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Effectively Closed Sets and Enumerations

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Abstract An effectively closed set, or Π^0_1 class, may viewed as the set of infinite paths through a computable tree. A *numbering*, or *enumeration*, is a map from ω onto a countable collection of objects. One numbering is reducible to another if equality holds after the second is composed with a computable function. Many commonly used numberings of Π^0_1 classes are shown to be mutually reducible via a computable permutation. Computable injective numberings are given for the family of Π^0_1 classes and for the subclasses of decidable and of homogeneous Π^0_1 classes. However no computable numberings exist for small or thin classes. No computable numbering of trees exists that includes all computable trees without dead ends.

1 Introduction

The general theory of numberings was initiated in the mid-1950s by Kolmogorov, and continued under the direction of Mal'tsev and Ershov [13]. A *numbering*, or *enumeration*, of a collection C of objects is a surjective map $F : \omega \to C$ [22]. In one of the earliest results, Friedberg constucted an injective computable numbering ψ of the Σ^0_1 or computably enumerable (*c.e.*) sets such that the relation "$n \in \psi(e)$" is itself Σ^0_1. More generally, we will say that a numbering ψ of collection of objects with complexity $\mathcal{C}$ (such as n-c.e., Σ^0_n, or Π^0_n) is *effective* if the relation "$x \in \psi(e)$" has complexity $\mathcal{C}$. We will also consider enumerations where the relation "$x \in \psi(e)$" has a different complexity than $\mathcal{C}$. (For example, there is a c.e., but not computable, numbering of the computable sets.)

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A numbering μ is *acceptable with respect to a numbering* ν, denoted $\nu \leq \mu$, iff there is a total computable function f such that $\nu = \mu \circ f$. If μ is acceptable with respect to all effective numberings, then μ is said to be *acceptable* [21]. The ordering $\leq$ gives rise to an equivalence relation $\equiv$, and two numberings in the same equivalence class are called *equivalent*. Furthermore, the structure $\mathcal{L}(C)$ of all numberings of C modulo $\equiv$ forms an upper semilattice under $\leq$. Here injective numberings occur only in the minimal elements and acceptable numberings occur only in the greatest element. This article is a study of effective numberings of families of effectively closed sets, also known as Π^0_1 classes.

A subset T of $\omega^{<\omega}$ is a *tree* if it is closed under initial segments. The set $[T]$ of infinite paths through T is defined by $X \in [T] \iff (\forall n) X\lceil n \in T$, where $X\lceil n$ denotes the initial segment $(X(0), X(1), \ldots, X(n-1))$. Let $\sigma^\frown\tau$ denote the concatenation of σ with τ and $\sigma^\frown i$ denote $\sigma^\frown(i)$ for $i \in \omega$. Then P is a Π^0_1 class if $P = [T]$ for some computable tree T. The string $\sigma \in T$ is a *dead end* if no extension $\tau^\frown i$ is in T. For any class P, T_P will denote the unique tree without dead ends so that $P = [T_P]$. P is said to be *decidable* if T_P is a computable tree. In general, even a Π^0_1 class $P \subseteq \{0,1\}^\omega$ does not necessarily contain a computable member, whereas a decidable Π^0_1 class must contain a computable member. P is said to be *special* if it does not contain a computable member.

Enumerations of Π^0_1 classes were given by Lempp [18] and Cenzer and Remmel [8,9], where index sets for various families of Π^0_1 classes were analyzed. For a given enumeration $\psi(e) = P_e$ of the Π^0_1 classes and a property R of sets, $\{e : R(P_e)\}$ is said to be an *index set*. For example, $\{e : P_e \text{ has a computable member}\}$ is a Σ^0_3 complete set. See [7] for many more results on index sets.

There are several equivalent definitions of Π^0_1 classes; in particular P is a Π^0_1 class if and only if $P = [T]$ for some primitive recursive tree T and also if and only if $P = [T]$ for some Π^0_1 tree T. Numberings based on primitive recursive trees and on Π^0_1 trees are both studied in the literature (see [8,9, 7,10]).

Certain types of Π^0_1 classes are of particular interest. Let P be a Π^0_1 class. We will say that P is *thin* if for every Π^0_1 subclass Q of P, there is clopen set U such that $Q = U \cap P$. We say that P is *homogenous* if, given distinct $\sigma, \tau \in T_P$ of the same length,

$$\sigma^\frown i \in T_P \iff \tau^\frown i \in T_P.$$

For $P \subseteq \{0,1\}^\omega$, P is homogeneous if and only if P is the class of separating sets $S(A,B)$ for two disjoint c.e. sets A, B, that is,

$$S(A,B) = \{C \subset \omega : A \subseteq C \text{ and } B \cap C = \emptyset\}.$$

P is *small* if there is no computable function ϕ such that, for all n, $card(T_P \cap \omega^{\phi(n)}) \geq n$. Let $\psi_P(n)$ be the least k such that $card(T_P \cap \omega^k) \geq n$; then P is *very small* if the function ψ_P dominates every computable function g – that is, $\psi_P(n) \geq g(n)$ for all but finitely many n.

In Section 2, several commonly used numberings of Π^0_1 classes are shown to be equivalent via a computable permutation. In Section 3, we give a Friedberg numbering of the Π^0_1 classes; this motivates a study of a general class of families of Π^0_1 classes, called *string verifiable families* in Section 4. In Sections 5-6, numberings for decidable, homogeneous, and thin classes are considered. Finally, in Section 7, motivated by some work by Binns [1], numberings for small classes are considered.

The partial computable $\{0,1\}$–valued functions are indexed as $\{\phi_e\}_{e\in\omega}$ and primitive recursive functions as $\{\pi_e\}_{e\in\omega}$. Partial computable functionals take natural numbers (m) and reals (x) as inputs and are indexed as Φ_e; we will write $\Phi^x_e(m)$ for the result of applying Φ_e to m and x. We generally follow the notation of Soare [24] for these functions. For example, $\phi_{e,s}$ denotes that portion ϕ_e defined by stage s, and $\phi_e(x)\downarrow$ means that ϕ_e is defined on x (and $\uparrow$ means undefined). Let $\langle \bullet, \bullet\rangle : \omega^2 \to \omega$ be a computable bijection such that $\langle 0, 0\rangle = 0$. $\overline{A}$ and $\mathcal{P}(A)$ denote the complement and power set of A, respectively.

We generally follow the notation of Cenzer [6] for Π^0_1 classes. In particular, for any $\sigma \in \{0,1\}^*$, $I(\sigma)$ is the *interval* of all infinite sequences extending σ. Now a c.e. open set is defined to be the complement of a Π^0_1 class. That is, if $P = [T]$, then $\omega^\omega - P = \bigcup_{\langle\sigma\rangle\notin T} I(\sigma)$. Thus for any c.e. set W, we define the c.e. open set generated by W to be

$$\mathcal{O}(W) = \bigcup\{I(\sigma) : \langle\sigma\rangle \in W\}.$$

Also let

$$\mathcal{O}(W)\restriction n = \{x\restriction n : x \in \mathcal{O}(W) \text{ and } (\forall j < n)\ x(j) \le n\}.$$

A tree $T \subseteq 2^{<\omega}$ and set $[T]$ are *clopen* if there is a nonempty finite $S \subseteq \omega^{<\omega}$ so that $T = \emptyset$ or $T = \{\sigma : \sigma \sqsubseteq \tau \text{ or } \tau \sqsubseteq \sigma \text{ for some } \tau \in S\}$.

A numbering $e \mapsto [T_e]$ of Π^0_1 classes is called a *tree* numbering and written $e \mapsto T_e$. If the set $\{(e,\sigma) : \sigma \in T_e\}$ is computable, then the numbering $\psi(e) = [T_e]$ is said to be a *computable* numbering.

2 Equivalent Numberings

In this section, we present several different numberings of Π^0_1 classes and show that certain of them are mutually equivalent via a computable permutation.

Numbering 1: Primitive Recursive Functions [8]

For each e, let π_e be the eth primitive recursive function from ω to ω and let

$$\sigma \in U_e \iff (\forall \tau \sqsubseteq \sigma)\ \pi_e(\langle\tau\rangle) = 1.$$

Then U_e is a (uniformly) primitive recursive tree for all e and if $\{\sigma : \pi(\langle\sigma\rangle) = 1\}$ is any primitive recursive tree, then U_e is that tree. Therefore the sequence $U_0, U_1, \ldots$ contains all primitive recursive trees and hence the mapping $\psi_1(e) = [U_e]$ is a computable numbering of the Π^0_1 classes.

Numbering 2: Computably Enumerable Sets [7]

Let

$$\psi_2(e) = \omega^\omega - \mathcal{O}(W_e).$$

This is an effective numbering since the relation "$x \in \psi_2(e)$" is Π^0_1 . This can actually be improved to a computable numbering, as follows.

For each e, recall that $W_{e,s}$ is the set of elements enumerated into the eth c.e. set W_e by stage s and let

$$\sigma \in S_e \iff (\forall \tau \sqsubseteq \sigma)\ \langle \tau \rangle \notin W_{e,|\sigma|}.$$

Then S_e is a (uniformly) primitive recursive tree for all e. Let $P = [T]$ be a Π^0_1 class, where T is a computable tree. It follows that for some e,

$$\sigma \in T \iff \langle \sigma \rangle \notin W_e.$$

Then $P = [S_e]$. It follows that the sequence $[S_0], [S_1], \dots$ contains all Π^0_1 classes and hence the mapping $\psi(e) = [S_e]$ is a computable numbering of the Π^0_1 classes. It is easy to see that in fact $[S_e] = \psi_2(e)$.

Numbering 3: Universal Π^0_1 Relation [16, p.73]

There is a universal Π^0_1 relation $U \subseteq \omega \times 2^\omega$ such that if $Q(x)$ is a Π^0_1 class, then there is an $e \in \omega$ such that $Q = \{x : U(e,x)\}$. U is defined in terms of the Kleene T-predicate, so that essentially

$$U(e,x) \iff \Phi^x_e(0) \uparrow .$$

(That is, the universal relation in [16] is given by $\Phi^x_e(\langle m_1, \dots, m_k \rangle) \uparrow$ with number variables $m_1, \dots, m_k$ and when $k = 0$ we have $\langle\rangle = 0$.

Define $\psi_3(e) = \{x : U(e,x)\}$ to obtain an effective numbering.

To see that this is a computable numbering, let

$$\sigma \in R_e \iff \Phi^\sigma_e(0) \uparrow .$$

so that $\psi_3(e) = [R_e]$ and the trees R_e are uniformly primitive recursive.

Numbering 4: The Halting Problem [17]

Consider the mapping given by

$$\psi_4(e) = \{x : \Phi^x_e(e)(e)\uparrow\}.$$

This is a computable numbering, since $\psi_4(e) = [T_e]$, where

$$\sigma \in T_e \iff \Phi^\sigma_e(e) \uparrow .$$

For any computable tree T, choose a so that $\Phi^\sigma_a(n)$ converges if and only if $\sigma \in T$. Then

$$\sigma \in T \iff \Phi^\sigma_a(a) \downarrow,$$

so that $[T] = \psi_4(a)$.

Numbering 5: Total Computable Functions

Here we will consider an effective, but not computable numbering ψ based on computable trees. This numbering will be used in connection with decidable classes in Section 4.

Let $\psi_5(e) = [T_e]$, where

$$\sigma \in T_e \iff (\forall \tau \preceq \sigma)[\phi_e(\langle \tau \rangle)\downarrow \longrightarrow \phi_e(\langle \tau \rangle) = 1].$$

This enumeration is uniformly Π^0_1, but is not computable, since the relation $\phi_e(m)\downarrow$ is c.e. non-computable. Clearly each $\psi_5(e)$ is a Π^0_1 class. If ϕ_e is total and T is a tree such that, for all σ, we have $\sigma \in T \iff \phi_e(\langle \sigma \rangle) = 1$, then $T_e = T$ and is a Π^0_1 class. Hence this enumeration has the crucial property that, for *every* computable tree T, there exists e such that $T = T_e$.

Proposition 2.1 *(a) For each pair i, j with $1 \leq i \leq 5$ and $1 \leq j \leq 4$, there is a computable function f such that $\psi_j = \psi_i \circ f$.*
(b) For each $j \leq 5$, there is a Δ^0_3 function f such that $\psi_5 = \psi_i \circ f$.

Proof ($\psi_1 \leq \psi_2$): Use the $S-m-n$ Theorem to define f so that

$$W_{f(e)} = \{n : \pi_e(n) \neq 1\}.$$

Then $\sigma \in U_e \iff \sigma \in S_{f(e)}$.

($\psi_2 \leq \psi_3$): Define f so that, for all m,

$$\Phi^x_{f(e)}(m) = (\text{least } n)\langle x \restriction n \rangle \in W_e).$$

Then $\psi_2(e) = \psi_3(f(e))$.

($\psi_3 \leq \psi_4$): Define f so that $\Phi^x_{f(a)}(n) = \Phi^x(0)$ for all n. Then $x \in \psi_3(e) \iff x \in \psi_4(f(e))$.

($\psi_4 \leq \psi_1$): Recall that $\psi_4(e) = [\{\sigma : \Phi^\sigma_e(e)\uparrow\}]$.

Define the primitive recursive function g so that for each e,

$$\pi_{g(e)}(\langle \sigma \rangle) = \begin{cases} 1, & \text{if } \Phi^\sigma_e(e)\uparrow; \\ 0, & \text{otherwise.} \end{cases}$$

Then $\psi_1(e) = \psi_4(g(e))$.

($\psi_1 \leq \psi_5$): Define the primitive recursive function f such that, for each e,

$$\phi_{f(e)}(\langle \sigma \rangle) = \begin{cases} 1, & \text{if } (\forall \tau \sqsubseteq \sigma)\pi_e(\langle \tau \rangle) = 1, \\ 0, & \text{otherwise.} \end{cases}$$

Then $\psi_1(e) = \psi_5(f(e))$.

The rest of the proof follows by composition.

Theorem 2.2 *For any computable numbering φ which is computably equivalent to ψ_2, there is a computable permutation p such that $\psi_2 = \varphi \circ p$.*

Proof The proof is a modification of an argument due to Jockusch [24, p. 25]. Let $\psi = \psi_2$. By assumption, there are computable functions f and g such that $\psi(f(e)) = \varphi(e)$ and $\varphi(g(e)) = \psi(e)$. Since the numbering ψ_2 is based on an enumeration of the partial computable functions, we can ensure by padding that f is injective. To modify g into an injective function g_1, it is sufficient to effectively compute from each e an infinite set S_e of indices such that $\varphi(g(i)) = \varphi(e)$ for all $i \in S_e$. We proceed as follows. Let A and B be computably inseparable c.e. sets and define computable functions k and ℓ such that, for all e and m:

$$\psi(k(e,m)) = \begin{cases} \psi(e), & \text{if } m \notin B, \\ \emptyset, & \text{if } m \in B \end{cases}$$

and

$$\psi(\ell(e,m)) = \begin{cases} \psi(e), & \text{if } m \in A, \\ 2^\omega, & \text{if } m \notin A \end{cases}$$

That is, we build a tree for $\psi(k(e,m))$ which exactly equals the tree for $\psi(e)$ for strings of length s until $m \in B_{s+1}$, in which case no strings of length $s+1$ are put into $\psi(k(e,m))$. To build the tree for $\psi(\ell(e,m))$, we put in all strings of length s until $m \in A_{s+1}$, in which case we include only the strings of length $s+1$ which are in $\psi(e)$.

Now let $C_e = \{k(e,m) : m \in A\}$ and $D_e = \{\ell(e,m) : m \in A\}$ and let $S_e = g(C_e \cup D_e)$. Then for $j = g(i) \in S_e$, it follows from the definition that $\psi(i) = \psi(e)$ and therefore $\varphi(g(i)) = \varphi(g(e))$. We will prove in two cases that either $g(C_e)$ is infinite or $g(D_e)$ is infinite.

Case I: Suppose that $\psi(e) \neq \emptyset$ and suppose by way of contradiction that $g(C_e)$ is finite. Then $S = \{m : g(k(e,m)) \in g(C_e)\}$ is a computable set. Now $A \subseteq g(C_e)$ by definition. On the other hand, if $j = g(k(e,m)) \in g(C_e)$ where $m \in A$, then $\varphi(j) = \psi(k(e,m)) = \phi(e) \neq \emptyset$. But for $m \in B$, $\varphi(g(k(e,m)) = \psi(k(e,m)) = \emptyset$, so that $S \cap B = \emptyset$. This contradicts the assumption that A and B are computably inseparable.

Case II: Suppose that $\psi(e) = \emptyset$. It follows as in Case I that $g(D_e)$ is infinite.

Thus we may assume without loss of generality that both f and g are one-to-one. Now define a sequence $\{e_n : n \in \omega\}$ and two partitions of ω as follows. Let $e_0 = 0$ and for each n, e_{n+1} is the least e such that $\varphi(e) \neq \varphi(e_i)$ for every $i \leq n$. Let $A_n = \{e : \psi(e) = \psi(e_n)\}$ and $B_n = \{e : \varphi(e) = \psi(e_n)\}$. Then $\omega = \bigcup_n A_n = \bigcup_n B_n$ and each sequence is pairwise disjoint. Furthermore, $f(B_n) \subseteq A_n$ and $g(A_n) \subseteq B_n$. The remainder of the proof follows as in the Myhill Isomorphism Theorem [24, p. 24].

A similar argument shows that if φ is a Δ^0_3 numbering of the Π^0_1 classes, then there is a Δ^0_3 permutation p with $\varphi = \psi_2 \circ p$. It follows that each of the computable numberings $\psi_1, \ldots, \psi_4$ are acceptable, that is, they occur in the greatest element of the semilattice $\mathcal{L}(\mathcal{P})$. In the next section, we will see that minimal elements exist in the semilattice– that is, injective numberings.

3 Injective Numberings

In this section, we construct a computable injective numbering of the Π^0_1 classes in 2^ω, by modifying Friedberg's original presentation. An alternative proof was sketched by Raichev [20].

Theorem 3.1 *There is a 1-1 computable numbering of all Π^0_1 classes in 2^ω.*

Proof The proof is a modification of the Friedburg construction [14] of an injective numbering for the c.e. sets of natural numbers.

Let $\{W_e\}_{e\in\omega}$ be a computable numbering of the c.e. subsets of $2^{<\omega}$. We will construct a computable numbering $\{Y_e : e \in \omega\}$ in stages $Y_{e,s}$ of a family of c.e. subsets of $2^{<\omega}$ so that $\{\mathcal{O}(Y_e)\}_{e\in\omega}$ is an injective numbering of the Σ^0_1 classes. It is important to note that an injective numbering of the c.e. subsets of $2^{<\omega}$ will not automatically yield an injective numbering of the Σ^0_1 classes, since each Σ^0_1 class will equal $\mathcal{O}(W)$ for many different c.e. sets W. However, if $\mathcal{O}(V) \neq \mathcal{O}(W)$ for two c.e. sets V and W, then there must be some interval $I(\sigma)$ which is included in, say $\mathcal{O}(V)$ but not included in $\mathcal{O}(W)$ and hence some stage s such that $\mathcal{O}(V_s)\restriction s \neq \mathcal{O}(W_s)\restriction s$ at stage s and at any later stage.

In the construction, we will use the notion of one Y-index i *following* a W-index e with the idea that in the end Y_i will equal W_e. At some point, however, we may decide that i will no longer follow e and we will say that i is *released.* If i is never released from e, then it is said to be a *loyal* follower and otherwise it is *disloyal.* Once released, an index remains free and is never again the follower of any e. At any particular stage, a Y-index that is not following any W-index is said to be *free.* A nonzero Y-index that has never followed any W-index is said to be *unused.*

To ensure that no c.e. set is excluded from the Y-sets, we will ensure that each W_e is infinitely often given the opportunity to be followed. To do this, at stage $s = \langle n, e_s\rangle$ all actions in the construction will be taken with respect to W_{e_s}. At each stage s, we initiate at most one new Y_i, so that after stage s, we have sets $Y_0, Y_1, \ldots, Y_{k_s}$ for some $k_s \leq s$. Let $Y_0 = \{\emptyset\}$, so that $\mathcal{O}(Y_0) = 2^\omega$.

Construction: There are three possible cases at each stage s.

Case 1: If i follows e_s and there exists $e < e_s$ such that $\mathcal{O}(W_{e,s})\restriction(i-1) = \mathcal{O}(W_{e_s,s})\restriction(i-1)$, then release i and go on to stage $s+1$.

Case 2: Suppose that Case 1 does not occur. If $\mathcal{O}(W_{e_s,s}) = \mathcal{O}(Y_{i,s-1})$, and either i follows some $e < e_s$, or i is free and either $i \leq e_s$ or i was previously *displaced* (see Case 3) by e_s and released, then go on to stage $s+1$ without taking any action.

Case 3: Suppose that Cases 1 and 2 do not occur. Now we ensure e_s has a follower. If it does not, choose the least unused $i \neq 0, 1$ to follow e_s. Now let $Y_{i,s} = W_{e_s,s}$.

If $Y_{j,s-1}$ for some $j \neq i$ satisfies $\mathcal{O}(Y_{j,s-1}) = \mathcal{O}(W_{e_s,s})$, then put some $\sigma_j \in 2^{<\omega}$, defined in what follows, into Y_j so that $\mathcal{O}(Y_{j,s}) \neq \mathcal{O}(W_{e_s,s})$.

Let $E_s = \{j \in \omega : j \neq i \ \&\ \mathcal{O}(Y_{j,s-1}) = \mathcal{O}(W_{e_s,s})\}$ be the set of indices of *equivalent* Y-open sets and suppose that $E_s = \{\epsilon_1 < \epsilon_2 < \ldots < \epsilon_{|E_s|}\}$. Now define $\mathrm{Str}(k,s) = \{\sigma \in 2^k : \sigma \notin \mathcal{O}(W_{e_s,s}) \restriction k\}$. Let $\ell(s)$ be the least k such that $|\mathrm{Str}(k,s)| > |E_s|$. Then $\ell(s)$ is the least level of $\mathcal{O}(W_{e_s,s})$ where there is enough room to give each equivalent $Y_{j,s-1}$ an additional string to distinguish $\mathcal{O}(Y_{j,s-1})$ from $\mathcal{O}(W_{e_s,s})$. (Notice that $\mathcal{O}(W_{e_s,s}) \neq 2^\omega$ by Case 2.) Suppose that $\mathrm{Str}(\ell(s),s) = \{\sigma_1 \prec \sigma_2 \prec \ldots \prec \sigma_{|\mathrm{Str}(\ell(s),s)|}\}$. Now put σ_j into Y_{ϵ_j} and release ϵ_j if it is a follower. We say that ϵ_j is *released* or *displaced* at stage s.

Verification: Given $e \in \omega$, let $\hat{e}$ be the least k such that $[\mathcal{O}(W_k) = \mathcal{O}(W_e)]$. We will show:

(i) $(\forall e)(\exists i)\ Y_i = W_{\hat{e}}$;
(ii) $i \neq j$ implies that $\mathcal{O}(Y_i)$ and $\mathcal{O}(Y_j)$ are not equal when both are clopen.
(iii) $i \neq j$ implies that $\mathcal{O}(Y_i)$ and $\mathcal{O}(Y_j)$ are not equal when both are *not* clopen.

Verification of (i). Fix e. First note that although $\hat{e}$ can have different followers at different stages, it cannot have an infinite number of disloyal followers. That is, if s and x are sufficiently large, then by the definition of $\hat{e}$, for all $j < \hat{e}$, $\mathcal{O}(W_{j,s}) \restriction x \neq \mathcal{O}(W_{\hat{e},s}) \restriction x$. Hence release can only occur in Case 1 a finite number of times. Furthermore, Case 2 ensures that release in Case 3 can only occur for any s when $e_s < \hat{e}$. Therefore, by the above, if $i > x$ is follower of $\hat{e}$ and $t > s$, then $\mathcal{O}(Y_{i,t-1}) = \mathcal{O}(W_{\hat{e},t-1}) \neq \mathcal{O}(W_{e_s,t})$. Hence i will not be released in Case 3. Therefore release can only occur in Case 3 a finite number of times.

Now let s be a stage after which $\hat{e}$ never loses a follower. If Case 3 occurs infinitely often after stage s for $\hat{e}$, then it has a permanent follower i so that $\mathcal{O}(Y_i) = \mathcal{O}(W_{\hat{e}})$. Therefore assume Case 3 occurs only finitely often. Since $\hat{e}$ never loses a follower, Case 1 cannot occur. Thus Case 2 must occur infinitely often. However there are only a finite number of i such that the hypothesis of Case 2 holds with $\mathcal{O}(W_{\hat{e},s}) = \mathcal{O}(Y_{i,s-1})$. To see this, consider the three sub-cases. First, each $e < e_s$ has only finitely many followers by the argument above; second, there are only finitely many $i \leq e_s$; and third, only a finite number of i are displaced by e_s, due to Case 3 occuring only a finite number of times. This contradiction shows that $\hat{e}$ has a permanent follower, as desired.

Verification of (ii). Suppose that $U = \mathcal{O}(Y_i) = \mathcal{O}(Y_j)$ is clopen and let $U = \mathcal{O}(W_{\hat{e}})$. It follows from compactness, that there is some finite s such that, for all $t \geq s$, $\mathcal{O}(Y_{i,t}) \restriction t = \mathcal{O}(Y_{j,t}) \restriction t = \mathcal{O}(Y_i)$. It follows from the verification of (i) above that there is a stage $t > s$ such that Case 3 applies to $\hat{e}$. But then at least one of $\mathcal{O}(Y_i), \mathcal{O}(Y_j)$ must change at stage t. This contradiction verifies (ii).

Verification of (iii). Assume both Y_i and Y_j are not clopen and $i \neq j$. It follows that $\mathcal{O}(Y_i)$ must change infinitely often, since of course $\mathcal{O}(Y_{i,s})$ is clopen for each s, and similarly for $\mathcal{O}(Y_j)$. Now i must eventually follow some W-index. If i is ever released, then it is free. Thereafter Y_i acquires members in Case 3 at stage s only when $W_{e_s,s} = Y_{i,s-1}$. This implies that Case 2 does not apply at stage s and thus $e_s < i$. But each $e < i$ can only displace i once, again by the hypothesis of Case 2. Thus if i is a disloyal follower, then in fact $\mathcal{O}(Y_i)$ is clopen. Thus we may assume that i is a loyal follower of e and j is a loyal follower of e'. Then $\mathcal{O}(W_e) = \mathcal{O}(W_{e'})$ but $e \neq e'$, since each e can have at most one loyal follower. Without loss of generality suppose $e < e'$.

Since $\mathcal{O}(W_e) = \mathcal{O}(W_{e'})$, there will be a stage s large enough so that $\mathcal{O}(W_e) \upharpoonright (i-1) = \mathcal{O}(W_{e'}) \upharpoonright (i-1)$. Then since i follows $e < e'$, i will be released in Case 1 at stage s, contradicting the assumption that i is a loyal follower.

This verification completes the proof.

The following generalization of Theorem 3.1 will be useful. Let $\mathcal{C}$ be the family of clopen subsets of 2^ω.

Theorem 3.2 *For any family $\mathcal{F}$ of Π^0_1 classes in 2^ω which has a computable numbering, there is a 1-1 computable numbering of $\mathcal{C} \cup \mathcal{F}$.*

Proof Let the computable enumeration P_e be given. We may assume that $\mathcal{C} \subseteq \mathcal{F}$ by simply enumerating the clopen sets as $\{Q_{2e} : e < \omega\}$ and letting $Q_{2e+1} = P_e$. Then the proof of Theorem 3.1 produces a 1-1 computable enumeration of $\mathcal{C} \cup \mathcal{F}$ as desired.

The problem of finding an injective enumeration of the Π^0_1 classes in ω^ω remains.

Suppose we modify each c.e. set in the standard numbering to enumerate an element only as long as it is larger than any previously enumerated element. Applying Friedberg's argument to this class of c.e. sets yields an effective injective numbering $e \mapsto C_e$ of the computable sets [26]. Furthermore each C_e still enumerates its elements in increasing order.

Now suppose $\{\chi_e\}_{e\in\omega}$ is a corresponding set of characteristic functions. One characterization of a Π^0_1 class P is that $P = \omega^\omega \setminus \mathcal{O}(W)$ for some computable set W [7]. As a result, $e \mapsto \omega^\omega \setminus \mathcal{O}(C_e) = \omega^\omega \setminus \mathcal{O}(\{n : \chi_e(n) = 1\})$ is an alternative *effective* numbering based on total computable functions (replacing noneffective Numbering 2).

It is known, for fixed $n > 0$, that there is a effective injective numbering of the n-c.e. sets [15]. We conjecture that, for each n, there is a numbering $e \mapsto N_e$ of n-c.e. sets such that there is an injective computable numbering $e \mapsto \omega^\omega \setminus \mathcal{O}(N_e)$ of all closed sets of this form. For $n = 1$ the result is given by Theorem 3.1.

We next show that Theorem 3.1 is not obtainable by any computable procedure that uniformly selects the minimal index of every Π^0_1 class.

Theorem 3.3 *There is no computable choice function for indices of Π^0_1 classes. (i.e. a computable function f such that $f(e)$ is an index of P_e and $P_i = P_e \Rightarrow f(i) = f(e)$)*

Proof Suppose that f exists. Let $a_0, a_1, \ldots$ be an enumeration of a noncomputable c.e. set A. Define a computable function g and trees $T_{g(e)}$ so that if $|\sigma| = n$, then

$$\sigma \in T_{g(e)} \leftrightarrow e \notin \{a_0, \ldots, a_n\}.$$

Then

$$P_{g(e)} = \begin{cases} \emptyset & \text{if } e \in A \\ 2^\omega & \text{otherwise} \end{cases}$$

For any $a \in A, e \in A \leftrightarrow f(g(e)) = f(g(a))$, making A computable.

It is still possible, however, that some interesting proper family of Π^0_1 classes may be enumerated by selecting minimal indices from the enumeration of all Π^0_1 classes.

4 String Verifiable Families of Classes

In this section, we examine a family of classes which can be computably enumerated. Let $F_0, F_1, \ldots$ be a computable enumeration of the finite subsets of $2^{<\omega}$, that is, for any n, $\sigma \in F_n \iff b_n(\langle\sigma\rangle) = 1$, where b_n is the binary expression for the natural number n. Let E denote the family of finite sequences of positive integers of even length. Let $P_0 = [T_0], P_1 = [T_1], \ldots$ be some computable enumeration of the Π^0_1 classes in 2^ω.

Definition 4.1 A **string function** is a computable function $f : 2^{<\omega} \to E$.

A family $\mathcal{H}$ of trees (or, more generally, of subsets of $2^{<\omega}$) is **string verifiable** if there is a string function $h : 2^{<\omega} \to E$ so that for all T, $T \in \mathcal{H}$ if and only if the following condition is satisfied for all $\sigma \in T$, where $h(\sigma) = (m_1, m_2, \ldots, m_{2n})$ and $D_i = F_{m_i}$ for $i = 1, \ldots, n$:

There exists $i < n$ such that $D_{2i+1} \subseteq T$ and $T \cap D_{2i+2} = \emptyset$ – that is, $[T] \in S(D_{2i+1}, D_{2i+2})$ (the family of separating sets of D_{2i+1} and D_{2i+2}).

Note that the family of trees itself is string verifiable among the family of all subsets of $2^{<\omega}$, via the function $h(\sigma) = (a, b)$, where $F_a = \{\tau : \tau \sqsubset \sigma\}$ and $F_b = \emptyset$.

Example 4.2 (a) *The Homogeneous Trees.* A tree T is said to be *homogeneous* if

$$\forall\sigma, \tau \in T)[|\sigma| = |\tau| \Rightarrow (\forall i)(\sigma^\frown i \in T \iff \tau^\frown i \in T)].$$

Define the string verification function h as follows. Let A_1, A_2, A_3, A_4 enumerate $\mathcal{P}(\{0,1\})$ and let $B_1, \ldots, B_{2^{|\sigma|}}$ enumerate the strings of length σ. Let $h(\sigma)$ enumerate in order the set of $m_{2\langle j,k\rangle+1}$ and $m_{2\langle j,k\rangle+2}$ for $1 \le j \le 4$ and $1 \le k \le 2^{|\sigma|}$, where

$$F_{m_{2\langle j,k\rangle+1}} = \{\tau^\frown i : i \in A_j, \tau \in B_k\}$$

and

$$F_{m_{2\langle j,k\rangle+2}} = \{\tau : \tau \notin B_k\} \cup \{\tau^\frown i : i \notin A_j, \tau \in B_k\}.$$

That is, $h(\sigma)$ verifies that T is homogeneous by selecting the unique set $B_k = \{|\tau| = |\sigma| \ \&\ \tau \in T\}$ and the unique set A_j such that for $\tau \in B_k$, $\tau^\frown i \in T_e \iff i \in A_j$.

(b) *The Extendible Trees.* Recall that a closed set P is *decidable* if the $P = [T]$ for some computable tree T without dead ends. For the purpose of string verification, let us say that a tree T is *extendible* if T has no dead ends. This means that that for any $\sigma \in T$, either $\sigma^\frown 0 \in T$ or $\sigma^\frown 1 \in T$. In general, $Ext(T) = \{\sigma : I(\sigma) \cap [T] \neq \emptyset\}$ is the set of *extendible* nodes of T and T is extendible if and only if $T = Ext(T)$. Thus we let $h(\sigma) = (m_1, m_2, m_3, m_4)$, such that $F_{m_1} = \{\sigma^\frown 0\}$, $F_{m_2} = F_{m_4} = \emptyset$ and $F_{m_3} = \{\sigma^\frown 1\}$. That is, h verifies that T has no dead ends by either showing that $\sigma^\frown 0 \in T$ if $F_{m_1} \subset T$ or that $\sigma^\frown 1 \in T$ if $F_{m_3} \subset T$.

Definition 4.3 (a) A Π^0_1 class P **satisfies a finite set of relations** $\mathcal{H}_i \subset \mathcal{P}(2^{<\omega})$ $(i \leq n)$ if there is a computable tree T such that $P = [T]$ and $\mathcal{H}_i(T)$ for each $i \leq n$.
(b) A Π^0_1 class P **strongly satisfies a finite set of relations** $\mathcal{H}_i \subset \mathcal{P}(2^{<\omega})$ $(i \leq n)$ if there is a primitive recursive tree T such that $P = [T]$ and $\mathcal{H}_i(T)$ for each $i \leq n$).

Definition 4.4 A family $\mathcal{F}$ of classes is **[strongly] string verifiable (s.v.)** if there is some finite set of string verifiable relations so that: $P \in \mathcal{F}$ if P [strongly] satisfies these relations.

Note that any string verifiable family of trees contains the empty tree, so that any string verifiable family of Π^0_1 classes contains the empty class. If $P = \emptyset$, then $P = [T]$ if and only if T is finite, so any tree T with $P = [T]$ is primitive recursive. So any strongly string verifiable family also contains the empty class.

Theorem 4.5 *(a) Any strongly string-verifiable family of Π^0_1 classes has a computable numbering.*
(b) Any string-verifiable family of Π^0_1 classes has an effective numbering.

Proof Suppose $\mathcal{F}$ is a [strongly] string-verifiable family of Π^0_1 classes satisfying string verifiable (tree) relations $\mathcal{H}_0, \mathcal{H}_1, ..., \mathcal{H}_m$, with corresponding string functions $h_0, h_1, ..., h_m$. For part (a), let the standard computable enumeration of the Π^0_1 classes in 2^ω be given by $P_e = [T_e]$, where the sequence T_e is uniformly primitive recursive (for example, the numbering ψ_2 given in Section 2). We will define a uniformly computable sequence S_e of trees such that the sequence $Q_e = [S_e]$ enumerates exactly the family of Π^0_1 classes strongly satisfying $\mathcal{H}_0, \mathcal{H}_1, ..., \mathcal{H}_m$.

For any $\sigma \in \{0,1\}^n$, we determine whether $\sigma \in S_e$ as follows. First check that $\sigma \in T_e$. If so, for each $\tau \in \{0,1\}^n$ and each $i \leq m$, compute $h_i(\tau) = (D_1, D_2, \ldots, D_{2j})$ and determine whether there exists $i < j$ such that $D_{2j+1} \subseteq T_e$ and $D_{2j+2} \cap T_e = \emptyset$. This process is computable since each D_ℓ is a canonical finite set. If the answer is yes, for every $\tau \in \{0,1\}^n$, then $\sigma \in S_e$ and otherwise, $\sigma \notin S_e$. It is clear that if T_e satisfies all of the relations $\mathcal{H}_0, \mathcal{H}_1, ..., \mathcal{H}_m$, then $T_e = S_e$. It follows that every Π^0_1 class in $\mathcal{F}$ occurs in the enumeration $Q_e = [S_e]$. On the other hand, if T_e fails any of the relations, then S_e is a finite set and $Q_e = \emptyset$. By assumption, $Q_e \in \mathcal{F}$ in this case as well, so that the sequence $\{Q_e : e < \omega\}$ enumerates exactly the family $\mathcal{F}$, as desired.

For part (b), let $P_e = \psi_5(e) = T_e$, the uniformly Π^0_1 enumeration which has the property that every computable tree occurs in the list $\{T_e : e < \omega\}$. We need to do the string verification in a Π^0_1 fashion and in particular to check that $D_{2j+2} \cap T_e = \emptyset$, which appears to be Σ^0_1. However, we can simply check that, if $\rho \in D_{2j+2}$ and $\phi_e(\rho)\downarrow$, then $\phi_e(\rho) = 0$. Then the sequence S_e is uniformly Π^0_1 and, if T_e has characteristic function ϕ_e and satisfies the string relations, it follows that $S_e = T_e$.

Now let us say that a Π^0_1 class P is *strongly decidable* if there is a primitive recursive tree T with no dead ends such that $P = [T]$ and that P is *strongly homogeneous* if there is a homogeneous primitive recursive tree T with no dead ends such that $P = [T]$.

Corollary 4.6 *(a) The family of decidable Π^0_1 classes in 2^ω has an effective numbering and the family of strongly decidable Π^0_1 classes in 2^ω has a computable numbering.*

(b) The family of homogeneous Π^0_1 classes in 2^ω has an effective numbering and the family of strongly homogeneous Π^0_1 classes in 2^ω has a computable numbering.

We can improve Theorem 4.5 to obtain a computable numbering of any string-verifiable family which includes the clopen sets. The following is reminiscent of the result of Pour-El and Putnam [19] that any family of c.e. sets containing all finite sets possesses an injective numbering.

Theorem 4.7 *If $\mathcal{F}$ is any string-verifiable family of Π^0_1 classes, then there is a computable numbering of $\mathcal{C} \cup \mathcal{F}$.*

Proof We modify the proof of Theorem 4.5 so that when the string-verifiable relations fail, we extend all nodes rather than making them dead ends. Once again, the construction is based on the enumeration ϕ_e of the partial computable functions. The construction is in stages, where at stage s we will have

$$n_{e,s} = max\{n : (\forall \sigma \in \{0,1\}^n)\phi_{e,s}(\langle\sigma\rangle)\downarrow\},$$

$$J_{e,s} = \{\sigma \in \{0,1\}^{n_{e,s}} : \phi_{e,s}(\langle\sigma\rangle) = 1\},$$

and

$$Q_{e,s} = \bigcup J_{e,s} = [S_{e,s}].$$

Then $Q_e = \bigcap_s Q_{e,s} = [S_e]$ will be the desired numbering. To ensure that this numbering is computable, we will determine whether $\sigma \in S_e$ at stage $|\sigma|$.

For this argument, we assume that $\phi_e(0) = 1$ for all e.

Construction At stage 0 we have $n_{e,0} = 0$, $J_{e,0} = S_{e,0} = \{\emptyset\}$ and $Q_{e,0} = 2^\omega$.

At stage $s+1$, we check to see whether $\phi_{e,s+1}(\langle\sigma\rangle)\downarrow$ for all $\sigma \in \{0,1\}^{n_s+1}$. If not, then $n_{e,s+1} = n_{e,s}$, $J_{e,s+1} = J_s$ and $S_{e,s+1} = S_{e,s} \cup \{\sigma^\frown i : \sigma \in S_{e,s}, i = 0,1\}$. If so, then we check to see that $\phi_{e,s+1}$ is the characteristic function of a tree on $\{0,1\}^{n_{e,s}+1}$ and we verify the string relations up to $\{0,1\}^{n_{e,s}+1}$. If this verification fails, then again $n_{e,s+1} = n_{e,s}$ and $J_{e,s+1} = J_{e,s}$. In this case, verification will also fail at all future stages, so that $Q_e = Q_{e,s}$ is a clopen set.

If the tree and string-verifications succeed, then $n_{e,s+1} = n_{e,s} + 1$, so that $J_{e,s+1} \subseteq \{0,1\}^{n_{e,s}+1}$ and $Q_{e,s+1}$ change as indicated above. In this case,

$$S_{e,s+1} = S_{e,s} \cup \{\sigma \in \{0,1\}^{s+1} : \sigma \restriction (n_{e,s+1}) \in J_{e,s+1}\}.$$

If ϕ_e is the characteristic function of the computable tree T_e, and if $P_e = [T_e] \in \mathcal{F}$, then it follows from the construction that $Q_e = P_e$, so that $Q_e \in \mathcal{F}$ and furthermore, any Π^0_1 class $P_e \in \mathcal{F}$ will thereby occur in the numbering. Otherwise, the construction will make Q_e a clopen set.

Corollary 4.8 *For any string verifiable family $\mathcal{F}$ of Π^0_1 classes, there a 1-1 computable numbering of $\mathcal{C} \cup \mathcal{F}$.*

Proof Let $\mathcal{F}$ be a string verifiable family. Then there is a computable numbering of $\mathcal{C} \cup \mathcal{F}$ by Theorem 4.7. It then follows from Theorem 3.2 that there is a 1-1 computable numbering of $\mathcal{C} \cup \mathcal{F}$.

Corollary 4.9 *There a 1-1 computable numbering of any string verifiable family of Π^0_1 classes containing all clopen classes.*

This corollary applies to the family of decidable classes, which we shall return to in the next section. However, clopen sets are not necessarily homogeneous, so we need a different argument for the homogeneous classes.

Theorem 4.10 *There is a 1-1 computable numbering of the homogeneous Π^0_1 classes.*

Proof It is well-known [7] that a Π^0_1 class P is homogeneous if and only if $P = S(A,B)$ for two c.e. sets A, B. Friedberg's construction of a 1-1 enumeration W_e of the c.e. sets may be modified to obtain an effective 1-1 enumeration of all ordered pairs $\langle A_e, B_e \rangle$ of disjoint r.e. sets. (See [3] for details.) Now let $P_e = S(A_e, B_e)$ to obtain a 1-1 enumeration of the homogeneous Π^0_1 classes. Furthermore, $S(A_e, B_e) = [T_e]$, where

$$\sigma \in T_e \iff (\forall n < |\sigma|)[(n \in A_{e,s} \longrightarrow \sigma(n) = 1) \ \&\ (n \in B_{e,s} \longrightarrow \sigma(n) = 0)].$$

This shows that the numbering is computable.

5 Decidable Classes

Since decidability is string-verifiable and every clopen set is decidable, it follows from Corollary 4.9 that the decidable classes have a 1-1 computable numbering.

This result could not be obtained by using the standard numbering of the Π^0_1 classes and modifying each tree as it becomes known that is has a dead end. (For example, simply extend each such node with, say, all ones.) This is because, as a consequence to the following theorem, P being decidable is insufficient to ensure that the unique tree T_P without dead ends shows up in a computable tree numbering.

Theorem 5.1 *In any computable numbering of computable trees in* $2^{<\omega}$ *there is a computable tree without dead ends outside the image of the numbering.*

Proof Let $\{T_e : e < \omega\}$ be a uniformly computable sequence of trees. Now use a diagonalization argument to construct a tree T such that for all n, $T \cap \{0,1\}^{n+1} \neq T_n \cap \{0,1\}^{n+1}$, as follows. At stage 0 let $T \cap \{0,1\}^0 = \{\emptyset\}$. At stage $n+1$ we are given $T \cap \{0,1\}^n \neq \emptyset$. Therefore there are at least 2 subtrees of $\{0,1\}^{n+1}$ without dead ends extending $T \cap \{0,1\}^n$. Define $T \cap \{0,1\}^{n+1}$ to be an extension which is different from $T_n \cap \{0,1\}^{n+1}$.

Corollary 5.2 *For any computable numbering* $P_e = [T_e]$ *of the* Π^0_1 *classes in* 2^ω*, there is a decidable* Π^0_1 *class* P *such that* $P \neq [T_e]$ *for any* T_e *without dead ends.*

Proof Let $P = [T]$ where T is the computable tree without dead ends provided by Theorem 5.1. Suppose that $P = [T_e]$ for some e. Since T has no dead ends, it follows that $T = T_P$ and if T_e also had no dead ends, then $T_e = T_P = T$. But by the construction, $T \cap \{0,1\}^{e+1} \neq T_e \cap \{0,1\}^{e+1}$, so that $T \neq T_e$.

It follows from this corollary that in the standard numbering, $\{e : T_e$ has no dead ends$\} \neq \{e : P_e = [T_e]$ is decidable$\}$. In fact both have distinct complexities. By Konig's Lemma, $\mathrm{Ext}(P_e) = \{\sigma \in 2^{<\omega} : I(\sigma) \cap P_e \neq \emptyset\}$ is Π^0_1. So $\{e : T_e$ has no dead ends$\} = \{e : T_e = \mathrm{Ext}(P_e)\}$ is Π^0_1. However, $\{e : P_e$ is decidable$\} = \{e : P_e = [T]$ for some computable T without dead ends$\} = \{e : (\exists a)\ \phi_a$ is a characterstic function for $\mathrm{Ext}(P_e)\}$ is Σ^0_3. An alternate proof of Corollary 5.2 is as a corollary of the following.

Theorem 5.3 *For any acceptable numbering* ψ *of the* Π^0_1 *classes,* $\{e : \psi(e)$ *is decidable*$\}$ *is* Σ^0_3 *complete.*

Proof It suffices to prove this for the standard numbering (ψ_2). We will make use of the well-known [24] Σ^0_3 completeness of $\{e : W_e$ is computable$\}$. It is easy to see that $\{e : \psi_2(e)$ is decidable$\}$ is Σ^0_3. For the completeness, define the uniformly computable trees $T_{f(e)}$ so that

(i) $0^n \in T_{f(e)}$ for all n;
(ii) $0^n 1^s \in T_{f(e)} \iff n \notin W_{e,s}$.

It follows that $0^n 1 \in Ext(T_{f(e)}) \iff n \notin W_e$, so that if $\psi_{f(e)}$ is decidable, then W_e is computable. On the other hand, $Ext(T_{f(e)}) = \{0^n : n \in \omega\} \cup \{0^n 1^s : s \in \omega, n \notin W_e\}$, so that if W_e is computable, then $\psi(f(e))$ is decidable. Thus W_e is computable if and only if $\psi(f(e))$ is decidable.

Note that in [8], a Π^0_1 class $P_e = [T_e]$ in the standard numbering is said to be decidable if T_e has no dead ends, which we now see is probably not the right approach.

6 Thin Classes

In the literature, a Martin-Pour El theory is a consistent c.e. propositional theory with additional 'thinness' conditions. The conditions imposed have varied depending upon the context and motivation of the authors, but include: (1) few c.e. extensions, (2) essentially undecidable, and (3) well-generated. Some authors have chosen to only impose (1) [5], while others (1) and (2) [6], [9], and finally others (1), (2), and (3) [12], [11], [4]. The complete consistent extensions of these theories correspond to thin, perfect thin (or equivalently, special thin [6]), and homogenous thin classes, respectively. This section is devoted towards demonstrating the nonexistence of computable numberings of the first two cases by modifying the classical Martin-Pour El construction of a perfect thin class. Recently Solomon [25] also modified this theorem to construct a homogeneous thin class and therefore we conjecture that no computable numberings exist for these classes.

A perfect class may be defined by a function $g : 2^{<\omega} \to 2^{<\omega}$ such that for all σ, τ, $\sigma \sqsubset \tau$ implies $g(\sigma) \sqsubset \tau$; let us say that g is *extension preserving.* Let $G(x) = \bigcup_n g(x \upharpoonright n)$. Then $G(2^\omega)$ is a perfect class. If g is defined in uniformly computable, extension-preserving stages g_s (with corresponding $G : 2^\omega \to 2^\omega$), so that $g_s(\sigma) \sqsubseteq g_{s+1}(\sigma)$, then we have $G(2^\omega) = \cap_e G_e(2^\omega)$, so that $G(2^\omega)$ is a Π^0_1 class.

Theorem 6.1 (Martin–Pour-El) *For any computable extension-preserving function $g : 2^{<\omega} \to 2^{<\omega}$, there exists a perfect thin Π^0_1 class $P \subseteq G(2^\omega)$.*

Proof Let $\{P_e = [T_e] : e \in \omega\}$ be the standard numbering of the Π^0_1 classes and $\{\phi_e : e \in \omega\}$ be the standard numbering of the $\{0,1\}$-valued partial computable functions. We will construct a computable tree S, corresponding Π^0_1 class $P = [S]$, and a surjective homeomorphism $F : 2^\omega \to P$. F will be constructed by means of an extension-preserving map $f : 2^{<\omega} \to S$, with corresponding map $F : 2^\omega \to 2^\omega$ defined by $F(x) = \bigcup_n f(x \upharpoonright n)$. We will define f in stages to obtain uniformly computable, extension-preserving functions f_s so that $f = \lim_s f_s$. To ensure that P is thin, we will meet the following requirement for each e:

$$\mathrm{Thin}(e): \quad (\forall \sigma \in \{0,1\}^{e+1})(\forall \tau) \;\; [(f(\sigma) \in T_e \wedge \sigma \sqsubseteq \tau) \;\to\; f(\tau) \in T_e]$$

To see that Thin(e) makes P thin, let $U = \{I(f(\sigma)) : |\sigma| = e+1 \;\&\; f(\sigma) \in T_e\}$ and observe that if $P_e \subset P$, then $P_e = P \cap U$.

Construction. Let $f_0 = g$. At stage $t+1$, we define f_{t+1} as follows. Look for $e < t+1$, $\sigma \in \{0,1\}^{e+1}$, and $\tau \sqsupset \sigma$ with $|\tau| \leq t+1$ such that $f_t(\sigma) \in T_e$, but $f_t(\tau) \notin T_e$. If no such e, σ, and τ exist, then $f_{t+1} = f_t$. Otherwise take the least such e and the lexicographically least σ and τ for that e. For all $\rho \in 2^{<\omega}$, let $f_{t+1}(\sigma^\frown\rho) = f_t(\tau^\frown\rho)$; for $\rho \sqsubseteq \sigma$ (with $\rho \neq \sigma$) or ρ incomparable with σ, let $f_{t+1}(\rho) = f_t(\rho)$.

Verification. It is easy to see by induction on $|\sigma|$ that for each σ, $f_s(\sigma)$ converges to a limit $f(\sigma)$. Then by induction on e, each requirement Thin(e) is satisfied. To see that f is injective, suppose towards a contradiction that $f(\sigma) = f(\tau)$ for $\sigma \neq \tau$. By the constuction, σ and τ must be comparable.

Assume, without loss of generality, that $\tau = \sigma^\frown\rho$ ($\rho \neq \emptyset$). By induction it is clear that for all t, $f_t(\sigma) \neq f_t(\sigma^\frown\rho) = f_t(\tau)$. Let $F_e(x) = \cup_n f_e(x \upharpoonright n)$, so that $P = \cap_e F_e(2^\omega)$. Since $f_0 = g$, it follows that $P \subseteq G(2^\omega)$.

Theorem 6.2 *Any computable numbering of Π^0_1 classes in 2^ω of Lebesgue measure zero omits some perfect thin class from its image.*

Proof Let $P_e = [T_e]$, where $\{T_e : e \in \omega\}$ is uniformly computable. We will construct a computable extension-preserving function $g : 2^{<\omega} \to 2^{<\omega}$ such that for all e and all $\sigma \in \{0,1\}^{e+1}$, $g(\sigma) \notin T_e$. Then letting $G(x) = \bigcup_n g(x \upharpoonright n)$ we will ensure that $G(2^\omega) \cap P_e = \emptyset$. Replacing f_0 by g in Theorem 6.1, we obtain a perfect thin class P such that $P \cap P_e = \emptyset$ (and hence certainly $P \neq P_e$), for all e.

We define $g : 2^{<\omega} \to 2^{<\omega}$ recursively, as follows. Define $g(\emptyset) = \emptyset$. Then for each $\sigma \in \{0,1\}^e$, compute the shortest and lexicographically least extension τ of $g(\sigma)$ such that $\tau \notin T_e$. Since $[T_e]$ has measure zero, it is nowhere dense and thus such a τ always exists. Then let $g(\sigma^\frown i) = \tau^\frown i$ for $i \in \{0,1\}$.

Corollary 6.3 *There is no computable numbering of all thin or of all perfect thin Π^0_1 classes.*

Proof All thin classes have Lebesgue measure zero [23]. Therefore if $e \mapsto P_e$ were a numbering of (perfect) thin classes then Theorem 6.2 would provide a (perfect) thin class P such that $P \neq P_e$ for all e, a contradiction.

7 Small Classes

Binns defined in [1] the notions of small and very small classes as a means of guaranteeing incompleteness in the lattice of the Medvedev and Muchnik degrees of subsets of ω^ω. A nonempty Π^0_1 class P is *small* if there is no computable function Φ such that for all n, $|T_P \cap \omega^{\Phi(n)}| \geq n$. Let $\Psi(n)$ be the least k such that $|T_P \cap \omega^k| \geq n$. A nonempty Π^0_1 class P is *very small* if the function Ψ dominates every computable function g; that is, $\Psi(x) \geq g(x)$ for all but finitely many x. Let A be a coinfinite c.e. set, say $\overline{A} = \{a_0 < a_1 < \ldots\}$. Recall that A is *hypersimple* if there is no computable function f such that $f(n) \geq a_n$ for all n and it is *dense simple* if $n \mapsto a_n$ dominates every computable function. In this section we will use these sets to show that no computable numbering exists for the small or very small classes.

First modify Shoenfield's Thickness Lemma [24, p. 131] as follows. Some definitions are needed. For $B \subseteq \omega$, let $B^{[y]} = \{\langle y, z\rangle \in B : z \in \omega\}$ and say that B is piecewise computable if $B^{[y]}$ is computable for all y. For $B \subseteq A \subseteq \omega$, we say that B is a *thick* subset of A if for all y, $B^{[y]} \setminus A^{[y]}$ is finite.

Lemma 7.1 (Thickness Lemma) *For any uniformly c.e. sequence $\{W_i : i \in \omega\}$ of noncomputable c.e. sets and any piecewise computable c.e. set B, there is a thick c.e. subset A of B so that $W_n \not\leq_T A$ for all n.*

Proof The proof as in [24] is modified to ensure that the length and restraint functions and the requirements incorporate the pair $\langle i, k\rangle$ in place of the single argument i to make the argument go through with each W_i in conjuction with each functional Φ_k.

We obtain the following corollary.

Corollary 7.2 *For any uniformly c.e. sequence $\{W_n : n \in \omega\}$ of noncomputable c.e. sets, there is a high c.e. set A such that for all i, $W_i \not\leq_T A$.*

Proof This follows from the modified thickness lemma above by the same argument found in [24, p. 133].

Corollary 7.3 *(a) There is no uniformly c.e. numbering of all high c.e. sets.*
(a) There is no uniformly c.e. numbering of all noncomputable c.e. sets.

In fact, it follows that there is no uniformly c.e. numbering of the high or noncomputable c.e. degrees. Inasmuch as the computable sets resemble the decidable Π^0_1 classes, we conjecture that there is no effective numbering of all nondecidable Π^0_1 classes.

The degree of a Π^0_1 class P is defined to be the degree of T_P and is thus always a c.e. degree (since T_P is a co-c.e. set).

We will use the following two classic results.

(1) [Martin] Any high degree contains a maximal (and hence dense simple) set [24, pp. 211–217].

(2) [Dekker] Any noncomputable c.e. degree contains a hypersimple set [24, p. 81].

Proposition 7.4 *A c.e. degree is high if and only if it contains a very small Π^0_1 class $P \subseteq 2^\omega$.*

Proof ($\longrightarrow$): Suppose $\boldsymbol{a}$ is high, and let $A \in \mathbf{a}$ be a maximal set, and let p be the principal function of $\omega - A$, so that p dominates every computable function. Now let $P_A = \{0^\omega\} \cup \{0^n 1 0^\omega : n \notin A\}$. Then P_A is a Π^0_1 class and for each n, the least k such that $|T_P \cap \{0,1\}^n| \geq k$ is precisely $p(n)+1$ for $n > 0$ and hence dominates every computable function.

($\longleftarrow$): Let $\boldsymbol{a}$ be a c.e. degree and suppose that $T_P \in \boldsymbol{a}$ for some very small P. Then the function $f(n) = (\text{least } k)[|T_P \cap \{0,1\}^k| \geq n]$, which dominates every computable function, is computable from T_P. It follows from Martin's Theorem [24, p. 208] that T_P is high.

Proposition 7.5 *A c.e. degree is noncomputable if and only if it contains an infinite, small Π^0_1 $P \subseteq 2^\omega$.*

Proof ($\longrightarrow$): Suppose $\boldsymbol{a}$ is a noncomputable c.e. degree, let $A \in \mathbf{a}$ be hypersimple, and p be the principal function of $\omega - A$, so that p is not dominated by any computable function. Then the Π^0_1 class P_A as defined in the proof of Proposition 7.4 will have degree $\boldsymbol{a}$ and will be small.

($\longleftarrow$): Suppose that P is an infinite Π^0_1 class and T_P is computable. Then the function $g(n) = (\text{least } k)[|T_P \cap \{0,1\}^k| \geq n]$ is computable and it follows that P is not small.

Theorem 7.6 *There is no effective (Π^0_1) numbering of all nondecidable, of all infinite small, or of all very small Π^0_1 classes in 2^ω.*

Proof Suppose, towards a contradiction, that $\{Q_n = [T_n] : n \in \omega\}$ is an effective numbering of Π^0_1 classes such that each Q_n is nondecidable. Then $W_n = \{\langle\sigma\rangle : \sigma \notin Ext(T_n)\}$ is a uniformly c.e. numbering of noncomputable c.e. sets. By Corollary 7.2, there is a high c.e. set A such that for all n, $W_n \not\leq_T A$. Therefore $\mathbf{A}$ is a high degree that contains a (very) small class not amongst the Q_i, a contradiction.

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