Equivalence structures and isomorphisms in the difference hierarchy

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1 Introduction

Computable model theory deals with the study of the effective properties of mathematical structures and the relationships between them. Perhaps the most basic kind of relationship between different structures is that of isomorphism. It is natural to study the isomorphism relations between structures in an effective context by investigating the following question: given that two structures are isomorphic, how complex must an isomorphism between them be?

In what follows, we restrict our attention to countable structures for computable languages. Hence, if our structure is infinite, we can assume its universe is the set of natural numbers ω. We recall some basic definitions. If $\mathfrak{A}$ is a structure with universe A for a language $\mathcal{L}$, then $\mathcal{L}^A$ is the result of expanding $\mathcal{L}$ by constants for every element of A. The *atomic diagram* of the structure $\mathfrak{A}$ is the set of all quantifier-free sentences of $\mathcal{L}^A$ true in the structure. The *elementary diagram* of $\mathfrak{A}$ is the set of all sentences of $\mathcal{L}^A$ true on $\mathfrak{A}$. A structure is *computable* if its atomic diagram is computable; it is *decidable* if its elementary diagram is computable. We call two structures *computably isomorphic* if there is a computable function that gives an isomorphism between them. A computable structure $\mathfrak{A}$ is *relatively computably isomorphic* to a (possibly noncomputable) structure $\mathfrak{B}$ if there is an isomorphism between them that is relatively computable in the atomic diagram of $\mathfrak{B}$. A computable structure $\mathfrak{A}$ is *computably categorical* if every other computable structure that is isomorphic to $\mathfrak{A}$ is computably isomorphic to $\mathfrak{A}$. $\mathfrak{A}$ is *relatively computably categorical* if every other structure that is isomorphic to $\mathfrak{A}$ is relatively computably isomorphic to $\mathfrak{A}$. Similar definitions arise for other natural definability classes of functions. For example, for any $n \in \omega$, a structure is Δ^0_n if its atomic diagram is; two structures are Δ^0_n*-isomorphic* if there is a Δ^0_n isomorphism between them; and a computable structure is Δ^0_n*-categorical* if every computable structure that is isomorphic to it is Δ^0_n-isomorphic to it.

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Among the simplest nontrivial structures to investigate is one consisting of nothing besides one equivalence relation. It is useful in the context of computability theory to split such an equivalence structure $\mathfrak{A}$ into two pieces $Inf^{\mathfrak{A}}$ and $Fin^{\mathfrak{A}}$, where $Inf^{\mathfrak{A}}$ consists of those elements with infinite equivalence classes and $Fin^{\mathfrak{A}}$ consists of those elements with finite equivalence classes. This is simply because it is natural to consider the different sizes of the equivalence classes of the elements in $Fin^{\mathfrak{A}}$ as coding information into the equivalence relation. The *character* of an equivalence structure $\mathfrak{A}$ is the set

$$\chi(\mathfrak{A}) = \{ \langle k, n \rangle : n, k > 0 \text{ and } \mathfrak{A} \text{ has at least } n \text{ equivalence classes of size } k \}.$$

This set provides a kind of skeleton for $Fin^{\mathfrak{A}}$. Any set of pairs K such that $\langle k, n{+}1 \rangle \in K$ implies $\langle k, n \rangle \in K$ is called a *character.* A character K is *bounded* if there is some finite k_0 such that for all $\langle k, n \rangle \in K$, $k < k_0$. The concepts of s-functions and s_1-functions introduced by Khisamiev [2] provide a means of computably approximating the characters of equivalence relations.

Definition 1. *A function $f : \omega^2 \to \omega$ is an* s-function *if and only if*

(a) for every $i, s \in \omega$, $f(i, s) \le f(i, s{+}1)$ and

(b) for every $i \in \omega$, $\lim_{s\to\infty} f(i, s)$ exists.

An s-function f is an s_1-function if for every $i \in \omega$, $\lim_{s\to\infty} f(i, s) < \lim_{s\to\infty} f(i{+}1, s)$.

The properties of computable equivalence relations have recently been studied by Calvert, Cenzer, Harizanov, and Morozov in [1]. There they give certain conditions under which a given character K can be the character of a computable equivalence structure. In particular, they show that if K is a bounded character and $\alpha \le \omega$, then there is a computable equivalence structure with character K and exactly α infinite equivalence classes. To prove the existence of computable equivalence structures for unbounded characters K, additional information in the form of s- and s_1-functions is needed. They show that if K is a Σ^0_2 character, $r < \omega$, and either

(a) there is an s-function such that

$$\langle k, n \rangle \in K \Leftrightarrow |\{ i : k = \lim_{s\to\infty} f(i, s) \}| \ge n$$

or

(b) there is an s_1-function such that for every $i \in \omega$, $\langle \lim_{s\to\infty} f(i, s), 1 \rangle \in K$,

then there is a computable equivalence structure with character K and exactly r infinite equivalence classes. Together with these positive results, they also construct an infinite Δ^0_2 set D such that for any computable equivalence structure $\mathfrak{A}$ with unbounded character and no infinite equivalence classes, $\{ k : \langle k, 1 \rangle \in K \}$ is not a subset of D.

The classification of objects by means of the arithmetic hierarchy has sometimes been felt to be a little crude. Because of this, other hierarchies have been introduced as a finer means of classification. One of the most common means of studying the fine structure of the class of Δ^0_2 objects is the difference hierarchy of Ershov. A set A is n-c.e. if there is a computable approximation A_s such that for all $x \in \omega$,

(i) $x \notin A_0$,

(ii) $x \in A$ if and only if $\exists^\infty s(x \in A_s)$, and

(iii) $|\{\, s : A_{s+1}(x) \neq A_s(x) \,\}| \leq n$.

A is ω-c.e. if instead of property (iii) above, there is a computable function g such that for all x $|\{\, s : A_{s+1}(x) \neq A_s(x) \,\}| \leq g(x)$.

We explore some results inspired by [1] in the context of the difference hierarchy by first showing that if K is any Δ^0_2 character, then there is a d.c.e equivalence relation with character K and no infinite equivalence classes in Theorem 4 below. It follows as a corollary that the construction of the set D in [1] fails for d.c.e. equivalence relations

Calvert, Cenzer, Harizanov, and Morozov also prove several results on the categoricity of computable equivalence relations. They show that if $\mathfrak{A}$ is a computable equivalence structure with only finitely many infinite equivalence classes, then it is relatively Δ^0_2 categorical. They also show that a computable equivalence structure is computably categorical if and only if either

(i) it has only finitely many finite equivalence classes or

(ii) it has only finitely many infinite equivalence classes, there is an upper bound on the size of all finite equivalence classes, and there is at most one $k \in \omega$ with infinitely many equivalence classes of size k.

The simplest computable structure that is Δ^0_2-categorical but not computably categorical is one consisting of infinitely many equivalence classes of sizes 1 and 2 and no other equivalence classes.

In order to examine such questions in the context of the difference hierarchy, one must choose an appropriate notion of an α-c.e. function. Since this has a certain amount of interest in its own right, we spend some time examining some different possibilities for this concept and show that they really are different. After distinguishing three different possibilities for α-c.e. categoricity, we show in Theorem 13 below that each of them leads to a nondegenerate hierarchy of computable equivalence relations. That is, for each notion n-c.e. categoricity, there are $(n+1)$-c.e. isomorphic computable equivalence relations that fail to be n-c.e. isomorphic for each n. Closely related results have recently been obtained independently by Khoussainov, Stephan, and Yang [3] for structures consisting of finite graphs, which we discuss below. In what follows, we generally use the notation of the standard reference Soare, [4], in particular we adopt his convention of writing $[s]$ after any formula or term to indicate that all the functionals involved therein are taken with their values at stage s. If z is an element of a computable equivalence structure $\mathfrak{X}$, we write $[z]^{\mathfrak{X}}$ for the equivalence class of z using $E^{\mathfrak{X}}$.

2 D.c.e. equivalence structures

We start with a useful technical fact.

Lemma 1. *Every n-c.e. set is the union of a finite number of d.c.e. sets*

Proof. Let A be an n-c.e. set and let A_s be a computable approximation which witnesses that A is n-c.e.. For each $k \geq$, let $B_i = \{x : \exists 0 \leq i_1 < i_2 < \cdots i_k(A_{i_j}(x) \neq (A_{i_j+1}(x)\}$. Clearly B_i is c.e. for all i. Then it is easy to see that if $n = 2m$ for some m, then

$$A = (B_1 - B_2) \cup \cdots \cup (B_{2m-1} - B_{2m}),$$

and if $n = 2m + 1$ for some m, then

$$A = (B_1 - B_2) \cup \cdots \cup (B_{2m-1} - B_{2m}) \cup B_{2m+1}.$$

$\square$

Proposition 2. *If $n \geq 1$, then for any unbounded n-c.e. character $\mathcal{K}$ there exists a computable s_1-function f such that for every $i \in \omega$, $\langle \lim_{s\to\infty} f(i, s), 1\rangle \in \mathcal{K}$.*

Proof. If $n = 1$, then $K = \{n : \langle n, 1\rangle : n \in \mathcal{K}\}$ is an infinite c.e. set. Hence K has a computable infinite subset so that it clearly has the required s_1-function. If $n = 2$, then $K = V{-}W$ for some c.e. sets V and W. Since K is infinite, we can clearly enumerate V and W in such a way that

1. for every s, $|(V{-}W)[s]| \geq s$ and
2. If $|(V{-}W)[s]{-}(V{-}W)[s{+}1]| = k$, then there exist at least k (new) elements in $(V{-}W)[s{+}1]$ all of which are greater than any element in $(V{-}W)[s]$.

This given, we can construct our desired computable s_1-function as follows. Let $f(x, 0) = x$ for every $x \in \omega$. At stage $s{+}1$, if for every $x \leq s$, $f(x, s) \in (V{-}W)[s{+}1]$, then define , $f(x, s{+}1) = f(x, s)$ for every x. Else let $y \leq s$ be least such that $f(y, s) \notin (V{-}W)[s{+}1]$. For every $x < y$, let $f(x, s{+}1) = f(x, s) \in (V{-}W)[s{+}1]$. If $y \leq x \leq s$, let $f(x, s{+}1)$ be the least element in $(V{-}W)[s{+}1]$ that is greater than or equal to $f(x, s)$ and greater than $f(x', s{+}1)$ for every $x' < x$. For the remaining $x > s$, let $f(x, s{+}1) = f(x, s) + x - s$.

We can show f is our desired s_1-function as follows. Evidently, f satisfies the criteria for being nondecreasing in the stage and increasing in the argument. So we need only show that all the limits converge to values in $V - W$. Suppose that for every $x < y$, $f(x, s)$ eventually comes to a limit $f(x) \in V{-}W$. After all of these have reached their limits at some stage s, then $f(y)$ is only ever moved to the least available number in $V{-}W$ when its previous value is removed from $V{-}W$. Since there are infinitely many elements that enter $V{-}W$ and never leave, eventually $f(y,s)$ comes to a limit $f(y) \in V{-}W$.

Since, by the lemma above, no matter what n is, K is an n-c.e. set which is the union of finitely many d.c.e. sets. Since at least one of these d.c.e. sets must be infinite, if $\mathcal{K}$ is unbounded, the full result now follows.

□

Theorem 3. *There exists an unbounded ω-c.e. character $\mathcal{K}$ such that there is no computable s_1-function f such that for every $i \in \omega$, $\langle \lim_{s\to\infty} f(i,s), 1\rangle \in \mathcal{K}$.*

Proof. We construct an infinite ω-c.e. set K that satisfies the following requirements:

$$\mathrm{R}_e \quad \phi_e \text{ is an } s_1\text{-function} \Longrightarrow \exists x \lim_{s\to\infty} \phi_e(x,s) \notin K,$$

where ϕ_e is the e-th partial computable function. Then $\mathcal{K} = K \times \{1\}$ is our desired unbounded character.

The basic strategy for satisfying requirement R_e is simple. We start out at stage 0 with $K[0] = \omega$ and wait for a stage s such that $\phi_e(0,t)\!\downarrow\![s] = m$ for some greatest number t. We then remove m from $K[s{+}1]$ and wait for a stage $s' > s$ such that $\phi_e(0,t')\!\downarrow\![s'] = m'$ for some least $t' > t$. Notice that since an s_1-function must be nondecreasing in the second argument, it must be the case that $m' > m$ or else, requirement R_e is automatically satisfied. We then enumerate m back into $K[s'{+}1]$ and remove m' from $K[s'{+}1]$. If ϕ_e is actually an s_1-function, then eventually $\phi_e(0,t)$ reaches some limit m_0 which we remove from K and never re-enumerate, so that ϕ_e cannot be an s_1-function for $\mathcal{K}$ such that for every $i \in \omega$, $\langle \lim_{s\to\infty} f(i,s), 1\rangle \in \mathcal{K}$. Of course, ϕ_e could just fail to converge to a limit on 0, either because some value never gets defined, or because the limit tends to ∞.

Notice that the strategy for one requirement produces a 3-c.e set, so that it may seem puzzling at first why we should need an ω-c.e. set to defeat all s_1-functions. The reason is that for any odd number n, if we use an n-c.e. character $\mathcal{K}$ to defeat ϕ_e, then some other function ϕ_a could just converge more slowly than ϕ_e and converge to values on which we have already made n changes. So we cannot uniformly bound n. This problem is naturally avoided with an ω-c.e. set. We can choose a distinct witness x for each p.c. function ϕ_e and ensure that $\lim_{s\to\infty} \phi_e(x,s) \notin K$ by using at most $e{+}2$ changes on values of $\phi_e(x,s)$. If ϕ_e is actually an s_1-function such that for every $i \in \omega$, $\langle \lim_{s\to\infty} f(i,s), 1\rangle \in \mathcal{K}$, then for any x and s, $\phi_e(x,s) \geq x$, so that if we use some $x \geq 2e{+}1$ as our witness to defeat ϕ_e, we will ensure that K is still infinite when all requirements are satisfied. With this in mind, let $K[s] = \omega - \{\, \phi_e(2e{+}1, s) \,:\, e < s \,\}$. Since $\phi_e(2e{+}1, s) \geq 2e{+}1$ for every e, if $2e - 1 \leq n < 2e{+}1$, then n can only be the value of some $\phi_a(2a{+}1)$ for $a < e$. For each such a, the number of stages s at which $K[s]$ changes on n because of ϕ_a is at most 2. Hence the number of times

K changes on n is at most $2(e-1)+1 = 2e-1 \leq n$. This means K is ω-c.e. If ϕ_e is actually an s_1-function, then $\phi_e(2e+1, s)$ must come to a limit, and, once this happens, this number can never again be in $K[s]$. This means ϕ_e cannot be an s_1-function for $\mathcal{K}$ such that for every $i \in \omega$, $\langle \lim_{s\to\infty} f(i,s), 1\rangle \in \mathcal{K}$. Finally, note that at most $e-1$ numbers less than or equal to $2e$ can be removed from K permanently, since $\phi_e(2a+1, s) > 2e$ for any $a \geq e$. Therefore, K must be infinite.

$\square$

Theorem 4. *If $\mathcal{K}$ is a Δ^0_2 character, then there exists a d.c.e. equivalence structure with no infinite equivalence classes and character $\mathcal{K}$.*

Proof. Let $\mathcal{K}$ be a Δ^0_2 character. Without loss of generality, we can assume that the set $\mathcal{K}_1 = \{\, x \,:\, \langle x, 1\rangle \in \mathcal{K} \,\}$ is infinite, since otherwise we can clearly construct a computable equivalence structure with character $\mathcal{K}$.

In order to make our construction easier, we can slow down our approximation to $\mathcal{K}$ so that the symmetric difference of $\mathcal{K}[s]$ and $\mathcal{K}[s+1]$ consists of exactly one element.

We can also assume without loss of generality that $\langle 1, n\rangle \notin \mathcal{K}$ for any n. That is, the number of equivalence classes of size 1 is just one piece of information and if we know that number, then we can always add an appropriate number of equivalences classes at the end. Recall that the ordered pairs of natural numbers have a standard ordering, namely, $\langle x_1, y_1\rangle < \langle x_1, y_1\rangle$ if and only if either $x_1 + y + 1 < x_2 + y_2$ or $x_1 + y + 1 = x_2 + y_2$ and $x_1 < y_1$.

Stage 0**:** We start with every element of ω associated only to itself. (That is, $E[0] = \{\, \langle y, y\rangle \,:\, y \in \omega \,\}$.)

Stage $s+1$**:** Suppose $\langle x, n\rangle \in \mathcal{K}[s+1] - \mathcal{K}[s]$. For all $\langle y, m\rangle > \langle x, n\rangle$, if there exist m equivalence classes of size y, we reduce the number of equivalence classes of size y by removing the most recent equivalence classes added of size y. Next, we add one equivalence class of size x by choosing the least number not *currently* associated with any other number and choosing the least $x-1$ numbers that have *never* been associated with any other number and associating all of these together to form a new equivalence class.

If, on the other hand, $\langle x, n\rangle \in \mathcal{K}[s] - \mathcal{K}[s+1]$, then we select the most recent equivalence class added of size x and remove it.

Because we only ever associate previously used numbers with completely new ones when forming a new equivalence class, the structure created by this procedure is d.c.e. Let k_s be the unique element on which $\mathcal{K}$ changes its value at stage s. A stage s is a *true* stage if for all $t \geq s$, $\mathcal{K} \upharpoonright k_s = \mathcal{K}[t] \upharpoonright k_s$. Because $\mathcal{K}$ is Δ^0_2, there exist infinitely many true stages. At any true stage s, a new equivalence class is formed that is never removed and, furthermore, if $\langle y, m\rangle \in \mathcal{K}$ and $\langle y, m\rangle < \langle x, n\rangle$, then there are m equivalence classes of size y in $\mathcal{K}[s]$, none of which are ever removed. The theorem now follows.

$\square$

From Theorems 3 and 4 we immediately have the following:

Corollary 5. *There is a d.c.e. equivalence structure $\mathfrak{A}$ with no infinite equivalence classes, an ω-c.e. character, and no s_1-function such that for every $i \in \omega$, $\langle \lim_{s\to\infty} f(i,s), 1\rangle \in \mathcal{K}(\mathfrak{A})$.*

3 Functions in the difference hierarchy

Ershov's difference hierarchy provides a finer means of classifying the isomorphisms between Δ^0_2-categorical structures. Before turning to the categoricity of equivalence structures in the difference hierarchy, we investigate the properties of two different ways in which the difference hierarchy can be used to classify the complexity of Δ^0_2 functions.

Definition 2. Suppose f is the limit as $s \to \infty$ of a computable function $f(x,s)$. If $n > 0$, f is an *n-c.e. function* if for all $x \in \omega$,

$$|\{\, s : f(x,s) \neq f(x,s+1) \,\}| < n.$$

f is an *ω-c.e. function* if there is a computable function g such that for all $x \in \omega$,

$$|\{\, s : f(x,s) \neq f(x,s+1) \,\}| \leq g(x).$$

Notice that a 1-c.e. function is just a computable function. It is not true that for $n > 0$, if f is n-c.e., its range must be an n-c.e. set. In fact, it is not hard to see that the following must be true:

Proposition 6. *For any nonempty Σ^0_2 set A, there is a 2-c.e. function whose range is A.*

Proof. Let $x \in A \leftrightarrow \exists y \forall z\, R(x,y,z)$, where R is a computable relation. Since $A \neq \emptyset$, there is some $a \in A$. Let $f(\langle x,y\rangle, s) = x$ if $\forall z \leq s\, R(x,y,z)$; and let $f(\langle x,y\rangle, s) = a$ otherwise. If $\forall z\, R(x,y,z)$, then for all s, $f(\langle x,y\rangle, s) = x$. If $\exists z\, \neg R(x,y,z)$, then with z_0 the least such, we have $f(\langle x,y\rangle, s) = x$ for all $s < z_0$, and $f(\langle x,y\rangle, s) = a$ for all $s \geq z_0$. Hence f is 2-c.e. Clearly the range of $\lim_{s\to\infty} f(x,s)$ is A. ◻

Choosing $A = \emptyset''$ in the proposition, we have the following:

Corollary 7. *There is a 2-c.e. function whose range is not Δ^0_2.*

On the other hand, it is also easy to see that there are Δ^0_2 functions with computable ranges that fail to be ω-c.e. functions. Let A be a Δ^0_2 set that is not in any ω-c.e degree. List A as $\{a_0 < a_1 < ...\}$, $\omega - A$ as $= \{b_0 < b_1 < ...\}$, and define $f(a_n) = 2n$ for members of A and $f(b_n) = 2n{+}1$ for members of A's complement. The function f is Δ^0_2, and the range of f is ω. Suppose f were ω-c.e. with approximation $f(x,s)$ and bounding function g. Let $A_s = \{\, x : f(x,s) \text{ is an even number} \,\}$. Since $f(x,s)$ can only take on at most $g(x)$

different values, it can only be counted as even at most $g(x)$ different times. Hence $|\{ s : x \in A_{s+1} - A_s \}| \leq g(x)$. But then $A = \lim_{s\to\infty} A_s$ would be an ω-c.e. set: a contradiction.

We are also interested in the following notion:

Definition 3. *A function f is* graph-α-c.e. *if the graph of f is an α-c.e. set.*

Any α-c.e. function is graph-α-c.e. since if we let $A_0 = \emptyset$ and $A_{s+1} = \{ \langle x, y\rangle : f(x,s) = y \}$, then $\lim_{s\to\infty} A_s =$ the graph of f. This is about the most that can be said about the relationship between these two notions, as the following result shows.

Proposition 8. *(i) For every $n \in \omega$, there exists an $(n+1)$-c.e. function that is not graph-n-c.e.*

(ii) There is a graph-2-c.e. function that is not an ω-c.e. function.

Proof. Fix n and let $\{ A_k : k \in \omega \}$ be a computable enumeration of the graphs of all partial graph-n-c.e. functions. Write ϕ_k for the partial function with graph A_k. To construct an $(n+1)$-c.e. function f that is not graph-n-c.e., we start by letting $f(k,0) = 2k$ for every $k \in \omega$. Suppose we have reached a stage s such that $\phi_k(k,s) = 2k$ and $\phi_k(j,t)$ is defined for all $j \leq k$ and all $t \leq s$. Then let $f(k,s+1) = 2k+1$. If at any later stage s', $\phi_k(k,s') = 2k+1$, then let $f(k,s'+1) = 2k$. Continue switching between these values of $f(k,t)$ to make $f(k,t) \neq \phi_k(k,t)$ as long as A_k has changed on $\langle k, 2k\rangle$ at most n times. Note that $f(k,t)$ will only change value at most $n+1$ times. Furthermore, $f(k,t+1) \neq \phi_k(k,t)$ at any stage at which A_k has changed on $\langle k, 2k\rangle$ at most n times. Hence if $\lim_{s\to\infty} \phi_k(k,s) = \lim_{s\to\infty} f(k,s)$, ϕ_k must have changed more than n times on $\langle k, 2k\rangle$. This establishes (i).

For (ii), a similar procedure works. Let ϕ_e be an enumeration of ω-c.e. functions with $\psi_e(x)$ the corresponding bounding function for the number of times $\phi_e(x,s)$ can change value. For every $e \in \omega$, let $\langle e, 1\rangle \in A_0$. Suppose we have reached a stage s, such that $\langle e, \phi_e(e,j)\rangle \in A$ and $\phi_e(j,t)$ is defined for all $j \leq e$ and all $t \leq s$; $\psi_e(e)\downarrow$. If $\phi_e(e,t)$ has changed value at most $\psi_e(e)$ times, then remove $\langle e, \phi_e(e,j)\rangle$ from A and let $\langle e, \phi_e(e,j) \cdot e\rangle \in A_{s+1}$. For each $e \in \omega$, the graph-2-c.e. function defined by A_s takes on the values $1, e^2, e^3, \ldots$ and at any stage at which $\phi_e(e,t)$ has changed value at most $\psi_e(e)$ times, $\langle e, \phi(e,t) \cdot e\rangle \in A_{t+1}$. Hence, as above, if ϕ_e is ω-c.e. with bounding function ψ_e, then $\langle e, \lim_{s\to\infty} \phi_e(e,s)\rangle \notin \lim_{s\to\infty} A_s$, which establishes (ii). $\square$

Of course, the distinction between the complexity of a function and the complexity of its graph collapses when we only classify the function by its place in the arithmetical hierarchy since any function with a Σ^0_n graph is Σ^0_n and any function with a Π^0_n graph is Π^0_n.

If R is an α-c.e. relation, then the converse of R, $R^{-1} = \{ \langle y, x\rangle : \langle x, y\rangle \in R \}$ is also α-c.e. Hence, if f is a one-to-one graph-α-c.e. function, then so is f^{-1}. This makes this notion somewhat more suitable for studying isomorphisms than that of n-c.e. functions, as the following proposition indicates.

Proposition 9. *There exists a* 2*-c.e. bijection* $f : \omega \to \omega$ *such that* f^{-1} *is not* ω*-c.e.*

Proof. Let $\{ \langle \phi_n, \psi_n \rangle : n \in \omega \}$ be a computable enumeration of all pairs of partial computable functions $\phi_n : \omega \times \omega \to \omega$ and $\psi_n : \omega \to \omega$. We intend to use each column $\omega^{[n]}$ to ensure that if ψ_n is a function bounding the changes for $\lim_{s\to\infty} \phi_n(x, s)$, then $\lim_{s\to\infty} \phi_n(x, s) \neq f^{-1}$. The construction proceeds in stages, using a parameter w^n for each n to keep track of whether or not it appears possible that $\lim_{s\to\infty} \phi_n(x, s) = f^{-1}$. For convenience sake, we assume the enumeration of p.c. functions to be slowed down so that at most one new function gets a new value at each stage.

Construction:

Stage 0**:** For all n, let $w^n[0] = \langle n, 1\rangle$ and let $f(\langle n, 1\rangle)[0] = \langle n, 1\rangle$.

Stage s+1**:** First, we diagonalize against any ϕ_n which is threatening to be f^{-1}. We say n *needs attention* at stage s+1 if there exists some greatest $y \leq s$ such that for every $z \leq y$, $\phi_n(\langle n, 1\rangle, z)$ converges at s, and

1. $\psi_n(\langle n, 1\rangle)$ converges at s,
2. x is in the range of $f[s]$,
3. $\phi_n(\langle n, 1\rangle, y) = f^{-1}(\langle n, 1\rangle)$, and
4. $|\{ z < y : \phi_n(\langle n, 1\rangle, z{+}1) \neq \phi_n(\langle n, 1\rangle, z) \}| < \psi_n(\langle n, 1\rangle)$.

If any number needs attention at stage s+1, let n be the least such.

Second, we continue to extend f so that it is a bijection on ω. Let k be the least number in $\omega^{[n]}$ not yet in the domain of f and let l be the least number not yet in the range of f. Let $f(w^n[s])[s{+}1] = l$, let $f(k) = \langle n, 1\rangle$, and let $w^n[s{+}1] = k$. For each $y < s$ such that $f(y)$ is not yet defined at s, choose the least number z not yet in the range of f and let $f(y) = z$.

This ends the construction.

A straightforward inductive argument shows that f only changes value on numbers that are chosen as $w^n[s]$ for some $n, s \in \omega$ and that, in each such case, $f(w^n[s])[s] = \langle n, 1\rangle$ and previously unused numbers are chosen for $w^n[s{+}1]$ and $f(w^n[s])[s{+}1])$. From this it follows that f is 2-c.e. and one-to-one. f is evidently onto since we continually choose the least unused number to add to f's range.

It also follows by induction on s that at each stage s+1, if y is the greatest number such that for every $z \leq y$, $\phi_n(\langle n, 1\rangle, z)$ converges at s, then either $|\{ z < y : \phi_n(\langle n, 1\rangle, z{+}1) \neq \phi_n(\langle n, 1\rangle, z) \}| \not< \psi_n(\langle n, 1\rangle)$ or $\phi_n(\langle n, 1\rangle, y) \neq f^{-1}(\langle n, 1\rangle)[s{+}1]$. This shows that if $\lim_{s\to\infty} \phi_n(x, s)$ is ω-c.e. with bounding function ψ_n, then $\lim_{s\to\infty} \phi_n(x, s) \neq f^{-1}$.

∎

4 Categoricity in the difference hierarchy

The considerations in the previous section yield some different possibilities for extending categoricity notions for computable structures to the difference hierarchy.

Definition 4. Let $\alpha \leq \omega$.

1. We say the structure $\mathfrak{A}$ is *weakly α-c.e. isomorphic* to the structure $\mathfrak{B}$ if there are α-c.e. functions f and g such that f is an isomorphism from $\mathfrak{A}$ to $\mathfrak{B}$ and g is an isomorphism from $\mathfrak{B}$ to $\mathfrak{A}$.

2. We say the structure $\mathfrak{A}$ is *α-c.e. isomorphic* to the structure $\mathfrak{B}$ if there is an α-c.e. function f such that f^{-1} is α-c.e. and f is an isomorphism from $\mathfrak{A}$ to $\mathfrak{B}$.

3. We say the structure $\mathfrak{A}$ is *graph-α-c.e. isomorphic* to the structure $\mathfrak{B}$ if there is a graph-α-c.e. function f such that f is an isomorphism from $\mathfrak{A}$ to $\mathfrak{B}$.

If $\mathfrak{A}$ is a computable structure and for any computable $\mathfrak{B}$ isomorphic to $\mathfrak{A}$, $\mathfrak{B}$ is [weakly] α-c.e isomorphic to $\mathfrak{A}$, then we say $\mathfrak{A}$ is *[weakly] α-c.e. categorical.* If $\mathfrak{A}$ is a computable structure and for any computable $\mathfrak{B}$ isomorphic to $\mathfrak{A}$, $\mathfrak{B}$ is graph-α-c.e. isomorphic to $\mathfrak{A}$, then we say $\mathfrak{A}$ is *graph-α-c.e. categorical.* If $\mathfrak{A}$ is a computable structure and for any structure $\mathfrak{B}$ isomorphic to $\mathfrak{A}$, there is an isomorphism from $\mathfrak{A}$ to $\mathfrak{B}$ that is graph-α-c.e. relative to the atomic diagram of $\mathfrak{B}$, then we say $\mathfrak{A}$ is *relatively graph-α-c.e. categorical.*

By Proposition 9, there are weakly 2-c.e. isomorphic structures that are not ω-c.e. isomorphic.

We examine now exactly which computable equivalence structures are categorical in these senses. The trivial case consisting of exactly one infinite equivalence class is evidently categorical in all the above senses. In fact, by Calvert-Cenzer-Harizanov-Morozov [1], Corollary 3.3, any equivalence relation consisting of only finitely many sizes of finite equivalence classes, at most one of which has an infinite number of equivalence classes, and only finitely many infinite equivalence classes is computably categorical. By Theorem 3.14 in [1], any structure with finitely many infinite equivalence classes and infinitely many finite sizes of equivalence classes cannot even be Δ^0_2 categorical. We examine the other possibilities in what follows by introducing some canonical equivalence structures.

Definition 5. The *basic* 1/2 *equivalence structure* is the structure $\mathfrak{S} = \langle \omega, E \rangle$ where for all n and m, nEm if and only if there exists $k \in \omega$

(a) $n = m = 3k$ or

(b) $n, m \in \{3k+1, 3k+2\}$.

Proposition 10. *Let $\mathfrak{S}$ be the basic $1/2$ equivalence structure. There exists a computable equivalence structure $\mathfrak{D}$ such that $\mathfrak{D} \cong \mathfrak{S}$ via some 2-c.e. isomorphism, but $\mathfrak{D} \not\cong \mathfrak{S}$ via any 1-c.e. isomorphism.*

Proof. The idea of the proof is to construct $\mathfrak{D}$ in such a way that any isomorphism from $\mathfrak{S}$ to $\mathfrak{D}$ projects a computable subset of the set of numbers divisible by 3 onto the set $\{ 2x : x \notin K \}$, where K is the halting problem. Since this latter set is not c.e., no such isomorphism can be a computable function.

The construction proceeds in stages over which we gradually enumerate an approximation $E^{\mathfrak{D}}[s]$ so that $E^{\mathfrak{D}}$ is a computable set and an approximation to a 2-c.e. function f that will be an isomorphism between $\mathfrak{S}$ and $\mathfrak{D}$. To achieve this, we keep track of the two least numbers not in the domain of $f[s]$ that are congruent to each of 0 and 1 modulo 3, using parameters y_s and z_s respectively. For convenience sake, we assume that exactly one new number k_s enters K at each stage s.

Stage 0**:** Let $E^{\mathfrak{D}}[0] =$ the identity relation.

Stage $s{+}1$**:** Let d_s^1 be the least odd number not in the range of f and let y_s and z_s be the two least numbers not in the domain of $f[s]$ that are congruent to each of 0 and 1 modulo 3 respectively.

(a) If $2k_s$ is not in the range of $f[s]$, then let $f(z_s, t) = 2k_s$ and $f(z_s{+}1, t) = d_s^1$ for all $t \leq s{+}1$, and associate $2k_s$ with d_s^1 in $E^{\mathfrak{D}}[s{+}1]$. For all $t \leq s{+}1$, let $f(y_s, t) = d_s^1{+}2$.

(b) If $2k_s$ is already in the range of f, then let x be the (unique) element such that $f(x, s) = 2k_s$. First, correct f by setting $f(x, s{+}1) = d_s^1$. Then for all $t \leq s{+}1$, let $f(z_s, t) = 2k_s$ and $f(z_s{+}1, t) = d_s^1{+}2$ and put $2k_s$ and $d_s^1{+}2$ into the same equivalence class in $E^{\mathfrak{D}}[s{+}1]$. For all $t \leq s{+}1$, let $f(y_s, t) = d_s^1{+}4$.

Let $f = \lim_{s \to \infty} f[s]$. Because f only changes value on a previously assigned number from an even number to an odd number, f is 2-c.e., so it is certainly well defined. Odd numbers are used as values only once, either in contiguous blocks of 2 in case (a) or blocks of 3 in case (b). Even numbers are either only used as values once, or, in case (b), they are used as values a second time after being given up as the value of the argument they were originally assigned to. Hence, f is one-to-one. Also, f is total, since z_s and y_s both tend to infinity.

$E^{\mathfrak{D}} = \bigcup \{ E^{\mathfrak{D}}[s] : s \in \omega \}$ is c.e., and its field is ω. Furthermore, d_s^1 is increasing in s, and any new pair $\langle x, y \rangle$ enumerated into $E^{\mathfrak{D}}[s]$ must have at least one of x and y greater than or equal to d_s^1. Hence $E^{\mathfrak{D}}$ is computable.

Hence $\mathfrak{D}$ is a computable equivalence structure isomorphic to $\mathfrak{S}$ via f, which is a 2-c.e. function. Suppose that g is a computable isomorphism from $\mathfrak{S}$ to $\mathfrak{D}$. But then $\overline{K} = \{ x : \exists y\, (y \text{ is divisible by 3 and } g(y) = 2x \}$ would be a c.e. set. This is a contradiction, which establishes the result.

□

There are a couple of ways to extend Proposition 10.

Definition 6. The *basic* $1/2/\cdots/n$ *equivalence structure* is the structure $\mathfrak{S} = \langle \omega, E\rangle$ where for all s and t, sEt if and only if there exists $k \in \omega$

(a) $s = t = k\binom{n+1}{2}$ or

(b) there is an r with $2 \le r \le n$ and $k\binom{n+1}{2} + \binom{r}{2} \le s, t < k\binom{n+1}{2} + \binom{r+1}{2}$.

The basic $1/2/\cdots/n$ equivalence structure has infinitely many equivalence classes of size i for $i = 1, \dots n$.

Theorem 11. *Let $\mathfrak{S}$ be the basic $1/2/\cdots/n$ equivalence structure where $n \ge 2$. Then there exists a computable equivalence structure $\mathfrak{D}$ such that $\mathfrak{D} \cong \mathfrak{S}$ via some n-c.e. isomorphism, but $\mathfrak{D} \not\cong \mathfrak{S}$ via any $(n-1)$-c.e. isomorphism.*

Proof. To make our proof somewhat easier, we first prove the following lemma which is of interest in its own right.

Lemma 12. *Let $n \ge 2$, let $\mathfrak{S}$ be the basic $1/2/\cdots/n$ equivalence structure, and let $\mathfrak{D}$ be a computable equivalence structure such that $\mathfrak{D} \cong \mathfrak{S}$ and for each $i = 1, \dots, n$, there is an infinite computable set $R_i \subseteq \{x : |[x]^{\mathfrak{D}}| = i\}$. Then there is an n-c.e. isomorphism $f : \mathfrak{S} \to \mathfrak{D}$.*

Proof. We construct the desired isomorphism f via a standard back and forth argument which will use a certain parameter $t_s \ge s$.

Stage 0: Let t_0 be the least t such so that for $i = 1, \dots, n$, there exist elements of $w_1, \dots w_n$ such that $|[w_i]^{\mathfrak{D}[t]}| = i$. Then let $f(0,0) = w_0$ and for $i = 2, \dots, n$, define $f[0]$ so that it maps the set of n such that $\binom{r}{2} \le n < \binom{r+1}{2}$ onto the set of elements in $[w_i]^{\mathfrak{D}[t_{s+1}]}$ in order of magnitude. Finally if 0 is not in the range of $f[0]$ up to this point, let $x \ge \binom{n+1}{2}$ such that $|[x]^{\mathfrak{S}}| = |[0]^{\mathfrak{D}[t_0]}|$ and define $f[0]$ on $[x]^{\mathfrak{S}}$ so that it maps onto the elements of $[0]^{\mathfrak{D}[t_0]}$ in order of magnitude.

Stage $s{+}1$**:** Suppose that at stage $s{+}1$, we have defined $f[s]$ and t_s so that

(a) $\{0, \dots, s\binom{n+1}{2}\}$ is contained in the domain of $f[s]$ and $\{0, \dots, s\}$ is contained in the range of $f[s]$,

(b) for all x in the domain of $f[s]$, all the elements of $[x]^{\mathfrak{S}}$ are contained in the domain of $f[s]$ and $|[x]^{\mathfrak{S}}| = |f(x)[s]^{\mathfrak{D}[t_s]}|$.

Then we exend $f[s]$ to $f[s{+}1]$ in two steps, first a corrrection step followed by an extension step.

Substage I. (Correction Step) First, we process the elements in the domain of $f[s]$ in order of magnitude. For each such x such that $|[x]^{\mathfrak{S}}| = i$ but $|[f(x)[s]]^{\mathfrak{D}[s+1]}| > i$, we find the least element of $y \in R_i$ such that no element of $[y]^{\mathfrak{D}}$ has been used in the construction up to this point and define $f[s{+}1]$ so that it maps the elements in $[x]^{\mathfrak{S}}$ onto the elements of

$[y]^{\mathfrak{D}}$ in increasing order of magnitude. Note that since R_i is infinite, we can find such a y. Since $y \in R_i$, we know that $|[y]^{\mathfrak{D}}| = i$ so that we can enumerate $E^{\mathfrak{D}}$ until we find a u_x such that $|[y]^{\mathfrak{D}}[u_x]| = i$.

Substage II. (Extension Step) Let t_{s+1} be the least stage which is greater than or equal to 1 plus the maximum of t_s and all the u_x's that were needed in substage I such that, for $i = 1, \ldots, n$, there are elements of $w_1, \ldots w_n$ where $|[w_i]^{\mathfrak{D}[t_{s+1}]}| = i$ and none of the elements of $[w_i]^{\mathfrak{D}[t_{s+1}]}$ have been used in the construction up to this point. Then if $(s+1)\binom{n+1}{2}$ is not in the domain of $f[s+1]$ up to this point, we set $f((s+1)\binom{n+1}{2}), s+1) = w_1$. Similarly, for all $i = 2, \ldots, n$ such that the set of m such that

$$(s+1)\binom{n+1}{2} + \binom{r}{2} \leq m < (s+1)\binom{n+1}{2} + \binom{r+1}{2}$$

are not in the domain of $f[s+1]$ up to this point, we define $f[s+1]$ so that it maps the set of m such that $(s+1)\binom{n+1}{2} + \binom{r}{2} \leq m < (s+1)\binom{n+1}{2} + \binom{r+1}{2}$ onto the set of elements in $[w_i]^{\mathfrak{D}[t_{s+1}]}$ in order of magnitude. Finally, we process the elements $y \leq s+1$ which are not in the range of $f[s+1]$ up to this point, if any, in order. For each such y, we find the least element x such that $|[x]^{\mathfrak{S}}| = |[y]^{\mathfrak{D}[t_{s+1}]}|$ and x has not be used in the construction up to this point and define $f[s+1]$ so that it maps the elements of $[x]^{\mathfrak{S}}$ onto the elements of $[y]^{\mathfrak{D}[t_{s+1}]}$ in order of magnitude.

This completes the construction of f.

Note that $f(x, s) \neq f(x, s+1)$ only because of actions taken at correction substages. Moreover, we will never have to change the value of $f(x)$ after stage $s+1$, since we have guaranteed that $f(x, s+1)$ has an equivalence class of size i in $\mathfrak{D}$. It follows that f is 2-c.e.. Our actions at the correction stages ensure that for all x, $|[x]^{\mathfrak{S}}| = |f(x)]^{\mathfrak{D}}|$. For each y, it is easy to see that we change the value of $f^{-1}(y, s)$ only if $|[y]^{\mathfrak{D}[s]}| < |[y]^{\mathfrak{D}[s+1]}|$. Since the size on any equivalence class in $\mathfrak{D}$ is at most n, this means that there can be at most $n-1$ stages such that $f^{-1}(y, s) \neq f^{-1}(y, s+1)$ and hence f^{-1} is n-r.e. Our actions at the extension stages ensure that f maps $\mathcal{S}$ onto $\mathfrak{D}$. Since $f[s]$ is one-to-one at each stage, it follows that f is an n-c.e. isomorphism from $\mathcal{S}$ onto $\mathfrak{D}$. $\square$

We now construct our desired computable equivalence structure $\mathfrak{D}$ which is n-c.e. isomorphic to $\mathfrak{S}$ but not $(n-1)$-c.e. isomorphic to $\mathfrak{S}$. Let $\phi_e(x, s)$ denote e-th p.c. function. Then we must meet the following set of requirements for all $e \geq 0$ and $i = 1, \ldots, n$:

R_e: if f_e equals the $lim_{s\to\infty}\phi_e(x, s)$, then f_e is not a $(n-1)$-c.e. isomorphism from $\mathfrak{S}$ onto $\mathfrak{D}$ and

S_i: there exists an infinite c.e. set contained in the set of elements x of $\mathfrak{D}$

whose equivalence class is of size i for $i = 1, \dots, n$.

To meet the requirements $S_1, \dots, S_n$ is easy. That is, we consider the set

$$A = \{x : (\exists k)(2k)\binom{n+1}{2} \leq x < (2k+1)\binom{n+1}{2}\}.$$

Then we define $E^{\mathfrak{D}}$ so that it is identical with $E^{\mathfrak{S}}$ on A. Let $\omega - A = \{b_0 < b_1 < \dots\}$. We shall use the element b_{2e} to meet requirement R_e. Again, the idea is simple, namely, everytime that $\phi_e(x,s)$ converges and $|[b_{2e}]^{\mathfrak{D}[s]}| = |[\phi_e(x,s)]^{\mathfrak{S}}|$, we add a new element to $[b_{2e}]^{\mathfrak{D}[s+1]}$ which will force either $\phi_e(x,t)$ to change after stage s or force f_e to not be an isomorphism. Since we can force $n-1$ changes via this strategy, it will follow that there can be no $(n-1)$-c.e. isomorphsim mapping $\mathfrak{D}$ onto $\mathfrak{S}$.

Our construction will proceed in stages.

Stage 0: Define $E^{\mathfrak{D}}[0]$ so that it is identical with $E^{\mathfrak{S}}$ on A. We will not change $E^{\mathfrak{D}}$ on A, so that the requirements $S_1, \dots, S_n$ will automatically be satisfied. Define $E^{\mathfrak{D}}[0]$ to be the identity outside of A.

Stage s+1: Let $\omega - A = \{b_0 < b_1 < \dots\}$. Assume that we have defined $E^{\mathfrak{D}}[s]$ so that

(a) $[b_{2e}]^{\mathfrak{D}[s]} \cap [b_{2f}]^{\mathfrak{D}[s]} = \emptyset$ if $e \neq f$,

(b) for all x, $|[x]^{\mathfrak{D}[s]}| \leq n$,

(c) $|[b_{2e}]^{\mathfrak{D}[s]}| = 1$ for $e > s$, and

(d) for $e \leq s$, $|[b_{2e}]^{\mathfrak{D}[s]}| = k \geq 2$ if and only if there are at least $k-1$ stages $0 \leq t_{e,1} < \dots < t_{e,k-1} < s$ such that $\phi_e(x, t_{e,i})[s]$ is defined for $i = 1, \dots, k-1$ and $\phi_e(b_{2e}, t_{e,i})[s] \neq \phi_e(b_{2e}, t_{e,i+1})[s]$ for $i = 1, \dots, k-2$.

Then, for each $e \leq s+1$, we see whether there is a $t \leq s$ such that

(i) $\phi_e(b_{2e}, r)[s]$ is defined for all $r < t$,

(ii) $\phi_e((b_{2e}, t)[s] \uparrow$ but $\phi_e((b_{2e}, t)[s+1] \downarrow$, and

(iii) $|[\phi_e((b_{2e}, t)[s+1]]^{\mathfrak{S}}| = |[b_{2e}]^{\mathfrak{D}[s]}| = i < n$.

If so, then take the least $j > s$ such that b_{2j+1} has not been used in the construction up to this point and add b_{2j+1} to the equivalence class of b_{2e} in $E^{\mathfrak{D}}[s{=}1]$.

This ends the construction.

It is easy to see that $\mathfrak{D}$ is a computable equivalence relation, since the only time that additional pairs get added to $E^{\mathfrak{D}}[s+1]$ is if one of the two elements is greater than or equal to $s+1$. Also, it is easy to see that all equivalence classes have size less than or equal to n. Our action at stage 0 ensures all the requirements of $S_1, \ldots, S_n$ are met, so that $\mathfrak{D}$ is isomorphic to $\mathfrak{S}$ and, hence, by Lemma 12, $\mathfrak{D}$ is n-c.e. isomorphic to $\mathfrak{S}$. Finally, it is to see, from our actions to meet the requirement R_e, that if f_e is an isomorphism from $\mathfrak{D}$ to $\mathfrak{S}$, then there have to be n values $s_1 < \cdots < s_n$ and corresponding stages $t_1 < \cdots < t_n$ such that $\phi_e(b_{2e}, s_i)[t_i] \downarrow$ and $[\phi_e(b_{2e}, s_i)[t_i]^{\mathfrak{D}[t_i]}| = i]$ and, hence, f_e is not $(n-1)$-c.e.. Thus there is no $(n-1)$-c.e. isomorphism from $\mathfrak{D}$ onto $\mathfrak{S}$. $\square$

A second way to extend Proposition 10 is to consider graph-n-c.e. isomorphisms. Graph-1-c.e. and 1-c.e. isomorphisms are the same since both are just computable isomorphisms. We can adapt the idea of the proof of part (i) of Proposition 8 to establish a similar result for numbers bigger than 2. However, in contrast to the situation in Proposition 8, we cannot use the method to diagonalize against graph-n-c.e. isomorphisms, since the number of times even a graph-2-c.e. function can change its value on a given argument cannot, in general, be computably bounded as long as previous values are never repeated.

Theorem 13. *Let $\mathfrak{S}$ be the basic $1/2$ equivalence structure. For every $n > 0$ there exists a computable equivalence structure $\mathfrak{D}_n$ such that $\mathfrak{D}_n \cong \mathfrak{S}$ via some $(n+1)$-c.e. isomorphism, but $\mathfrak{D} \ncong \mathfrak{S}$ via any weakly-n-c.e. isomorphism.*

Proof. Fix n and let $\{\phi_k : k \in \omega\}$ be a computable enumeration of of all partial n-c.e. functions. We use the number $3k$ as a witness to ensure that ϕ_k cannot be an isomorphism from $\mathfrak{S}$ to $\mathfrak{D}_n$ by ensuring that if $\phi_k(3k)\downarrow$, then it belongs to an equivalence class consisting of two elements in $\mathfrak{D}_e$. Since $3k$ belongs in a single-element equivalence class in $\mathfrak{S}$, this ensures that ϕ_k is not an isomorphism. While we do this, we continually build and correct an $(n+1)$-c.e. isomorphism f from $\mathfrak{S}$ to $\mathfrak{D}_n$.

Stage 0: Let $E^{\mathfrak{D}_n}[0] =$ the identity relation.

Stage $s+1$: We proceed in substages for each $k \leq s$. First, suppose that $\phi_k(3k)[s] = z$ and that z belongs to a single-element equivalence class in $\mathfrak{D}_n$. Let z_s^k be the least element not yet in the range of f and put z_s^k into the equivalence class of z at stage $s+1$. If $f(3k, s) = z$, then let $f(3k, s+1) = z_s^k+1$. Choose y_s^k to be the least element congruent to 1 modulo 3 that is not yet in the domain of f, let $f(y_s^k, s+1) = z$ and $f(y_s^k+1, s+1) = z_s^k$.

After all these substages are complete, let x_s be the least multiple of 3 not yet in the domain of f, let y_s be the least number congruent to 1 modulo 3 not yet in the domain of f, and let z_s be the least number not yet in the range of f. Let $f(x_s, s+1) = z_s$, $f(y_s, s+1) = z_s+1$, $f(y_s+1, s+1) = z_s+2$, and put z_s+1 and z_s+2 into the same equivalence class in $\mathfrak{D}_n$.

It follows immediately from the construction that no n-c.e. function ϕ_k can be an isomorphism from $\mathfrak{S}$ to $\mathfrak{D}_n$ since at any sufficiently large stage s we ensure that $\phi_k(3k)$ belongs to a two-element equivalence class in $\mathfrak{D}_n$ at stage $s+1$.

Let $f = \lim_{s\to\infty} f[s]$. Notice f never changes values on any number that is not a multiple of 3. f can only change values on a number $3k$ at a stage $s{+}1$ if $\phi_k(3k)[s] = z$ and z belongs to a single-element equivalence class at stage s. Since ϕ is n-c.e., and we always enlarge z's equivalence class at the next stage, this can only happen at most n times. Hence f changes from its original value at most n times and so is $(n{+}1)$-c.e. The parameters x_s, y_s, z_s, y_s^k, and z_s^k are all strictly increasing in s. From this it follows straightforwardly from the construction that f is one-to-one. Since x_s and y_s are always chosen least, f is total. Since z_s is always chosen least and always remains in f's range thereafter, f is onto. It should be clear from the construction that we continually update the sizes of the equivalence classes of the elements in the range of f to ensure that f is a homomorphism. The only changes of previously defined values of f^{-1} occur when for numbers z, k, and l $f^{-1}(z, s) = 3k$, but $f^{-1}(z, s{+}1) = 3l{+}1$. Hence f^{-1} is 2-c.e. so that f is an $n{+}1$-c.e. isomorphism, not merely a weakly $(n{+}1)$-c.e. isomorphism.

Finally, as in Proposition 10, $\mathfrak{D}_n$ is computable, since numbers x and y are only newly associated with each other in $\mathfrak{D}_n[s{+}1]$ if at least one of x and y is greater than z_{s+1}. This establishes the result. □

It is also not hard to adapt the proof of part (ii) of Proposition 8 to the context of computable equivalence relations.

Theorem 14. *Let $\mathfrak{S}$ be the basic 1/2-equivalence structure. There exists a computable equivalence structure $\mathfrak{D}$ such that $\mathfrak{D} \cong \mathfrak{S}$ via some graph-2-c.e. isomorphism, but $\mathfrak{D} \not\cong \mathfrak{S}$ via any weakly ω-c.e. isomorphism.*

Proof. Let $\{\, \langle \phi_e, \psi_e \rangle \,:\, e \in \omega \,\}$ be a computable enumeration of all partial ω-c.e. functions with the number of changes on each value of ϕ_e bounded by ψ_e.

Stage 0: Let $f[0] = \emptyset$, and $E^{\mathfrak{D}}[0] =$ the identity relation.

Stage $s{+}1$: We proceed in substages for each $e \leq s$. Requirement $\mathbf{R}_e$ *needs attention* if $\psi_e(3e)\!\downarrow$, $\phi_e(3e, s)\!\downarrow$, $|\{\, t < s \,:\, \phi_e(3e, t) \neq \phi_e(3e, t{+}1) \,\}| < \psi_e(3e)$, and there does not exist any $z \neq \phi_e(3e, s)$ such that $zE^{\mathfrak{D}}\phi_e(3e)[s]$. Choose z_s^e to be the least number not yet in the range of f and put z_s^e into the equivalence class of $\phi_e(3e)$ at stage $s{+}1$. If $\langle 3e, \phi_e(3e, s)\rangle \in f[s]$, then remove it and enumerate $\langle 3e, z_s^e{+}1\rangle \in f[s]$. Choose y_s^e to be the least element congruent to 1 modulo 3 that is not yet in the domain of f and enumerate $\langle y_s^e, \phi_e(3e, s)\rangle$ and $\langle y_s^e{+}1, z_s^e\rangle$ into $f[s{+}1]$. After all these substages are complete, let x_s be the least multiple of 3 not yet in the domain of f, let y_s be the least number congruent to 1 modulo 3 not yet in the domain of f, and let z_s be the least number not yet in the range of f. Enumerate the pairs $\langle x_s, z_s\rangle$, $\langle y_s, z_s{+}1\rangle$, and $\langle y_s{+}1, z_s{+}2\rangle$ into $f[s{+}1]$, and put $z_s{+}1$ and $z_s{+}2$ into the same equivalence class in $E^{\mathfrak{D}}[s{+}1]$.

Let $f = \lim_{s\to\infty} f[s]$.

f can only change values on a number $3e$ if $\psi_e(3e)\downarrow$, and it only changes values on $3e$ at most $\psi_e(3e)$ times. The parameters x_s, y_s, z_s, y_s^e, and z_s^e are all strictly increasing in s. From this it follows straightforwardly that f is well-defined and one-to-one. Once a pair $\langle x, z\rangle$ is removed from f, it is never enumerated again, since any new $\langle x, z'\rangle$ that enters f at a later stage has $z' > z$. Hence f is graph-2-c.e. Sinced z_s is always chosen least, and always remains in f's range thereafter, f is onto. Clearly f is a homomorphism. $\mathfrak{D}$ is computable, since new pairs $\langle x, y\rangle$ are only enumerated into $E^{\mathfrak{D}}[s+1]$ if at least one of x and y is greater than z_{s-1}. No ω-c.e. function can be an isomorphism from $\mathfrak{D}$ to $\mathfrak{S}$, since such a function would have to be one of the ϕ_e in our enumeration. But at any stage where $\phi_e(3e, s)$ and $\psi_e(3e)\downarrow$, we ensure that $\phi_e(3e)$ is mapped to a number whose equivalence class is not a singleton in $\mathfrak{D}$. Since $3e$'s equivalence class is a singleton in $\mathfrak{S}$, ϕ_e cannot be an isomorphism. This establishes the result.

□

Corollary 15. *The basic $1/2$ equivalence structure $\mathfrak{S}$ is not weakly ω-c.e. categorical.*

There is nothing particularly special about the numbers 1 and 2 here.

Definition 7. Suppose a and b are distinct positive integers. The *basic a/b-equivalence structure* is the structure $\mathfrak{S}_{a/b} = \langle \omega, E\rangle$ where for all n and m, nEm if and only if there exists $k \in \omega$

(a) $(a+b)k \leq m, n < (a+b)k+a$ or

(b) $(a+b)k+a \leq m, n < (a+b)(k+1)$.

$\mathfrak{S}_{a/b}$ consists of infinitely many equivalence classes of sizes a and b and no other equivalence classe.

It is easy to see that the previous two results can be extended to the case of $\mathfrak{S}_{a/b}$ by simply using the strategies in the proofs above on a and b rather than 1 and 2. By considering direct sums, we can extend these results to many other computable equivalence structures.

Definition 8. If $\mathfrak{A}$ and $\mathfrak{B}$ are two equivalence structures with domain ω, then the *direct sum of $\mathfrak{A}$ with $\mathfrak{B}$* is the equivalence structure $\mathfrak{A} \oplus \mathfrak{B}$ with domain ω such that $mE^{\mathfrak{A}\oplus\mathfrak{B}}n$ if and only if there exist $k, l \in \omega$ such that either

(a) $m = 2k$, $n = 2l$ and $kE^{\mathfrak{A}}l$ or

(b) $m = 2k+1$, $n = 2l+1$ and $kE^{\mathfrak{B}}l$.

The essential feature of a direct sum is of course that any homomorphism with the direct sum as the domain is the disjoint union of two homomorphisms with the summands as their respective domains. This makes it easy to see that the following is true:

Theorem 16. *Let $\mathfrak{A}$ be any equivalence structure for which there exist at least two distinct integers m and n such that in $\mathfrak{A}$ there are infinitely many equivalence classes of sizes m and n. Then there exists a computable equivalence structure $\mathfrak{S}$ that is Δ^0_2-isomorphic to $\mathfrak{A}$ such that*

(a) for every $n > 0$, there exists a computable equivalence structure $\mathfrak{C}_n$ such that $\mathfrak{C}_n \cong \mathfrak{S}$ via some $(n+1)$-c.e. isomorphism, but $\mathfrak{C}_n \not\cong \mathfrak{S}$ via any weakly n-c.e. isomorphism and

(b) there exists a computable equivalence structure $\mathfrak{C}$ such that $\mathfrak{C} \cong \mathfrak{S}$ via some graph-2-c.e. isomorphism, but $\mathfrak{C} \not\cong \mathfrak{S}$ via any ω-c.e. isomorphism.

Proof. Let $\mathfrak{A}$ be as stated. Then $\mathfrak{A}$ is Δ^0_2-isomorphic to $\mathfrak{A} \oplus \mathfrak{S}_{m/n}$ since it is possible to check for each element of $\mathfrak{A}$ whether or not it belongs to an equivalence class of size m or n by using either a Σ^0_2 or a Π^0_2 relation. Hence, since there are infinitely many such elements in each case, we can simply alternate the side of the direct sum to which we send each of the equivalence classes of these two sizes and send all other elements x to $2x$.

Since the results in Theorems 13 and 14 hold with $\mathfrak{S}_{m/n}$ in place of $\mathfrak{S}_{1/2}$, we can just use the identity on $\mathfrak{A}$ to get appropriate isomorphisms between $\mathfrak{A} \oplus \mathfrak{S}_{m/n}$ and each of the $\mathfrak{A} \oplus \mathfrak{D}_n$, as well as $\mathfrak{A} \oplus \mathfrak{D}$ where $\mathfrak{D}_n$ and $\mathfrak{D}$ are as in the results for $\mathfrak{S}_{m/n}$.

□

Corollary 17. *If $\mathfrak{A}$ is any equivalence structure for which there exist at least two distinct integers m and n such that in $\mathfrak{A}$ there are infinitely many equivalence classes of sizes m and n, then $\mathfrak{A}$ is not weakly ω-c.e. categorical.*

Khoussainov, Stephan, and Yang [3] show that for each $n > 0$, there exists a computable structure consisting of finite graphs that is weakly n+1-c.e. categorical but fails to be graph-n-c.e. categorical. We will show that any such result must fail for computable equivalence relations by showing that any computable equivalence structures that fails to be computably categorical also fails to be weakly ω-c.e. categorical. By the results of [1], the only case remaining to consider here are equivalence structures $\mathfrak{A}$ such that $\mathfrak{A}$ has finite equivalence classes of infinitely many different sizes, not more than one size of which has infinitely many equivalence classses of that size.

Theorem 18. *If $\mathfrak{A}$ is any equivalence structure for which there exist infinitely many integers n such that there exists an equivalence class of size n, then there exists a computable equivalence structure $\mathfrak{C}$ such that $\mathfrak{C} \cong \mathfrak{A}$ via some graph-2-c.e. isomorphism, but $\mathfrak{C} \not\cong \mathfrak{A}$ via any weakly ω-c.e. isomorphism.*

Proof. For clarity's sake, we first describe the basic strategy of the result assuming that $\mathfrak{A}$ has no infinite equivalence classes. We must construct a computable equivalence structure $\mathfrak{C}_1$ that it is graph-2-c.e. isomorphic to $\mathfrak{A}$ and, whenever ϕ and ψ are a pair of computable functions such that ψ witnesses that $\lim_{s\to\infty} \phi(n, s)$ is ω-c.e., it is the case that $\lim_{s\to\infty} \phi(n, s)$ is not an isomorphism from $\mathfrak{A}$ to $\mathfrak{C}_1$. The basic strategy is that of the previous theorems: we pick a witness x and wait for $\phi(x)$ to converge. We then add elements to $\phi(x)$'s equivalence class in $\mathfrak{C}_1$ to ensure that its size is greater than that of x in $\mathfrak{A}$. The significant difference here from the situation in the previous result is that there is no way to know what different sizes of equivalence classes $\mathfrak{A}$ might have. Since $\mathfrak{A}$ has no infinite classes, this is always Δ_2^0 information, but it is in general not computable. Hence, not only do we fail to have any idea in advance how big the equivalence class of x is, but we cannot choose in advance the size of the equivalence class of $\phi(x)$. We solve this problem in the following way. Since $\mathfrak{A}$ is computable, if we believe at some stage s that the size of x's equivalence class is k, then we can search for a y such that there exist at least $k+1$ distinct z with $yE^{\mathfrak{A}}z$. There is guaranteed to be such a y since there are infinitely many different sizes of equivalence classes. Then we just add enough elements to $[\phi(x)]^{\mathfrak{C}_1}$ to make it the same size as $[y]^{\mathfrak{A}}$. Of course, we must assign $f(y) = \phi(x)$ and assign a new large number to be $f(x)$ when we do this. Later, as $[x]^{\mathfrak{A}}$ and $[y]^{\mathfrak{A}}$ grow, we must add new elements to $[f(x)]^{\mathfrak{C}_1}$ and $[f(y)]^{\mathfrak{C}_1}$, respectively. The only problem arises if $[x]^{\mathfrak{A}}$ grows to a size k_1 that equals that of $[y]^{\mathfrak{A}}$. If this happens, then $|[y]^{\mathfrak{A}}|$ can no longer serve as the size to witness $|[\phi(x)^{\mathfrak{C}_1}]| \neq |[x]^{\mathfrak{A}}|$. In this case, we must search for a y_1 such that there exist at least k_1+1 distinct z with $y_1E^{\mathfrak{A}}z$. Once we find such a number, we choose enough extra new elements to make $|[\phi(x)^{\mathfrak{C}_1}| = |[y_1]^{\mathfrak{A}}|$, let $f(y_1) = \phi(x)$, and assign the other elements of $[y_1]^{\mathfrak{A}}$ to the other elements of $[\phi(x)^{\mathfrak{C}_1}]$. Of course, now f is no longer one-to-one, since $f(y) = f(y_1) = \phi(x)$ so we must redefine all $f(z)$ for $zE^{\mathfrak{A}}y$ to new, large numbers and relate them to each other in $\mathfrak{C}_1$. Notice that f is still graph-2-c.e., since we only assign a new value under f for an element z of $[y]^{\mathfrak{A}}$, and we always choose new numbers for $f(z)$. We continue this process, performing such a reassignment at any stage at which $|[f(x)]^{\mathfrak{C}_1}| = |[\phi(x)]^{\mathfrak{C}_1}|$.

Since, by our simplifying assumption, $[x]^{\mathfrak{A}}$ is finite, we must eventually hit on a larger size for $[\phi(x)]^{\mathfrak{C}_1}$, which prevents ϕ from being an isomorphism. It is easy to see that by intializing all lower priority requirements each time we act, this strategy can succeed. In fact, as long as there are only a finite number of infinite equivalence classes, no problems arise, since we can pick representatives for these classes in advance and thereby avoid ever picking an element with an infinite equivalence class as a witness. This strategy runs into a significant problem if there are infinitely many infinite equivalence classes, since it then relies on finding a given element with a finite equivalence class to use as a suitable witness. To solve this problem, we must attack with a whole sequence of contiguous witnesses x, $x+1$, $x+2$, and so on until we encounter one with a finite equivalence class. In order to avoid infinite injury to the strategies for weaker requirements, we only select a new witness at a stage at which all previous witnesses have appeared to fail. Eventually, we will hit upon a witness with a finite equivalence class, and, at some point thereafter, we will succeed permanently on this witness.

To perform the construction, we proceed in stages s, each of which has a substage for each $n < s$. At Stage 0, we do nothing.

Stage $s+1$: We act in substages, in increasing order for each $n < s$.

Requirement n *needs to be initialized* if its first witness $x_n(0)$ is undefined. In this case we pick $x_n(0)[s]$ to be the least number such that no y previously mentioned in the construction is associated with $x_n(0)[s]$ by $E^{\mathfrak{A}}$. We let $f(x_n(0))[s]$ be the least number greater than any mentioned before in the construction and go immediately to the next substage $(n+1)$.

If requirement n does not need to be initialized, then we first update all the equivalence classes and function values associated with it. For each i such that $x(i)\downarrow[s]$, we ensure that $\big(\big|[x_n(i)]\big| = \big|[f(x_n(i)]\big|\big)[s]$. For each y such that $yE^{\mathfrak{A}}x_n(i)$ and $f(y)\uparrow$, we choose a new number z greater than any yet mentioned in the construction, set $f(y) = z$ and associate z with $f(x_n(i))[s]$ in $E^{\mathfrak{C}}$.

Requirement n *needs attention* if for every i such that $x(i)\downarrow[s]$, either

1. **$\big(\big|[x_n(i)]\big| = \big|[\phi_n(x_n(i)]\big|\big)[s]$, and the number of times which $\phi_n(x_n(i))$ has changed value is less than or equal to $\psi_n(x_n(i))$ for all such i; or**
2. **$\big|[x_n(i)]\big|[s] > \big|[x_n(i)]\big|[s^-]$, where s^- is the last previous stage at which requirement n needed attention.**

Suppose requirement n needs attention. Let k be the greatest number such that $x_n(k)\downarrow[s]$ and choose $x_n(k+1)$ to be the least number that is not in the equvalence class of any $x_j(i)[s+1]$ for $j \le n$. Let $f(x_n(k+1))[s+1]$ be some number greater than any yet mentioned in the construction. For every y with $yE^{\mathfrak{A}}x_n(k+1)$, we set $f(y)$ equal to a new number greater than any yet mentioned in the construction. If f was already mapping $x_n(k+1)$ to some other number at stage s, say $f(x_n(k+1))[s] = z$, we search for some y such that $f(y)\uparrow[s]$ and $\big(\big|[y]^{\mathfrak{A}}\big| \ge \big|[z]^{\mathfrak{C}}\big|\big)[s+1]$. We set $f(y)$ and all the members of its equivalence class equal to the members of z's equivalence class, picking new number greater than any yet mentioned to make up the difference if $\big|[y]^{\mathfrak{A}}\big|$ is actually greater than $\big|[z]^{\mathfrak{C}}\big|$.

Next for any i such that $\big(\big|[x_n(i)]\big| = \big|[\phi_n(x_n(i)]\big|\big)[s]$, and the number of times which $\phi_n(x_n(i))$ has changed value is less than or equal to $\psi_n(x_n(i))$, we search for a number y greater than any yet mentioned in the construction such that $\big|[y]^{\mathfrak{A}}\big| > \big|[x_n(i)]^{\mathfrak{A}}\big|$. We set $f(y) = \phi_n(x_n(i))$, add enough elements greater than any yet mentioned in the construction to the equivalence class of $\phi_n(x_n(i))$ to make $\big(\big|[\phi_n(x_n(i)]^{\mathfrak{C}}\big| = \big|[y]^{\mathfrak{A}}\big|\big)[s+1]$, and set $f(x_n(i))$ and all the members of its equivalence class in $\mathfrak{A}$ equal to members of a new equivalence class in $\mathfrak{C}$ made up of numbers greater than any yet mentioned in the construction.

Finally, if n has needed attention, we end the substage by undefining all parameters for requirements $m > n$.

After all substages have been completed, we extend the domain of f to all $x \leq s$ by assigning mapping each element in order to the least available elements not in f's range and extending $E^{\mathfrak{C}}$ appropriately to ensure f is an isomorphism.

Notice that as in the proofs of the previous theorems, since we always extend $E^{\mathfrak{C}}$ to pairs containing at least one number that is greater than any yet mentioned in the construction, $\mathfrak{C}$ is computable. Since f always maps elements to new large numbers, no pair $\langle x, f(x)\rangle$ ever appears in the graph of f more than once. Hence f is 2-c.e.

We verify that this construction works and that each requirement needs attention at only finitely many stages by induction. Assume that we eventually reach a stage after which no requirement $m < n$ needs attention. Then there is some greatest number y that is a witness for a requirement less than the n-th. As long as every current witness for requirement n has an infinite equivalence class in $\mathfrak{A}$, we eventually reach a stage such that a new witness is chosen for requirement n because of condition 2 in the our set of conditions for when a requirement needs attention. Since we always pick the next available number that is not involved in satisfying a higher-priority requirement, we eventually find some witness $x_n(k)$ greater than y that has a finite equivalence class in $\mathfrak{A}$. But then condition 1 under the defintion of needing attention guarantees that if $\phi_n(x_n)$ and $\psi_n(x_n)$ are defined and $\psi_n(x_n)$ bounds the number of changes on $\phi_n(x_n)$, then $[\phi_n(x_n)]^{\mathfrak{C}}$ is larger than $[x_n]^{\mathfrak{A}}$. Once this happens, requirement n can never need attention again. This shows that every requirement is satisfied and needs attention at most finitely-many times. Since every requirement needs attention only finitely often, f is total, which completes the proof of the result. □

Notice that a very similar construction works to achieve the same result in the case that $\mathfrak{A}$ has infinitely many equivalence classes of at least one finite size k and infinitely many infinite equivalence classes. In this case we can arrange things so that for each n we eventually find a witness x_n with an equivalence class in $\mathfrak{A}$ of size k so that if $\phi_n(x_n)$ is defined, then $[\phi_n(x_n)]^{\mathfrak{C}}$ is either infinite or has some finite size greater than k. The changes are straightforward.

Theorem 19. *If $\mathfrak{A}$ is any equivalence structure for which there exist an integer n such that there exist infinitely many equivalence classes of size n and infinitely many infinite equivalence classes, then there exists a computable equivalence structure $\mathfrak{C}$ such that $\mathfrak{C} \cong \mathfrak{A}$ via some graph-2-c.e. isomorphism, but $\mathfrak{C} \not\cong \mathfrak{A}$ via any weakly ω-c.e. isomorphism.*

Corollary 20. *A computable equivalence structure is computably categorical if and only if it is weakly-ω-c.e. categorical.*

Proof. Any computably categorical equivalence structure is of course *a fortiori* ω-c.e. categorical. By Calvert, Cenzer, Harizanov, and Morozov, [1], Corollary 3.3, if a computable equivalence structure has either

(i) finitely-many finite equivalence classes or

(ii) at most one finite size with infinitely-many equivalence classes of that size, a bound on the number of finite sizes of equivalence classes, and finitely-many infinite equivalence classes,

then that structure is computably categorical. Hence, if a computable equivalence structure fails to be computably categorical, it must have infinitely many finite equivalence classes and, in addition, there must be either

1. at least two different finite sizes with infinitely-many equivalence classes of each of those sizes, in which case it is not weakly ω-c.e. categorical by Corollary 17; or, failing that,
2. an unbounded number of different finite sizes of equivalence classes, in which case it is not weakly ω-c.e. categorical by Theorem 18; or, failing both of these,
3. infinitely-many infinite equivalence classes and at least one finite size with infinitely-many equivalence classes of that size, in which case it is not weakly ω-c.e. categorical by Theorem 19.

□

5 Some categorical structures

Although we have shown that there is no difference between ω-c.e. categoricity and computable categoricity for computable equivalence structures, the same is not true for graph-n-c.e. categoricity.

First we shall consider the structure $\mathfrak{S}_{I,\alpha}$ which consists of infinitely many equivalence classes of sizes i for each $i \in I$ and α infinite equivalence classes. More formally, we make the following definition.

Definition 9. Suppose $\alpha \leq \omega$ and I is a set of positive integers with $|I| \geq 1$ and least element i_0. The *basic* I, α*-equivalence structure* is the structure $\mathfrak{S}_{I,\alpha} = \langle \omega, E \rangle$ where for all n and m, nEm if and only if either

1. there exists $i \in I$ and $j, k, l \in \omega$ such that $n = \langle 2i, j \rangle$, $m = \langle 2i, k \rangle$, and $l \cdot i \leq j, k < (l{+}1) \cdot i$; or
2. there exists $i \in \omega - I$ and $j, k, l \in \omega$ such that $n = \langle 2i, j \rangle$, $m = \langle 2i, k \rangle$, and $l \cdot i_0 \leq j < (l{+}1) \cdot i_0$.
3. there exists $i < \alpha$ and $j, k \in \omega$ such that $n = \langle 2i{+}1, j \rangle$, and $m = \langle 2i{+}1, k \rangle$.
4. there exists $i \geq \alpha$ and $j, k, l \in \omega$ such that $n = \langle 2i{+}1, j \rangle$, $m = \langle 2i{+}1, k \rangle$, and $l \cdot i_0 \leq j < (l{+}1) \cdot i_0$.

Theorem 21. *If I is a nonempty finite set of positive integers and $\alpha \leq \omega$, then the basic I, α-equivalence structure $\mathfrak{S}_{I,\alpha}$ is relatively graph-2-c.e. categorical.*

Proof. Suppose $\mathfrak{A}$ is an equivalence structure that is isomorphic to $\mathfrak{S}_{I,\alpha}$. Let $I = \{i_0 < ... < i_n\}$. We construct a $E^{\mathfrak{A}}$-graph-2-c.e. isomorphism $f : \mathfrak{S}_{I,\alpha} \to \mathfrak{A}$. by defining a $E^{\mathfrak{A}}$-2-c.e. set F that is the graph of f in stages. Let $A[s] = \{ y \in A : y < s \}$ and $E^{\mathfrak{A}[s]} = \{ \langle y, z \rangle : y < s \wedge z < s \wedge zE^{\mathfrak{A}}y \}$. Then, for any natural number $y < s$, $[y]^{\mathfrak{A}[s]} = \{ z : z < s \wedge zE^{\mathfrak{A}}y \}$. Clearly, $[y]^{\mathfrak{A}[s]}$ and $E^{\mathfrak{A}[s]}$ are finite sets with indices computable in $E^{\mathfrak{A}}$.

Stage 0**:** $F[0] = \emptyset$.

Stage $s{+}1$**:**

Substage I. We first update our isomorphism between $Inf^{\mathfrak{S}_{I,\alpha}}$ and $Inf^{\mathfrak{A}}$. If there exists some $y < s$ and $z < s$ such that y is the image under $f[s]$ of some number with an infinite equivalence class in $\mathfrak{S}_{I,\alpha}$, $yE^{\mathfrak{A}}z$, and z is not in the range of $f[s]$, then we take the least number x in this infinite equivalence class that is not in the domain of $f[s]$ and enumerate $\langle x, z \rangle$ into $F[s{+}1]$.

Substage II. Next, suppose there exists some pair $\langle x, y \rangle \in F[s]$ such that $|[x]^{\mathfrak{S}_{I,\alpha}}| = i$ and $|[y]^{\mathfrak{A}[s]}| = j$, and $i < j$. We may assume that either $j \in I$ or $j > i_n$ since if $j < i_0$ or if $i_m < j < i_{m+1}$, then we can simply wait until more elements are added to $[y]^{\mathfrak{A}}$. Then we must correct f before extending it to any new values. We correct all pairs that need correction in order of their first coordinates. Let $\langle x, y \rangle$ be such a pair.

(a) We first remove all pairs $\langle w, z \rangle$ with $wE^{\mathfrak{S}_{I,\alpha}}x$ from $F[s{+}1]$.

(b) We then add new images for the elements of $[x]^{\mathfrak{S}_{I,\alpha}}$ by searching for the least $t > s$ such that there is an equivalence class $[y']^{\mathfrak{A}[t]}$ of size i that was not in $E^{\mathfrak{A}[s]}$. Such a class will always be available since the structures are isomorphic, so that there are infinitely many equivalence classes of size i in $\mathfrak{A}$. Unfortunately, $|[y']^{\mathfrak{A}}|$ itself may actually be greater than i. To avoid picking the wrong class infinitely often, we need to always pick the equivalence class at stage t that has the smallest allowable members to send $[x]$ to so that eventually we will pick one of the right size. We guarantee this as follows: if there exists any z such that $|[f(z)]^{\mathfrak{A}[s]}| = |[f(z)]^{\mathfrak{A}[t]}| = i$, and all elements of $[z]^{\mathfrak{S}_{I,\alpha}}$ are greater than x, then let x' be the least such number so that the size of $[f(x')]^{\mathfrak{A}}$ has appeared to be i for at least as long as any other such number at stage t. We first remove all pairs $\langle z, w \rangle$ with $zE^{\mathfrak{S}_{I,\alpha}}x'$ from $F[s{+}1]$. Then assign the elements of $[x]^{\mathfrak{S}_{I,\alpha}}$ to the equivalence class of $f(x')[s]$ by enumerating the appropriate pairs into $F[s{+}1]$. Finally, we assign the elements of $[x']^{\mathfrak{S}_{I,\alpha}}$ to $[y']^{\mathfrak{A}[t]}$.

(c) If $j \in I$, we then map the least equivalence class of size j in $\mathfrak{S}_{I,\alpha}$ that has not been previously assigned values under f by enumerating the appropriate pairs to do this into $F[s{+}1]$.

(d) Otherwise, $|[y]^{\mathfrak{A}}| = \omega$. In this case, we take the next unused infinite equivalence class in $\mathfrak{S}_{I,\alpha}$ and map the first j of its elements to those of $[y]^{\mathfrak{A}[s]}$. Such a class will always be available since the structures are isomorphic.

Substage III. After all corrections are made, we extend f.

(a) First, we choose the least equivalence classes of each of the sizes $i \in I$ in $\mathfrak{S}_{I,\alpha}$ that are not currently assigned images under f and search for the least $t \geq s$ such that $\mathfrak{A}[t]$ has equivalence classes of those sizes consisting of elements not currently in the range of f. We then enumerate pairs into $F[s{+}1]$ to map the unassigned classes in $\mathfrak{S}_{I,\alpha}$ to these new classes of the same size in $\mathfrak{A}[t]$.

(b) If there exists a number y such that $|[y]^{\mathfrak{A}[s]}| = j > i_n$ that is not in the range of $f[s]$, then we take the next unused infinite equivalence class in $\mathfrak{S}_{I,\alpha}$ and map the first j of its elements to those of $[y]^{\mathfrak{A}[s]}$. As above, such a class will always be available since the structures are isomorphic.

This completes the construction.

First, notice that the correcting action in substage II.b above only enumerates a pair $\langle x, y\rangle$ into $F[s{+}1]$ if this pair has never been in $F[t]$ for any $t \leq s$. This is the only time that f changes value on an argument, so F is a 2-c.e. set. Since the correcting action only changes F when some equivalence class increases in size, the value of f on any argument can only change at most $\max(I)$ times, so that f comes to a limit on each argument. The extension of f that takes place at the end of each stage therefore shows that f is a total graph-2-c.e. function. Since the equivalence class of $[y']^{\mathfrak{A}}[t]$ in substage II.b is not of size i in $\mathfrak{A}[s]$, any number x' that is mapped to an element of $[y']^{\mathfrak{A}}$ by $f[s]$ must belong to an equivalence class of smaller size than i, and so it must change its value on or before stage t. Hence the function f is one-to-one.

Since the only equivalence classes in either $\mathfrak{S}_{I,\alpha}$ or $\mathfrak{A}$ that have cardinality greater than $\max(I)$ are the infinite ones, the action taken in substage III.b guarantees that any number with an infinite equivalence class in $\mathfrak{A}$ has a number with an infinite equivalence class in $\mathfrak{S}_{I,\alpha}$ mapped to it. The action taken in substage I keeps f consistent on the infinite equivalence classes, and so the action taken in substage III.b together with the correcting action that takes place in substage II.d guarantee that every infinite class in $\mathfrak{S}_{I,\alpha}$ is mapped to one in $\mathfrak{A}$.

If $[x]^{\mathfrak{S}_{I,\alpha}} = i \in I$, then this fact is computable, so that x is never mapped to any y such that $|[y]^{\mathfrak{A}}| < i$. By the extension of f that takes place in substage III.a, x is eventually mapped to some y with $|[y]^{\mathfrak{A}}| \geq i$. If $|[y]^{\mathfrak{A}}| > i$, then the correcting action later forces x to be mapped to a new number with an apparently smaller equivalence class. Since the structures are isomorphic and there are infinitely-many equivalence classes of size i, there is always some y so that $|[y]^{\mathfrak{A}}| = i$, and, eventually, such a number must become available for

x to be assigned to it. In substage II.b, we always map $[x]^{\mathfrak{S}_{I,\alpha}}$ to the class in $\mathfrak{A}[t]$ that has appeared to have size i for the longest time; hence, we eventually select an image of size i. The action taken in substage III guarantees that each y eventually gets put into the range of f. If $|[y]^{\mathfrak{A}}|$ is finite, then the correcting action in substage II guarantees that at most $|[y]^{\mathfrak{A}}|$ different numbers get mapped to y over the course of the construction and that whatever number is eventually mapped to y has an equivalence class of the correct size. Hence f is a graph-2-c.e. isomorphism, as required.

□

Corollary 22. *If $\mathfrak{A}$ is a computable equivalence structure with bounded finite character (and any number of infinite equivalence classes), then $\mathfrak{A}$ is relatively graph-2-c.e. categorical.*

Proof. If $\mathfrak{A}$ has only finitely-many finite-sized equivalence classes, then it is computably categorical. Otherwise, let I_0 be the set of all $i \in \omega$ such that there are only finitely many nonempty equivalence classes of size i in $\mathfrak{A}$. I_0 is finite, since $\mathfrak{A}$ has bounded character. Then $\mathfrak{A}$ is isomorphic to the direct sum of a finite equivalence structure consisting of all the elements with equivalence classes of sizes in I_0 and some basic I, α equivalence structure $\mathfrak{S}_{I,\alpha}$. Such a structure is graph-2-c.e categorical by Theorem 21. □

Theorem 21 makes it possible to finish characterizing the relationship between ω-c.e. categoricity and graph-2-c.e. categoricity for computable equivalence relations.

Corollary 23. *Any weakly ω-c.e. categorical computable equivalence structure is graph-2-c.e. categorical. There is a graph-2-c.e. categorical computable equivalence structure that is not ω-c.e. categorical.*

Proof. Since weakly ω-c.e. categorical computable equivalence structures are computably categorical, the first assertion is trivial. By Theorem 21, the basic $1/2$ equivalence structure $\mathfrak{S}_{1/2}$ is graph-2-c.e. categorical. By Theorem 14, it is not weakly ω-c.e. categorical. □

A computable equivalence structure need not have bounded finite character in order to be graph-2-c.e. categorical.

Theorem 24. *If $\mathfrak{A}$ is a computable equivalence structure such that for every $i \leq \omega$, $\mathfrak{A}$ has only finitely many equivalence classes of size i, then $\mathfrak{A}$ is relatively graph-2-c.e categorical.*

Proof. Let $\mathfrak{A}$ be a computable equivalence structure with only finitely many equivalence classes of each size $i \leq \omega$ and let $\mathfrak{B}$ be a computable equivalence structure isomorphic to $\mathfrak{A}$. Without loss of generality, we can in fact assume that $\mathfrak{A}$ has no infinite equivalence classes since there will always be a $E^{\mathfrak{B}}$-computable mapping from any finite number of infinite equivalence classes in $\mathfrak{A}$ to the same number of classes in $\mathfrak{B}$. Also, by Corollary 22, we can assume $\mathfrak{A}$ has unbounded finite character.

As in the proof of Theorem 21, we construct an $E^{\mathfrak{B}}$-graph-2-c.e. isomorphism $f : \mathfrak{A} \to \mathfrak{B}$. by defining a $E^{\mathfrak{B}}$-2-c.e. set F that is the graph of f in stages. Let $B[s] = \{ y \in B : y < s \}$ and $E^{\mathfrak{B}[s]} = \{ \langle y, z \rangle : y < s \wedge z < s \wedge zE^{\mathfrak{B}}y \}$. Then, for any natural number $y < s$, $[y]^{\mathfrak{B}[s]} = \{ z : z < s \wedge zE^{\mathfrak{B}}y \}$. Again, $[y]^{\mathfrak{B}[s]}$ and $E^{\mathfrak{B}[s]}$ are finite sets with indices computable in $E^{\mathfrak{B}}$. We make analogous definitions for $\mathfrak{A}$. For each s, $[x]^{\mathfrak{A}[s]}$ and $E^{\mathfrak{A}[s]}$ have computable indices. Notice that this definition ensures that elements enter equivalence classes in increasing order during the construction.

An important difference here from the situation in Theorem 21 is that for each $x \in \omega$, we knew the size of $[x]^{\mathfrak{S}_{I,\alpha}}$, whereas here have in general no knowledge in advance of the size of $[x]^{\mathfrak{A}}$. The key problem we have is the following. Suppose at stage $s+1$ we assign $f(x) = y$ because $|[x]^{\mathfrak{A}[s]}| = |[y]^{\mathfrak{B}[s]}|$ and then at a later stage s', the size of y's equivalence class increases, although at this stage we can still find a new number z so that $|[x]^{\mathfrak{A}[s']}| = |[z]^{\mathfrak{B}[s']}|$. So far there is no problem, since, just as in the proof of Theorem 21, we can remove $\langle x, y \rangle$ from the graph of f and add $\langle x, z \rangle$. But now, suppose that at a later stage s'', the size of x's equivalence class increases so that $|[x]^{\mathfrak{A}[s'']}| \neq |[z]^{\mathfrak{B}[s'']}|$, but $|[x]^{\mathfrak{A}[s'']}| = |[y]^{\mathfrak{B}[s'']}|$. Now it is impossible to re-enumerate $\langle x, y \rangle$ into the graph of f since this set is supposed to be relatively $E^{\mathfrak{B}}$-2-c.e., not $E^{\mathfrak{B}}$-3-c.e. However, since both $|[x]^{\mathfrak{A}}|$ and $|[y]^{\mathfrak{B}}|$ have increased since stage s, there must be new elements x' and y' in each of these sets respectively. We can use the extra leeway these new elements give us to map $[x]^{\mathfrak{A}[s'']}$ onto $[y]^{\mathfrak{B}[s'']}$ at stage $s''+1$ without repeating any of the actual values we assigned at stage $s+1$. Fortunately, because there are never infinitely-many equivalence classes of any particular size i, we do no have to change f's value on any x except when the size of either $[x]^{\mathfrak{A}}$ or $[f(x)^{\mathfrak{B}}]$ changes. We cannot do this in proof of Theorem 25 below, which forces our isomorphism there to be merely graph-ω-c.e.

Stage 0**:** $F[0] = \emptyset$.

Stage $s+1$**:** We first correct f, then extend it.

Substage I. Suppose there exists some pair $\langle x, y \rangle \in F[s]$ such that $|[x]^{\mathfrak{A}[s]}| \neq |[y]^{\mathfrak{B}[s]}|$. We correct all pairs that need correction in order of their first coordinates. Let $\langle x, y \rangle$ be such a pair. We search for the least $t \geq s$ such that there exists a y' such that $|[x]^{\mathfrak{A}}[t]| = |[y']^{\mathfrak{B}[t]}|$ and for all z, if $f(z)[s] \in [y']^{\mathfrak{B}[t]}$, then either $zE^{\mathfrak{A}[t]}x$ or $|[z]|^{\mathfrak{A}[t]}| \neq |[y']^{\mathfrak{B}[t]}|$. Notice that if $s < t'$, $[x]^{\mathfrak{A}} = [x]^{\mathfrak{A}[t']}$, and $\{ z < t' : |[z]|^{\mathfrak{B}} = |[x]^{\mathfrak{A}}| \} = \{ z : |[z]|^{\mathfrak{B}} = |[x]^{\mathfrak{A}}| \}$, then $t \leq t'$, so this search terminates. There are two cases.

(a) Suppose there is such a y' so that there does not exist any z such that $f(z)[s] \in [y']^{\mathfrak{B}[t]}$. Remove all pairs $\langle z, f(z)[s] \rangle$ such that $f(z)[s] \in [y]^{\mathfrak{B}[s]}$ from $F[s+1]$. Let $[x]^{\mathfrak{A}[t]} = \{x_0 < x_2 < \ldots < x_k\}$ and let $[y']^{\mathfrak{B}[t]} = \{y_0 < y_2 < \ldots < y_k\}$. For each $j \leq k$, enumerate $\langle x_j, y_{k-j} \rangle$ into $F[s+1]$.

(b) Otherwise, suppose for all such y', there exists a z such that $f(z)[s] \in$

$[y']^{\mathfrak{B}[t]}$. Choose the least such y'. First, remove all pairs $\langle z, f(z)[s]\rangle$ such that $f(z)[s] \in [y]^{\mathfrak{B}[s]}$ from $F[s+1]$. Next, for all z such that $f(z)[s] \in [y']^{\mathfrak{B}[t]}$, remove $\langle z, f(z)[s]\rangle$ from $F[s+1]$. Finally, let $[x]^{\mathfrak{A}[t]} = \{x_0 < x_2 < \ldots < x_k\}$ and let $[y']^{\mathfrak{B}[t]} = \{y_0 < y_2 < \ldots < y_k\}$. For each $j \leq k$, enumerate $\langle x_j, y_{k-j}\rangle$ into $F[s+1]$.

Substage II. After all corrections are made, we extend f. For every equivalence class in $\mathfrak{A}[s]$ that is currently unassigned in $f[s]$, we pick the least representative x and search for the least $t \geq s$ such that there is some $y < t$ with $|[x]^{\mathfrak{A}[t]}| = |[y]^{\mathfrak{B}[t]}|$. Such a t is guaranteed to exist since $\mathfrak{A}$ is isomorphic to $\mathfrak{B}$ and all equivalence classes are finite. Let $[x]^{\mathfrak{A}[t]} = \{x_0 < x_2 < \ldots < x_k\}$ and let $[y]^{\mathfrak{B}[t]} = \{y_0 < y_2 < \ldots < y_k\}$. For each $j \leq k$, enumerate $\langle x_j, y_{k-j}\rangle$ into $F[s+1]$.

This completes the construction.

The function f is well-defined, since it can only change from a previously-defined value at substages I.a or I.b and the pair defining the previous value is removed from F at that point. Similarly, f is one-to-one, since the only case where $x \neq z$ and $f(x)[s+1] = f(z)[s]$ is in substage I.b and there $\langle z, f(z)[s]\rangle$ is removed from $F[s+1]$. Our actions in substage I ensure that the function f is structure-preserving and our actions in substage II ensure that f is total. Since there are only finitely many equivalence classes of any particular size and f is a total, injective homomorphism, it is evidently onto as well. Hence f is an isomorphism.

All that remains is to show that F is 2-c.e. relative to $E^{\mathfrak{B}}$. Clearly, the construction is computable in $E^{\mathfrak{B}}$. Suppose there exists some x and $s_0 < s_1 < s_2$ so that $f(x)[s_0] = f(x)[s_2]$, but $f(x)[s_0] \neq f(x)[s_1]$. Let $y = f(x)[s_0]$, let $k = |[x]^{\mathfrak{A}[s_0]}|$ and let $m = |[x]^{\mathfrak{A}}[s_2]|$. Since $|[x]^{\mathfrak{A}}[s_0]| = |[y]^{\mathfrak{B}[s_0]}|$, $|[x]^{\mathfrak{A}}[s_2]| = |[y]^{\mathfrak{B}[s_2]}|$, and at least one class had to increase in size, $k < m$. Let $[x]^{\mathfrak{A}[s_2]} = \{x_0 < x_2 < \ldots < x_m\}$ and $[y]^{\mathfrak{B}[s_2]} = \{y_0 < y_2 < \ldots < y_m\}$. There exists $j \leq k < m$ such that $x = x_j$. But then $f(x)[s_0] = y_{k-j} \neq y_{m-j} = f(x)[s_2]$. This is a contradiction so that f can never return to a previous value. Hence F is 2-c.e. relative to $E^{\mathfrak{B}}$.

□

Corollary 20 shows there is no nondegenerate hierarchy of computable equivalence structures in the case of α-c.e. categoricity. Hence there is no way to produce analogues of the results of Koussainov, Stephan, and Yang [3] in the case of computable equivalence structures.

Calvert, Cenzer, Harizanov, and Morozov, [1] show that a computable equivalence structure is not relatively Δ^0_2 categorical if and only if it has unbounded character and infinitely many infinite equivalence classes. Hence, the following result, together with Theorem 21, shows that a computable equivalence structure is relatively Δ^0_2 categorical if and only if it is graph-ω-c.e. categorical.

Theorem 25. *Any computable equivalence structure with a finite number of infinite equivalence classes is relatively graph-ω-c.e categorical.*

Proof. Let $\mathfrak{A}$ be a computable equivalence structure with only finitely many infinite equivalence classes and let $\mathfrak{B}$ be a computable equivalence structure isomorphic to $\mathfrak{A}$. Without loss of generality, we can in fact assume that $\mathfrak{A}$ has no infinite equivalence classes, since there will always be a $E^{\mathfrak{B}}$-computable mapping from any finite number of infinite equivalence classes in $\mathfrak{A}$ to the same number of classes in $\mathfrak{B}$. Also, by Corollary 22, we can assume $\mathfrak{A}$ has unbounded finite character.

The situation here is similar to that in Theorem 24, but we have the additional problem that, as in Theorem 21, we have to make sure that if there are infinitely many classes of some particular size in $\mathfrak{A}$, then all of them eventually get mapped to class of the same size in $\mathfrak{B}$. It is not too hard to incorporate the strategy that solves this problem in the proof of Theorem 21 into the construction by giving smaller numbers priority in selecting which equivalence class they enter. This does, however, cause the isomorphism to be relatively graph-ω-c.e. rather than graph-2-c.e. The details are similar to those in the proof of Theorem 24.

Stage 0**:** $F[0] = \emptyset$.

Stage $s{+}1$**:** We first correct f, then extend it.

Substage I. Suppose there exists some pair $\langle x, y\rangle \in F[s]$ such that $|[x]^{\mathfrak{A}[s]}| \neq |[y]^{\mathfrak{B}[s]}|$. We correct all pairs that need correction in order of their first coordinates. Let $\langle x, y\rangle$ be such a pair. We search for the least $t \geq s$ such that there exists a y' such that $|[x]^{\mathfrak{A}[t]}| = |[y']^{\mathfrak{B}[t]}|$ and for all z, if $f(z)[s] \in [y']^{\mathfrak{B}[t]}$, then either $zE^{\mathfrak{A}[t]}x$ or $|[z]|^{\mathfrak{A}[t]}| \neq |[y']^{\mathfrak{B}[t]}|$. Such a stage t clearly exists if there are infinitely many equivalence classes in $\mathfrak{A}$ with the same size as $[x]^{\mathfrak{A}}$. Otherwise there exists some t' such that $s < t'$, $[x]^{\mathfrak{A}} = [x]^{\mathfrak{A}[t']}$ and $\{\, z < t' : |[z]|^{\mathfrak{B}} = |[x]^{\mathfrak{A}}| \,\} = \{\, z : |[z]|^{\mathfrak{B}} = |[x]^{\mathfrak{A}}| \,\}$. As in the proof of Theorem 24, then, there must exist an appropriate $t \leq t'$. Either way, this search terminates. By the conditions on t, there must exist some y' such that $|[x]^{\mathfrak{A}[t]}| = |[y']^{\mathfrak{B}[t]}|$ and for all z, if $f(z)[s] \in [y']^{\mathfrak{B}[t]}$, then either $zE^{\mathfrak{A}[t]}x$ or $|[z]|^{\mathfrak{A}[t]}| \neq |[y']^{\mathfrak{B}[t]}|$. Hence, there must also exist some y' such that $|[x]^{\mathfrak{A}[t]}| = |[y']^{\mathfrak{B}[t]}|$ and for all z, if $f(z)[s] \in [y']^{\mathfrak{B}[t]}$, then either $zE^{\mathfrak{A}[t]}x$ or $|[z]|^{\mathfrak{A}[t]}| \neq |[y']^{\mathfrak{B}[t]}|$ or $x < z$. Choose the least such y' that has the least s' for which $[y']^{\mathfrak{B}[s']} = [y']^{\mathfrak{B}[t]}$. (As in the proof of Theorem 21, we intend to map $[x]^{\mathfrak{A}[t]}$ to the best equivalence class in $\mathfrak{B}$ that is not the image of some higher-priority equivalence class.) Remove all pairs $\langle z, f(z)[s]\rangle$ such that $f(z)[s] \in [y]^{\mathfrak{B}[s]}$ from $F[s{+}1]$. For all z such that $f(z)[s] \in [y']^{\mathfrak{B}[t]}$, remove $\langle z, f(z)[s]\rangle$ from $F[s{+}1]$. Finally, let $[x]^{\mathfrak{A}[t]} = \{x_0 < x_2 < \ldots < x_k\}$ and let $[y']^{\mathfrak{B}[t]} = \{y_0 < y_2 < \ldots < y_k\}$. For each $j \leq k$, enumerate $\langle x_j, y_{k-j}\rangle$ into $F[s{+}1]$.

Substage II. After all corrections are made, we extend f, using the same procedure as above. For every equivalence class in $\mathfrak{A}[s]$ that is currently unassigned in $f[s]$, we pick the least representative x. We search for the least

$t \geq s$ such that there exists a y' such that $|[x]^{\mathfrak{A}[t]}| = |[y']^{\mathfrak{B}[t]}|$ and for all z, if $f(z)[s] \in [y']^{\mathfrak{B}[t]}$, then either $zE^{\mathfrak{A}[t]}x$ or $|[z]|^{\mathfrak{A}[t]}| \neq |[y']^{\mathfrak{B}[t]}|$. By the conditions on t, there must exist some y' such that $|[x]^{\mathfrak{A}[t]}| = |[y']^{\mathfrak{B}[t]}|$ and for all z, if $f(z)[s] \in [y']^{\mathfrak{B}[t]}$, then either $zE^{\mathfrak{A}[t]}x$ or $|[z]|^{\mathfrak{A}[t]}| \neq |[y']^{\mathfrak{B}[t]}|$ or $x < z$. Choose the least such y' that has the least s' for which $[y']^{\mathfrak{B}[s']} = [y']^{\mathfrak{B}[t]}$. Remove all pairs $\langle z, f(z)[s]\rangle$ such that $f(z)[s] \in [y]^{\mathfrak{B}[s]}$ from $F[s{+}1]$. For all z such that $f(z)[s] \in [y']^{\mathfrak{B}[t]}$, remove $\langle z, f(z)[s]\rangle$ from $F[s{+}1]$. Finally, let $[x]^{\mathfrak{A}[t]} = \{x_0 < x_2 < \ldots < x_k\}$ and let $[y']^{\mathfrak{B}[t]} = \{y_0 < y_2 < \ldots < y_k\}$. For each $j \leq k$, enumerate $\langle x_j, y_{k-j}\rangle$ into $F[s{+}1]$.

This completes the construction.

The function f is well-defined, since it can only change from a previously-defined value in substage I and the pair defining the previous value is removed from F at that point. Similarly, f is one-to-one, since whenever $x \neq z$ and $f(x)[s{+}1] = f(z)[s]$, $\langle z, f(z)[s]\rangle$ is removed from $F[s{+}1]$. Our actions in substage I ensure that the function f is structure-preserving and our actions in substage II ensure that f is total. If there are only finitely many equivalence classes of the same size as $[y]^{\mathfrak{B}}$, then the fact that f is a total, injective homomorphism, shows y is in the image of f. Otherwise, suppose there are infinitely many equivalence classes with the same size as $[y]^{\mathfrak{B}}$. Eventually, $[y]^{\mathfrak{B}}$ is the equivalence class of that size that has been available for the longest time. Then, as soon as a new equivalence class of that size in $\mathfrak{A}$ appears, it will be assigned to $[y]^{\mathfrak{B}}$ at substage II. Since there are infinitely many such equivalence classes in $\mathfrak{A}$, one will eventually appear and, hence, y is in the image in this case as well. Thus f is onto.

All that remains is to show that F is ω-c.e. relative to $E^{\mathfrak{B}}$. Clearly, the construction is computable in $E^{\mathfrak{B}}$. Suppose there exists some x and $s_0 < s_1 < s_2$ so that $f(x)[s_0] = f(x)[s_2]$, but $f(x)[s_0] \neq f(x)[s_1]$. Let $y = f(x)[s_0]$, let $k = |[x]^{\mathfrak{A}[s_0]}|$ and let $m = |[x]^{\mathfrak{A}}[s_2]|$. Let $[x]^{\mathfrak{A}[s_2]} = \{x_0 < x_2 < \ldots < x_m\}$ and $[y]^{\mathfrak{B}[s_2]} = \{y_0 < y_2 < \ldots < y_m\}$. There exists $j \leq k \leq m$ such that $x = x_j$. If $k < m$, then $f(x)[s_0] = y_{k-j} \neq y_{m-j} = f(x)[s_2]$. This is a contradiction so that it must be the case that $k = m$. But then $f(x)$ must have changed value under I because some $[x']^{\mathfrak{A}}[s_1]$ with $x' < x$ was temporarily assigned to $[f(x)]^{\mathfrak{B}}[s_1]$ and then later $[x']$ was unassigned, giving $[x]$ a chance to be restored. A bound on the number of times this can happen can be computed recursively from x, in fact, it is less than 2^{x+1}. Hence f is relatively graph-ω-c.e., as required. $\square$

We conjecture that in fact this result can be improved to show that such a structure must be relatively graph-2-categorical. In that case, it would follow that for computable equivalence relations, relative Δ_2^0 categoricity is identical with relative graph-2-categoricity. This would provide an even sharper contrast with the case of finite graph, since there would only be three possibilities for a computable equivalence structure: weakly-ω-c.e. categorical, which is the same as computably categorical; not computably categorical, but still graph-2-computably categorical; and not even Δ_2^0 categorical.

References

[1] W. Calvert, D. Cenzer, V.Harizanov, A. Morozov, *Effective categoricity of equivalence structures*, Annals of Pure and Applied Logic 141 (2006), pp. 6178.

[2] N. G. Khisamiev, "Constructive abelian p-groups", *Siberian Advances in Mathematics 2 (1992)*, pp. 68-113.

[3] B. Khoussainov, F. Stephan, and Y. Yang, "Computable Categoricity and the Ershov Hierarchy", *unpublished*

[4] R. I. Soare, *Recursively Enumerable Sets and Degrees*, Springer-Verlag, Berlin, 1987.