptyset$. Then ϕ^{-1} may be extended to an isomorphism from $\mathcal{H}'$ to a finite subgroup $\mathcal{A}_1 \oplus F_2$. Since θ is Π^0_1, $\mathcal{A}_1 \oplus F_2 \vdash \theta(a, \overrightarrow{c}, \overrightarrow{d})$. Thus by the isomorphism, $\mathcal{H}' \vdash \theta(a', \overrightarrow{b}, \overrightarrow{d})$. Since this is true for any finite subgroup of $\mathcal{G}$, it follows that $\mathcal{G} \vdash \theta(a', \overrightarrow{b}, \overrightarrow{d})$. Therefore $\Psi(a', \overrightarrow{d})$ for the Scott formula Ψ.

But a' is not divisible, so it is not automorphic with a. This contradiction proves the theorem for the first case.

Suppose now that $\mathcal{G}$ that α is finite; we will assume for simplicity that $\alpha = 1$. Let d_i be the parameter of greatest order in any quasicyclic summand and let a be an element of that summand with $p \cdot a = d$. Let p^m be the order of a and let $\mathcal{F}_1$ be the cyclic subgroup generated by a; note that any other parameter in $\mathcal{F}_1$ is a multiple of d_i. Now choose an element g of order p^m, generating a subgroup $\mathcal{F}_2$ so that any element from $\overrightarrow{d}$ in $\mathcal{F}_1 \oplus \mathcal{F}_2$ is already in $\mathcal{F}_1$. This can be done since $\mathcal{H}$ has infinite period. Now let $a' = a + p^{m-1} \cdot g$, so that $p \cdot a' = d_i$. Then there is an automorphism of $\mathcal{F}_1 \oplus \mathcal{F}_2$ taking a to a' and preserving the parameters, defined by $\psi(j \cdot a + k \cdot g) = j \cdot a' + k \cdot g$. This automorphism may be extended to an automorphism of the finite subgroup $\mathcal{H}_1$ generated by $a, \overrightarrow{d}, \overrightarrow{b}$. Let $\overrightarrow{c}$ be the image of $\overrightarrow{b}$ under this extended automorphism.

We claim that $\theta(a', \overrightarrow{d}, \overrightarrow{c})$. Let $\mathcal{H}'$ be any finite subgroup of $\mathcal{G}$ containing $a', \overrightarrow{d}, \overrightarrow{c}$; we may assume that $\mathcal{H}' = \mathcal{H}_1 \oplus \mathcal{F}$ for some $\mathcal{F}$, so that there is an automorphism of $\mathcal{H}'$ taking $a, \overrightarrow{d}, \overrightarrow{b}$ to $a', \overrightarrow{d}, \overrightarrow{c}$. Since $\mathcal{H}'$ is a finite subgroup of $\mathcal{G}$, we have $\mathcal{H}' \vDash \theta(a, \overrightarrow{d}, \overrightarrow{b})$ and hence, by the automorphism, $\mathcal{H}' \vDash \theta(a', \overrightarrow{d}, \overrightarrow{c})$. Since this holds for any finite subgroup $\mathcal{H}'$, it follows that $\theta(a', \overrightarrow{d}, \overrightarrow{c})$ and hence $\Psi(a', \overrightarrow{d})$. But a' is not divisible, so cannot be automorphic with a. $\square$

In the paper [6], we defined a uniformly Σ^0_2 enumeration K_e of the Σ^0_2 characters and an enumeration $\mathcal{C}_e$ of the computable equivalence structures. For a total computable function $\phi_e : \omega \times \omega \to \omega$, let $\mathcal{G}_e$ be the structure with universe ω and with group operation ϕ_e.

Lemma 3.7 *[[6]] For any fixed infinite Σ^0_2 character K, $\{e : K_e = K\}$ is Π^0_3 complete.* $\square$

Theorem 3.8 *[[6]] Let $\mathcal{A}$ be a computable equivalence structure with unbounded character K and with infinitely many infinite equivalence classes. Suppose also that there exists a structure $\mathcal{B}$ with character K and with no infinite equivalence classes. Then $\{e : C_e \simeq \mathcal{A}\}$ is Π^0_4 complete.* $\square$

We can apply this to p-groups to get a similar result, using 2.9.

Theorem 3.9 *Let $\mathcal{G}$ be isomorphic to $\oplus_\omega Z(p^\infty) \oplus \mathcal{H}$, with $\mathcal{H}$ having infinite period and all elements of finite height. Suppose also that there is a computable copy of $\mathcal{H}$. Then $\{e : C_e \simeq \mathcal{G}\}$ is Π^0_4 complete.* $\square$

Proof: Fix such a group $\mathcal{G}$ with character K and let $\mathcal{C}$ be an equivalence structure with character K. It can be checked that $\{e : \mathcal{G}_e \simeq \mathcal{G}\}$ is a Π^0_4 set. For the completeness, we observe that the uniformity of the proof of Theorem 2.9 provides a computable function f such that C_a is isomorphic to C_b if and only if $\mathcal{G}_{f(a)}$ is isomorphic to $\mathcal{G}_{f(b)}$. Then $C_e \simeq C$ if and only if $G_{f(e)} \simeq \mathcal{G}$ and the completeness follows from Theorem 3.8. $\square$

This gives the following result for categoricity.

Theorem 3.10 *Suppose that the computable group $\mathcal{G}$ is isomorphic to $\oplus_\omega Z(p^\infty) \oplus \mathcal{H}$ for some group $\mathcal{H}$ with infinite period and all elements of finite height and suppose in addition that there is a computable group isomorphic to $\mathcal{H}$. Then $\mathcal{G}$ is not Δ^0_2 categorical.*

Proof: If $\mathcal{G}$ were Δ^0_2 categorical, then $\{e : \mathcal{G}_e \simeq \mathcal{G}\}$ has a Σ^0_4 definition. That is, let M be a complete c.e. set, let $+$ be $+^{\mathcal{G}}$ and let $+_e$ be $+^{\mathcal{G}_e}$. Then $\mathcal{G}_e \simeq \mathcal{G}$ if and only if

$$(\exists a)[a \in Tot^M \ \&\ (\forall m)(\forall n)(\phi^M_a(m+n) = \phi^m_a(m) +_e \phi^M_a(n))].$$

But this contradicts the Π^0_4 completeness from Theorem 3.9. □

Finally, all of the groups discussed above are certainly relatively Δ^0_3 categorical.

Theorem 3.11 *Let $\mathcal{G}$ be a computable group isomorphic to $\oplus_\alpha Z(p^\infty) \oplus \mathcal{H}$, where $\mathcal{H}$ has all elements of finite height. Then $\mathcal{G}$ is relatively Δ^0_3 categorical.*

Proof: The divisible part $D(\mathcal{G})$ can be defined by a Π^0_2 sentence. For The definition of the Scott sentences builds on that of Theorem 3.5. Given a finite pure subgroup $\mathcal{F}$ and a finite subgroup $\mathcal{D}$ of $D(\mathcal{G})$, we define the formula $\phi_{\overrightarrow{g},\mathcal{F},\mathcal{D}}$ as before to give the atomic diagram of $\mathcal{D} \oplus \mathcal{F}[\overrightarrow{g}]$. Any such formula satisfied by $\overrightarrow{g}$ will be a Scott formula. □

There is a stronger result for groups $\mathcal{G}$ where $D(\mathcal{G})$ is computable.

Theorem 3.12 *For any two isomorphic computable Abelian p-groups $\mathcal{G}_1$ and $\mathcal{G}_2$ of length $\leq \omega$ such that $D(\mathcal{G}_1)$ and $D(\mathcal{G}_2)$ are both computable, $\mathcal{G}_1$ and $\mathcal{G}_2$ are Δ^0_2 isomorphic.*

Proof: We will construct a computable subgroup $\mathcal{H}_1$ and $\mathcal{H}_2$ such that $\mathcal{G}_i = D(\mathcal{G}_i) \oplus \mathcal{H}_i$. $\mathcal{H}_i$ will be defined as the union of a computable sequence $\mathcal{A}_s$ of pure finite subgroups. $\mathcal{A}_0 = \{0\}$. Given A_s, find the least element $g \notin A_s$ such that $\langle A_s \cup \{g\}\rangle \cap D(\mathcal{G}_i) = \{0\}$ and let $A_{s+1} = \langle A_s \cup \{g\}\rangle$. Then for each s, $D(\mathcal{G}_i) \cap A_s = \{0\}$ and therefore $D(\mathcal{G}) \cap \mathcal{H}_i = \{0\}$. The factor group $\mathcal{H}_i$ is computable since $s \in \mathcal{H}_i$ if and only if $s \in \mathcal{A}_{s+1}$. We argue by induction on the order of g that any element $g \in \mathcal{G}$ belongs to $D(\mathcal{G}) \oplus \mathcal{H}_i$. For the initial case, suppose that $p \cdot g = 0$. Then either $g \in A_{g+1}$ or else $a + g = d \neq 0$ for some $d \in D(\mathcal{G})$. But in the latter case, $g = a - d \in D(\mathcal{G}) \oplus \mathcal{H}_i$ as desired. Now suppose all elements of order $< p^m$ belong to $D(\mathcal{G}) \oplus \mathcal{H}_i$ and let $p^m g = 0$. Then $h = p \cdot g = a + d$ for some $a \in \mathcal{H}_i$ and $d \in D(\mathcal{G})$. Choose d' so $d = pd'$ since d is divisible. Then we have $a = p \cdot (g - d')$ so that, since $\mathcal{H}_i$ is pure, $a = p \cdot a'$ for some $a' \in \mathcal{H}_i$. Now $p \cdot (g - d' - a') = 0$, so that $g - d' - a' \in D(\mathcal{G}) \oplus \mathcal{H}_i$ by the initial case and therefore $g \in D(\mathcal{G}) \oplus \mathcal{H}_i$ as desired.

It follows from Proposition 3.1 that $D(\mathcal{G}_1)$ and $D(\mathcal{G}_2)$ are computably isomorphic, and it follows from Theorem 3.5 that $\mathcal{H}_1$ and $\mathcal{H}_2$ are Δ^0_2 isomorphic. Now, the two corresponding isomorphisms may be combined into a Δ^0_2 isomorphism between $\mathcal{G}_1$ and $\mathcal{G}_2$. □

4 Groups of length $> \omega$

Index set results can tell us something about many of the groups of greater length. By using the calculations in Theorems 5.6, 5.15, and 5.16 of [7], we can prove the following.

Theorem 4.1 *Let $\mathcal{G}$ be an Abelian p-group.*

1. *If $\lambda(\mathcal{G}) = \omega \cdot N$ and $M \leq 2N - 1$, then $\mathcal{G}$ is not Δ^0_M-categorical.*
2. *If $\lambda(\mathcal{G}) > \omega \cdot N$ and $M \leq 2N - 2$, then $\mathcal{G}$ is not Δ^0_M-categorical.*
3. *If $\lambda(\mathcal{G}) = \omega \cot N + k$ where $k \in \omega$, and $\mathcal{H}$ is the reduced part of $\mathcal{G}$, the following hold:*
   (a) *If $\mathcal{H}_{\omega N}$ is isomorphic to $\mathbb{Z}_{p^k}$ and $M \leq 2N - 1$, then $\mathcal{G}$ is not Δ^0_M-categorical.*
   (b) *If $\mathcal{H}_{\omega N}$ is finite but not isomorphic to $\mathbb{Z}_{p^k}$ and $M \leq 2N - 1$, then $\mathcal{G}$ is not Δ^0_M-categorical.*
   (c) *If there is a unique $j < k$ such that $u_{\omega N+j}(\mathcal{H}) = \infty$, and $M \leq 2N$, then $\mathcal{G}$ is not Δ^0_M-categorical.*
   (d) *If there are distinct $i, j < k$ such that $u_{\omega N+i}(\mathcal{H}) = u_{\omega N+j}(\mathcal{H}) = \infty$ and $M \leq 2N + 1$, then $\mathcal{G}$ is not Δ^0_M-categorical.*

Corollary 4.2 *Let $\mathcal{G}$ be a computable Abelian p-group whose reduced part has a computable copy, and suppose that $\mathcal{H}$ is the reduced part of $\mathcal{G}$. Then if $\mathcal{H}$ has infinitely many elements of height at least ω, then $\mathcal{G}$ is not Δ^0_2-categorical.*

Results of Barker [5] give the following additional information.

Theorem 4.3 (Barker) *Let $\mathcal{G}$ be a countable reduced Abelian p-group with recursive Ulm invariants such that $\lambda(\mathcal{G}) = \omega\alpha + \omega + n$ and $\mathcal{G}_{\omega\alpha+\omega}$ is finite. Then $\mathcal{G}$ is relatively $\Delta^0_{2\alpha+2}$-categorical but not $\Delta^0_{2\alpha+1}$-categorical.*

Finally, we can prove the following result on relative categoricity:

Theorem 4.4 *Let $\mathcal{G}$ be a computable Abelian p-group of length greater than ω whose reduced part has no computable copy. Then $\mathcal{G}$ is not relatively Δ^0_2-categorical.*

Proof: The proof is the same as that of Theorem 3.6. □

5 Open Problems

The present paper does not completely characterize the relatively Δ^0_2-categorical or Δ^0_2-categorical Abelian p-groups. We give below an exhaustive list of the open cases.

Problem 5.1 *Let $\mathcal{G}$ be a computable Abelian p-group isomorphic to $\mathcal{D} \oplus \mathcal{H}$, where $\mathcal{D}$ is a direct sum of finitely many copies of the Prüfer group and $\mathcal{H}$ is reduced, with infinite period but all elements of finite height. Can $\mathcal{G}$ be Δ^0_2-categorical?*

Problem 5.2 *Let $\mathcal{G}$ be a computable Abelian p-group isomorphic to $\mathcal{D} \oplus \mathcal{H}$, where $\mathcal{D}$ is divisible and $\mathcal{H}$ is reduced, with Ulm length greater than ω, and $\mathcal{H}_\omega$ finite. Can $\mathcal{G}$ fail to be Δ^0_2-categorical? Can $\mathcal{G}$ fail to be relatively Δ^0_2-categorical?*

Problem 5.3 *Let $\mathcal{G}$ be a computable Abelian p-group whose reduced part has no computable copy. We have shown that $\mathcal{G}$ cannot be relatively Δ^0_2-categorical. Can it be Δ^0_2-categorical?*

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