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Degrees of difficulty of generalized r.e. separating classes

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Abstract Important examples of Π^0_1 classes of functions $f \in {}^\omega\omega$ are the classes of *sets* (elements of ${}^\omega 2$) which separate a given pair of disjoint r.e. sets: $\mathsf{S}_2(A_0, A_1) := \{ f \in {}^\omega 2 : (\forall i < 2)(\forall x \in A_i) f(x) \neq i \}$. A wider class consists of the classes of functions $f \in {}^\omega k$ which in a generalized sense separate a k-tuple of r.e. sets (not necessarily pairwise disjoint) for each $k \in \omega$: $\mathsf{S}_k(A_0, \ldots, A_{k-1}) := \{ f \in {}^\omega k : (\forall i < k)(\forall x \in A_i) f(x) \neq i \}$. We study the structure of the Medvedev degrees of such classes and show that the set of degrees realized depends strongly on both k and the extent to which the r.e. sets intersect. Let $\mathcal{S}^m_k$ denote the Medvedev degrees of those $\mathsf{S}_k(A_0, \ldots, A_{k-1})$ such that no $m+1$ sets among $A_0, \ldots, A_{k-1}$ have a nonempty intersection. It is shown that each $\mathcal{S}^m_k$ is an upper semi-lattice but not a lattice. The degree of the set of k-ary diagonally nonrecursive functions DNR_k is the greatest element of $\mathcal{S}^1_k$. If $2 \leq l < k$, then $\mathbf{0}_M$ is the only degree in $\mathcal{S}^1_l$ which is below a member of $\mathcal{S}^1_k$. Each $\mathcal{S}^m_k$ is densely ordered and has the splitting property and the same holds for the lattice $\mathcal{L}^m_k$ it generates. The elements of $\mathcal{S}^m_k$ are exactly the joins of elements of $\mathcal{S}^1_i$ for $\lceil \frac{k}{m} \rceil \leq i \leq k$.

Keywords Medvedev reducibility · degrees · separating class

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1 Background and summary

Turing reducibility is a way of comparing the complexity of functions $f, g \in {}^{\omega}\omega$, where $\omega := \{0, 1, \ldots\}$ is the set of natural numbers and ${}^{\omega}\omega$ is the set of total functions from ω into ω. $f \leq_T g$ means that there exists an algorithm which using information about g computes arbitrary values of f and is interpreted as signifying that f is no more complex than g. This algorithm may also be viewed as a partial recursive functional Φ such that $f = \Phi(g)$. Medvedev reducibility is an analogous way of comparing the complexity of two *sets* of functions: for $P, Q \subseteq {}^{\omega}\omega$, $P \leq_M Q$ iff there exists a partial recursive functional Φ such that $\Phi : Q \to P$. In particular, $f \leq_T g \iff \{f\} \leq_M \{g\}$. The notion arises from viewing P and Q as the sets of solutions to "problems" **P** and **Q**, for example, the set of functions $\mathsf{Col}_k(G) \subseteq {}^{\omega}k$ ($k := \{0, 1, \ldots, k-1\}$) which serve as k-colorings of an infinite graph G with node set ω or the set $\mathsf{CplExt}(T) \subseteq {}^{\omega}2$ of characteristic functions of sets of Gödel numbers of the complete extensions of a first-order theory T. Then $P \leq_M Q$ means that there is a partial recursive functional Φ which maps any solution to problem **Q** to a solution to problem **P** and thus signifies that **P** is no more difficult than **Q**.

Medvedev reducibility was introduced in [**?**] in 1955 and has been studied continuously ever since, albeit at a much lower level of intensity than its Turing counterpart. Recent surveys of the state of the theory are [**?**] and [**?**]; we discuss here only a few points that are essential background for the present work. Since $\leq_M$, like $\leq_T$, is reflexive and transitive, there is a natural notion of equivalence

$$P \equiv_M Q \iff P \leq_M Q \quad \text{and} \quad Q \leq_M P.$$

The equivalence classes are called *Medvedev degrees*:

$$\mathsf{dg}_M(P) := \{\, Q : P \equiv_M Q \,\};$$

they inherit a partial ordering: $\mathsf{dg}_M(P) \leq \mathsf{dg}_M(Q) \iff P \leq_M Q$. Recall that the Turing degrees form an upper semi-lattice with join (least upper bound) operation

$$\mathsf{dg}_T(f) \oplus \mathsf{dg}_T(g) := \mathsf{dg}_T(f \oplus g),$$

where $(f \oplus g)(2x) = f(x)$ and $(f \oplus g)(2x+1) = g(x)$, but they do not form a lattice. The Medvedev degrees, on the other hand, do form a distributive lattice with join and meet operations

$$\mathsf{dg}_M(P) \vee \mathsf{dg}_M(Q) = \mathsf{dg}_M(P \vee Q),$$

where $P \vee Q := \{\, f \oplus g : f \in P \text{ and } g \in Q \,\}$, and

$$\mathsf{dg}_M(P) \wedge \mathsf{dg}_M(Q) = \mathsf{dg}_M(P \wedge Q),$$

where $P \wedge Q := \{\, (0)^\frown f : f \in P \,\} \cup \{\, (1)^\frown g : g \in Q \,\}$, $\big((i)^\frown f\big)(0) = i$ and $\big((i)^\frown f\big)(x+1) = f(x)$. There is a largest degree $\mathsf{dg}_M(\emptyset)$ and a smallest degree $\mathbf{0}_M := \mathsf{dg}_M(P)$ for any set P that has a recursive element.

Although it will not concern us directly here, the reader should be aware that there is another natural and closely related notion of reducibility for sets of functions, known as *weak* or Mučnik reducibility: $P \le_w Q$ iff $(\forall g \in Q)(\exists f \in P) f \le_T g$ and there are corresponding notions $\equiv_w$ and $\mathsf{dg}_w(P)$. It is immediate that $P \le_M Q \implies P \le_w Q$, and $\le_M$ is sometimes viewed as the uniform version of $\le_w$.

In studying Turing degrees, one often restricts attention to a subset of all degrees, most notably the r.e. degrees $\mathsf{dg}_T(\chi_A)$ for χ_A the characteristic function of a recursively enumerable (r.e.) set $A \subseteq \omega$. In 1999 Simpson suggested that the natural analog of the r.e. Turing degrees are the classes

$$\mathcal{D}^w_k := \{\, \mathsf{dg}_w(P) : P \subseteq {}^\omega k \text{ is a } \Pi^0_1 \text{ class} \,\}$$

for $k \ge 2$. We consider here the related classes

$$\mathcal{D}_k := \{\, \mathsf{dg}_M(P) : P \subseteq {}^\omega k \text{ is a } \Pi^0_1 \text{ class} \,\}$$

for $k \ge 2$.

One aspect of this analogy is the close connection between r.e. sets and Π^0_1 "problems". For example, if the graph G mentioned in the first paragraph is r.e., then $\mathsf{Col}_k(G)$ is a Π^0_1 class, and if the first-order theory T is r.e. (recursively axiomatizable), then $\mathsf{CplExt}(T)$ is a Π^0_1 class. Most relevant to the present work is

$$\mathsf{S}(A,B) := \{\, f \in {}^\omega 2 : A \subseteq \{\, x : f(x) = 1 \,\} \subseteq \overline{B} \,\},$$

the class of *separating sets* of $A, B \subseteq \omega$; if these are r.e. sets, then $\mathsf{S}(A,B)$ is a Π^0_1 class.

It is immediate that the join and meet operations described above are well-defined for each $\mathcal{D}_k$, so these structures are also distributive lattices. Several recent papers have studied the structure of $\mathcal{D}_2$; a few results most relevant to the current study are:

1. ([**?**]) $\mathcal{D}_2$ has a largest element $\mathsf{dg}_M(\mathsf{DNR}_2)$, where

   $$\mathsf{DNR}_k := \{\, f \in {}^\omega k : \forall a\, f(a) \not\simeq \{a\}(a) \,\}$$

   is the set of k-ary *diagonally non-recursive* functions.
2. ([**?**]; Theorem 14) $\mathcal{D}_2$ is densely ordered; in fact
3. ([**?**]; Theorem 8) $\mathcal{D}_2$ has the *splitting property*: for any $\mathbf{p} < \mathbf{q}$ in $\mathcal{D}_2$, there exist $\mathbf{q}^+, \mathbf{q}^- \in \mathcal{D}_2$ such that $\mathbf{p} < \mathbf{q}^+, \mathbf{q}^- < \mathbf{q}$ and $\mathbf{q}^+ \vee \mathbf{q}^- = \mathbf{q}$.

Although these were formulated explicitly for $\mathcal{D}_2$, they are equally valid for all $\mathcal{D}_k$ since

Proposition 1.1 *For all $k \ge 2$, $\mathcal{D}_k = \mathcal{D}_2$.*

Proof Since ${}^\omega k \subseteq {}^\omega(k+1)$ it follows that $\mathcal{D}_k \subseteq \mathcal{D}_{k+1}$. For the converse it suffices to show that for all n, $\mathcal{D}_{2^n} \subseteq \mathcal{D}_2$. For each $f \in {}^\omega(2^n)$, let $f^* \in {}^\omega 2$ be the function such that for each x the sequence values $f(nx), \ldots, f(nx+n-1)$ is the binary representation of $f(x)$ (with leading 0's to make it of length n), and for $P \subseteq {}^\omega(2^n)$, $P^* := \{\, f^* : f \in P \,\}$. Then easily P^* is a Π^0_1 class iff P is and $P \equiv_M P^*$. □

The theme of this paper is that despite this fact, there are interesting and subtle differences among subclasses of the classes $\mathcal{D}_k$. This was already suggested by a result obtained in a different context long before the classes $\mathcal{D}_k$ were defined; in the current terminology it reads

Proposition 1.2 ([?]; Theorem 6)

$$\mathsf{dg}_M(\mathsf{DNR}_2) > \mathsf{dg}_M(\mathsf{DNR}_3) > \cdots > \mathsf{dg}_M(\mathsf{DNR}_k) > \cdots .$$

Note that generally

$$\mathsf{S}(A,B) = \{\, f \in {}^{\omega}2 : (\forall x \in \omega)\ [x \in A \implies f(x) \neq 0 \wedge x \in B \implies f(x) \neq 1]\,\},$$

and thus with $\mathsf{K}_i := \{\, a : \{a\}(a) \simeq i \,\}$, $\mathsf{DNR}_2 = \mathsf{S}(\mathsf{K}_0, \mathsf{K}_1)$. This suggests the following generalization.

Definition 1.3 For all $k \geq 2$, $m < k$ and $A_0, \ldots, A_{k-1} \subseteq \omega$,

i. $\mathsf{S}_k(A_0, \ldots, A_{k-1}) := \{\, f \in {}^{\omega}k : (\forall i < k)(\forall x \in \omega)\ [x \in A_i \implies f(x) \neq i]\,\}$;
ii. the sequence $A_0, \ldots, A_{k-1}$ is *at most m-intersecting* iff

$$\text{for any } i_0 < i_1 < \cdots < i_m < k, \quad \bigcap_{j \leq m} A_{i_j} = \emptyset;$$

iii. P is an (m,k)-*separating class* iff $P = \mathsf{S}_k(A_0, \ldots, A_{k-1})$ for some sequence $A_0, \ldots, A_{k-1}$ of r.e. sets which is at most m-intersecting;
iv. $\mathcal{S}_k^m := \{\, \mathsf{dg}_M(P) : P \text{ is an } (m,k)\text{-separating class}\,\}$, the set of (m,k)-*separating degrees*;
v. $\mathcal{S}_k := \mathcal{S}_k^{k-1}$.

Some immediate consequences of this definition are the following.

Proposition 1.4 *For all $k \geq 2$ and $m < k$,*

i. $\mathsf{dg}_M(\mathsf{DNR}_k) \in \mathcal{S}_k^1$;
ii. $\{\mathbf{0}_M\} = \mathcal{S}_k^0 \subseteq \mathcal{S}_k^1 \subseteq \cdots \subseteq \mathcal{S}_k^{k-1} = \mathcal{S}_k$;
iii. $\mathcal{S}_k^m$ *is a set of Π_1^0 Medvedev degrees.*

Proof For (i), $\mathsf{DNR}_k = \mathsf{S}_k(\mathsf{K}_0, \ldots, \mathsf{K}_{k-1})$ and these are clearly pairwise disjoint. The first equality of (ii) follows from the fact that if $A_0, \ldots, A_{k-1}$ is at most 0-intersecting, then each $A_i = \emptyset$ and $\mathsf{S}_k(\emptyset, \ldots, \emptyset) = {}^{\omega}k$. The other clauses are immediate. □

Some of the most quotable of our results are the following, for all $k \geq 2$ and $1 \leq m < k$.

4. $\mathcal{S}_k^m$ is an upper semi-lattice but not a lattice.
5. $\mathsf{dg}_M(\mathsf{DNR}_k)$ is the greatest element of $\mathcal{S}_k^1$, so for $k \neq l$, $\mathcal{S}_k^1 \neq \mathcal{S}_l^1$.
6. If $\lceil \frac{k}{m} \rceil \leq l \leq k$, then $\mathcal{S}_l^1 \subseteq \mathcal{S}_k^m$, but if $l < \lceil \frac{k}{m} \rceil$, then for all $n < l$, the only element of $\mathcal{S}_l^n$ which is even $\leq$ any element of $\mathcal{S}_k^m$ is $\mathbf{0}_M$.

7. For $q = \lceil \frac{k}{m} \rceil$, the elements of $\mathcal{S}_k^m$ are exactly those of the form $\mathbf{p}_q \vee \mathbf{p}_{q+1} \vee \cdots \vee \mathbf{p}_k$, where each $\mathbf{p}_i \in \mathcal{S}_i^1$.
8. Each $\mathcal{S}_k^m$ is densely ordered and has the splitting property; this holds also for the sublattice $\mathcal{L}_k^m$ of $\mathcal{D}_k$ generated by $\mathcal{S}_k^m$.

There is a large literature on Π_1^0 classes; a good survey is [**?**] and we recall here only a few most relevant facts. Any Π_1^0 class may be represented as the set $P = [T]$ of infinite paths through a recursive tree $T \subseteq {}^{<\omega}k$: $f \in P \iff \forall y\, (f \upharpoonright y) \in T$, where $f \upharpoonright y := \big(f(0), \ldots, f(y-1)\big)$. Associated with P is also a canonical tree $T_P := \{\, f \upharpoonright y : f \in P \text{ and } y \in \omega \,\}$. Also $P = [T_P]$; T_P is generally not recursive but only co-r.e. (Π_1^0) and has the advantage of having no *dead ends* or *leaves*, elements σ which have no proper extensions in T. It is sometimes convenient to represent T_P as the result of iterated *pruning* of leaves from T:

$$T_{P,0} := T; \quad T_{P,s+1} := \{\, \sigma \in T_{P,s} : (\exists i < k)\, \sigma^\frown(i) \in T_s \,\}.$$

Since by hypothesis T is finite branching, the König Infinity Lemma gives immediately that $T_P = \bigcap_{s \in \omega} T_{P,s}$. We shall also make use of the finite subtrees $T_{P,s}^s := \{\, \sigma \in T_{P,s} : |\sigma| \leq s \,\}$, where $|\sigma|$ is the length of σ. Our terminology and notation for recursion theory will generally follow [**?**].

2 Basic structure

We begin with some simple observations.

Proposition 2.1 *For all $k \geq 2$, $\mathsf{dg}_M(\mathsf{DNR}_k)$ is the greatest element of $\mathcal{S}_k^1$.*

Proof For any k-tuple $(A_0, \ldots, A_{k-1})$ of r.e. sets which is at most 1-intersecting — that is, pairwise disjoint — let a be an index of a partial recursive function F such that $F(x, y) \simeq i \iff x \in A_i$. Then using the standard S_n^m functions, $x \in A_i$ iff $S_1^1(a, x) \in \mathsf{K}_i$, and the recursive functional Φ defined by $\Phi(f)(x) = f(S_1^1(a, x))$ maps DNR_k into $\mathsf{S}_k(A_0, \ldots, A_{k-1})$ and hence witnesses that $\mathsf{S}_k(A_0, \ldots, A_{k-1}) \leq_M \mathsf{DNR}_k$. □

Definition 2.2 For any sets $\mathcal{D}$ and $\mathcal{E}$ of Medvedev degrees,

i. $\overline{\mathcal{E}} := \{\, \mathbf{d} \in \mathcal{D}_2 : (\exists \mathbf{e} \in \mathcal{E})\, \mathbf{d} \leq \mathbf{e} \,\}$;
ii. $\mathcal{D} \vee \mathcal{E} := \{\, \mathbf{d} \vee \mathbf{e} : \mathbf{d} \in \mathcal{D} \wedge \mathbf{e} \in \mathcal{E} \,\}$.

Corollary 2.3 $\mathcal{D}_2 = \mathcal{D}_k = \overline{\mathcal{S}}_2^1$.

Proof By (1) of Section 1 and the preceding proposition. □

However, it does not follow that $\mathcal{S}_2^1 = \mathcal{D}_2$ and we shall see that this is far from the case. For example, we show that $\mathcal{S}_2^1 \cap \overline{\mathcal{S}}_3^1 = \{\mathbf{0}_M\}$ and more generally all of the classes $\mathcal{S}_k^1$ are almost pairwise disjoint in this sense. Note that we cannot expect that $\overline{\mathcal{S}}_2^1 \cap \overline{\mathcal{S}}_3^1 = \{\mathbf{0}_M\}$, since $\overline{\mathcal{D}} \cap \overline{\mathcal{E}} = \{\mathbf{0}_M\}$ only when one of $\mathcal{D}$ or $\mathcal{E}$ is $\{\mathbf{0}_M\}$.

First a simple result in the other direction. As usual, $\lceil \frac{k}{m} \rceil$ is the *ceiling* of $\frac{k}{m}$, the smallest integer p such that $k \leq mp$.

Proposition 2.4 *For all $k, l \geq 2$ and $1 \leq m < k$, if $\lceil \frac{k}{m} \rceil \leq l \leq k$, then $\mathcal{S}_l^1 \subseteq \mathcal{S}_k^m$.*

Proof With k, l and m as in the hypothesis, fix a pairwise disjoint sequence $(A_0, \ldots, A_{l-1})$. Since $k \leq ml$ there exist $m' \leq m$ and $l' < l$, with $l' = 0$ if $m' = m$, such that $k = m'l + l'$. Then

$$\mathsf{S}_l(A_0, \ldots, A_{l-1}) \equiv_M \mathsf{S}_k(A_0, \ldots, A_{l-1}, \ldots, A_0, \ldots, A_{l-1}, A_0, \ldots, A_{l'-1}),$$

where there are m'-many repetitions of $A_0, \ldots, A_{l-1}$. The list on the right side is easily at most m-intersecting. The inequality $\geq_M$ follows from the fact that the left side is a subset of the right. For $\leq_M$, the recursive functional Φ defined by $\Phi(f)(x) = x \pmod{l}$ maps the right side into the left. □

Proposition 2.5 *For all $k \geq 2$ and $m < k$, $\mathcal{S}_k^m$ is closed under $\Cup$ and hence forms an upper semi-lattice. However, it is not closed under $\Cap$ and is not a sublattice of $\mathcal{D}_k$.*

Proof Given $m < k$, let $(A_0, \ldots, A_{k-1})$ and $(B_0, \ldots, B_{k-1})$ be sequences of r.e. sets which are at most m-intersecting. Then easily

$$\mathsf{S}_k(A_0, \ldots, A_{k-1}) \Cup \mathsf{S}_k(B_0, \ldots, B_{k-1}) = \mathsf{S}_k(A_0 \oplus B_0, \ldots, A_{k-1} \oplus B_{k-1}),$$

where $A \oplus B := \{ 2x : x \in A \} \cup \{ 2x + 1 : x \in B \}$, and the sequence on the right side is also at most m-intersecting. On the other hand, a simple modification of Proposition 7 of [?] establishes that for $\mathbf{p}, \mathbf{q} \in \mathcal{S}_k^m$, $\mathbf{p} \Cap \mathbf{q} \in \mathcal{S}_k^m$ only in the trivial cases $\mathbf{p} \Cap \mathbf{q} = \mathbf{p}$ or $\mathbf{p} \Cap \mathbf{q} = \mathbf{q}$. □

Next we establish the following representation theorem.

Theorem 2.6 *For all $k \geq 2$, $1 \leq m < k$ and $q = \lceil \frac{k}{m} \rceil$,*

$$\mathcal{S}_k^m = \mathcal{S}_q^1 \Cup \mathcal{S}_{q+1}^1 \Cup \cdots \Cup \mathcal{S}_k^1.$$

Proof This is trivial for $m = 1$, so we assume $m \geq 2$. The inclusion $\supseteq$ is immediate from Propositions **??** and **??**. For the converse inclusion, we introduce a refinement of the notion of m-intersecting: for $1 \leq m < k$ and $n \leq k$, a sequence $(A_0, \ldots, A_{k-1})$ is of *type* (m, n) iff there exists a set $G \subseteq k$ of cardinality n such that

1. $(\forall i \in G)(\forall j < k)\ i \neq j \implies A_i \cap A_j = \emptyset$;
2. $(A_i\ :\ i \in k \setminus G)$ is at most m-intersecting.

Let $\mathcal{S}_k^{m,n}$ denote the set of joins of finitely many degrees of the form $\mathsf{dg}_M(\mathsf{S}_k(A_0, \ldots, A_{k-1}))$ such that $A_0, \ldots, A_{k-1}$ are r.e. and $(A_0, \ldots, A_{k-1})$ is of type (m, n). Some easy consequences of the definition which we leave to the reader are

3. $\mathcal{S}_k^{m,0} = \mathcal{S}_k^m$;
4. $\mathcal{S}_k^{1,n} = \mathcal{S}_k^1$ for all $n \leq k$;
5. $\mathcal{S}_k^{m,n+1} \subseteq \mathcal{S}_k^{m,n}$.

For fixed k, $1 \leq m < k$ and $n \leq k-m$, let

$$l := \begin{cases} k-m-n, & \text{if } n < k-m; \\ 1, & \text{otherwise.} \end{cases}$$

We shall establish that for all $n \leq k-m$,

$$\mathcal{S}_k^{m,\,n} \subseteq \mathcal{S}_{k-m+1}^{\min\{l,m\},\,n+1} \Cup \mathcal{S}_k^{m-1,\,n}. \tag{6}$$

Fix a sequence $(A_0, \ldots, A_{k-1})$ of type (m,n) and a witnessing set

$$G := \{j_0, \ldots, j_{n-1}\} \subseteq k.$$

Suppose first that $n < k-m$. For each $F \subseteq k \setminus G$ of cardinality m, let $\{i_0^F, \ldots, i_{l-1}^F\}$ be the elements of $k \setminus (F \cup G)$. Set

$$\mathsf{S}_k^F(A_0, \ldots, A_{k-1}) := \mathsf{S}_{k-m+1}\Big(A_{i_0^F}, \ldots, A_{i_{l-1}^F}, A_{j_0}, \ldots, A_{j_{n-1}}, \bigcap_{i \in F} A_i\Big).$$

Because $(A_0, \ldots, A_{k-1})$ is at most m-intersecting, $\bigcap_{i\in F} A_i$ is disjoint from each of the other A_i ($i \notin F$) and $(A_{i_0^F}, \ldots, A_{i_{l-1}^F})$ is at most $\min\{l,m\}$-intersecting. Hence the sequence on the right side is of type $(\min\{l,m\}, n+1)$ and therefore $\mathsf{dg}_M(\mathsf{S}_k^F(A_0, \ldots, A_{k-1})) \in \mathcal{S}_{k-m+1}^{\min\{l,m\},\,n+1}$.

If $n = k-m$, there is a unique set $F = k \setminus G$ of cardinality m and we set

$$\mathsf{S}_k^F(A_0, \ldots, A_{k-1}) := \mathsf{S}_{k-m+1}\Big(A_{j_0}, \ldots, A_{j_{n-1}}, \bigcap_{i \in F} A_i\Big).$$

This sequence is pairwise disjoint, so by (4) again

$$\mathsf{dg}_M(\mathsf{S}_k^F(A_0, \ldots, A_{k-1})) \in \mathcal{S}_{n+1}^1 = \mathcal{S}_{k-m+1}^{\min\{l,m\},\,n+1}.$$

We next define a sequence $A_0^*, \ldots, A_{k-1}^*$ as follows. Fix a simultaneous enumeration $\langle A_{i,s} : i < k, s \in \omega \rangle$ of $A_0, \ldots, A_{k-1}$. Set

$$A_i^* := \Big\{ x : \exists s \ \Big(x \in A_{i,s} \wedge \big(\exists^{<(m-1)} j < k\big)\ \exists t \big[\big((t,j) \prec (s,i) \wedge x \in A_{j,t}\big)\big]\Big) \Big\},$$

where $\prec$ is the lexicographical ordering. Each A_i^* is r.e., $A_i^* \subseteq A_i$ and $(A_0^*, \ldots, A_{k-1}^*)$ is of type $(m-1, n)$. Thus it will suffice to show that

$$\mathsf{S}_k(A_0, \ldots, A_{k-1}) \equiv_M \bigvee_{\substack{F \subseteq k \setminus G \\ |F| = m}} \mathsf{S}_k^F(A_0, \ldots, A_{k-1}) \Cup \mathsf{S}_k(A_0^*, \ldots, A_{k-1}^*). \tag{7}$$

For the inequality $\geq_M$ it suffices to show that $\mathsf{S}_k(A_0, \ldots, A_{k-1})$ is separately above each component of the right side. $\mathsf{S}_k(A_0, \ldots, A_{k-1}) \geq_M \mathsf{S}_k(A_0^*, \ldots, A_{k-1}^*)$ because $\mathsf{S}_k(A_0, \ldots, A_{k-1}) \subseteq \mathsf{S}_k(A_0^*, \ldots, A_{k-1}^*)$. Fix $F \subseteq$

$k \setminus G$ of cardinality m. Then if $n < k - m$ it is easy to check that the following functional Φ maps $\mathsf{S}_k(A_0, \ldots, A_{k-1})$ into $\mathsf{S}_k^F(A_0, \ldots, A_{k-1})$:

$$\Phi(f)(x) = \begin{cases} p, & \text{if } f(x) = i_p^F \quad (p < l); \\ l + p, & \text{if } f(x) = j_p \quad (p < n); \\ k - m, & \text{if } f(x) \in F. \end{cases}$$

If $n = k - m$, we omit the first clause of the definition of Φ.

We address now the inequality $\leq_M$ of (7). An element of the right side of (7) is (essentially) a finite set of functions

$$\{\, f_F : F \subseteq k \setminus G \wedge |F| = m \,\} \cup \{g\},$$

with each $f_F \in \mathsf{S}_k^F(A_0, \ldots, A_{k-1})$ and $g \in \mathsf{S}_k(A_0^*, \ldots, A_{k-1}^*)$. We describe a recursive mapping from such a set to a function $h \in \mathsf{S}_k(A_0, \ldots, A_{k-1})$ as follows. Given x, and assuming $n < k - m$,

8. if for some (least) F, $f_F(x) = p < l$, then $h(x) := i_p^F$;
9. otherwise, if for some (least) F, $f_F(x) = l + p$ for $p < n$, then $h(x) := j_p$;
10. otherwise, $h(x) := g(x)$.

We need to show that $x \notin A_{h(x)}$. If $h(x) = i_p^F$ because $f_F(x) = p$, then $x \notin A_{i_p^F}$ because $f_F \in \mathsf{S}_k^F(A_0, \ldots, A_{k-1})$. The argument in case (9) is similar. Suppose now that $h(x)$ is defined by case (10). This means that for all F, $f_F(x) = k - m$ and therefore $x \notin \bigcap_{i \in F} A_i$. Since $g \in \mathsf{S}_k(A_0^*, \ldots, A_{k-1}^*)$, $x \notin A_{g(x)}^* = A_{h(x)}^*$. Suppose, towards a contradiction, that $x \in A_{h(x)}$. By the construction of $A_{h(x)}^*$ this happens only if for some distinct $i_0, \ldots, i_{m-2}$ different from $h(x)$, $x \in A_{i_j}$ ($j \leq m-2$). But then for $F := \{i_0, \ldots, i_{m-2}, h(x)\}$, $x \in \bigcap_{i \in F} A_i$, contrary to the case hypothesis. Hence $x \notin A_{h(x)}$ as required; this establishes (7) and therefore (6).

To complete the proof we show by induction on $k \geq 2$ that for all $1 \leq m < k$ and all $n \leq k - m$,

$$\mathcal{S}_k^{m,\,n} \subseteq \mathcal{S}_q^1 \vee \mathcal{S}_{q+1}^1 \vee \cdots \vee \mathcal{S}_k^1 \qquad \text{for } q = \left\lceil \frac{k-n}{m} \right\rceil + n. \tag{11}$$

This gives the desired result by (3). For $k = 2$, the only cases are $\mathcal{S}_2^{1,\,0}$ and $\mathcal{S}_2^{1,\,1}$ which are immediate by Proposition **??** and (4). Assume as induction hypothesis that the result holds for all $k' < k$. For $m = 1$, $q = k$ and the result follows by (4). Assume as secondary induction hypothesis that (11) holds for k and all $m' < m$. In particular, for $k' = k - m + 1$ and $m' = m - 1$,

$$\mathcal{S}_{k'}^{\min\{l,m\},\,n+1} \subseteq \mathcal{S}_{q_0}^1 \vee \cdots \vee \mathcal{S}_{k'}^1 \quad \text{for } q_0 := \left\lceil \frac{k' - (n+1)}{\min\{l, m\}} \right\rceil + (n+1);$$

$$\mathcal{S}_k^{m',\,n} \subseteq \mathcal{S}_{q_1}^1 \vee \cdots \vee \mathcal{S}_k^1 \quad \text{for } q_1 := \left\lceil \frac{k-n}{m'} \right\rceil + n.$$

Note that the hypothesis is satisfied since $n \leq k - m < k - m'$ and if $l > 1$, then $k' - l = n + 1$ so $n + 1 \leq k' - \min\{l, m\}$. Hence by (6), it suffices to show that both $q_0, q_1 \geq q$. This is immediate for q_1 and for q_0 we compute

$$\left\lceil \frac{k' - (n+1)}{\min\{l, m\}} \right\rceil + (n+1) \geq \left\lceil \frac{k' - (n+1)}{m} \right\rceil + (n+1)$$
$$= \left\lceil \frac{k - m - n}{m} \right\rceil + (n+1) = \left\lceil \frac{k - n}{m} \right\rceil + n = q. \square$$

The following examples illustrate the content of this result.

Corollary 2.7 *i.* $\mathcal{S}^2_3 = \mathcal{S}^1_2 \vee \mathcal{S}^1_3$;
ii. $\mathcal{S}^2_4 = \mathcal{S}^3_4 = \mathcal{S}^1_2 \vee \mathcal{S}^1_3 \vee \mathcal{S}^1_4$;
iii. $\mathcal{S}^2_5 = \mathcal{S}^1_3 \vee \mathcal{S}^1_4 \vee \mathcal{S}^1_5$;
iv. $\mathcal{S}^3_5 = \mathcal{S}^4_5 = \mathcal{S}^1_2 \vee \mathcal{S}^1_3 \vee \mathcal{S}^1_4 \vee \mathcal{S}^1_5$;
v. $\mathcal{S}^2_7 = \mathcal{S}^1_4 \vee \mathcal{S}^1_5 \vee \mathcal{S}^1_6 \vee \mathcal{S}^1_7$.

We show next that in a strong sense the representation of the Theorem is unique.

Definition 2.8 For any $k, l \geq 2$, $n > 0$ and $p \geq 1$,

i. a tree $T \subseteq {}^{\leq n}k$ is *p-fat* iff for each $\tau \in T$ with $|\tau| < n$ there exist $i_0 < \cdots < i_{p-1} < k$ such that for all $q < p$, $\tau^\frown(i_q) \in T$;
ii. for any $F : {}^n k \to l$ and $E \subseteq l$, let

$$T^F_E := \{\, \tau : (\exists \sigma \in {}^n k) F(\sigma) \in E \wedge \tau \subseteq \sigma \,\};$$

E is *p-dense* (with respect to F) iff there exists a p-fat tree $T \subseteq T^F_E$.

Proposition 2.9 *For any $k, l \geq 2$, $n > 0$, $1 \leq m < k$ and $F : {}^n k \to l$, if $k > lm$, then*

i. for some $j < l$, $\{j\}$ is $(m+1)$-dense;
ii. for each $j < l$, if $l \setminus \{j\}$ is $(k-m)$-dense, then $\{j\}$ is not $(m+1)$-dense.

Proof For (i) we proceed by induction on n. For $n = 1$ this is just the pigeon-hole principle. Given $F : {}^{n+1}k \to l$, define $G : {}^n k \to l$ by

$$G(\tau) = \text{least } j < l\, (\exists i_0 < \cdots < i_m < k)(\forall q \leq m)\ F(\tau^\frown(i_q)) = j;$$

such a j must exist again by the pigeon-hole principle. By the induction hypothesis there is a $j < l$ and an $(m+1)$-fat tree $T \subseteq T^G_{\{j\}}$. Then by construction

$$\{\, \tau^\frown(i) : \tau \in T \wedge F(\tau^\frown(i)) = j \,\}$$

is an $(m+1)$-fat subtree of $T^F_{\{j\}}$.

For (ii), suppose towards a contradiction that for some $j < l$ there exist both a $(k-m)$-fat tree $T \subseteq T^F_{l \setminus \{j\}}$ and an $(m+1)$-fat tree $U \subseteq T^F_{\{j\}}$. Recursively, again just by the pigeon-hole principle, there exist, $\tau_0 \subseteq \tau_1 \subseteq \cdots \subseteq \tau_n$ such that for each $q \leq n$, $|\tau_q| = q$ and $\tau_q \in T \cap U$. But then both $F(\tau_n) \neq j$ and $F(\tau_n) = j$, a contradiction. $\square$

Theorem 2.10 *For all $k, l \geq 2$ and $1 \leq m < k$, if $l < \lceil \frac{k}{m} \rceil$, then $\mathcal{S}_l \cap \overline{\mathcal{S}}_k^m = \{ \mathbf{0}_M \}$.*

Proof With k, l, and m as in the hypothesis, suppose that $\mathbf{p} \in \mathcal{S}_l$, $\mathbf{q} \in \mathcal{S}_k^m$ and $\mathbf{p} \leq \mathbf{q}$; we show that $\mathbf{p} = \mathbf{0}_M$. Fix sequences of r.e. sets $(A_0, \ldots, A_{k-1})$, which is at most m-intersecting, and $(B_0, \ldots, B_{l-1})$ such that $\mathbf{q} = \mathsf{dg}_M(\mathsf{S}_k(A_0, \ldots, A_{k-1}))$ and $\mathbf{p} = \mathsf{dg}_M(\mathsf{S}_l(B_0, \ldots, B_{l-1}))$, and a recursive functional (total by Lemma 1 of [**?**])

$$\Phi : \mathsf{S}_k(A_0, \ldots, A_{k-1}) \to \mathsf{S}_l(B_0, \ldots, B_{l-1}).$$

To show that $\mathbf{p} = \mathbf{0}_M$ we show that $\mathsf{S}_l(B_0, \ldots, B_{l-1})$ has a recursive element f. Since Φ is total on ${}^{<\omega}k$, by compactness for each $x \in \omega$, there exists n such that for all $\sigma \in {}^n k$, $\Phi(\sigma)(x) \downarrow$. Hence, there is a recursive function $x \mapsto n_x$ such that for each x, n_x is some (not necessarily the least) such n. For each $\sigma \in {}^{n_x}k$ set $F_x(\sigma) := \Phi(\sigma)(x)$ and

$$f(x) := \text{least } j < l \; [\{ j \} \text{ is } (m+1)\text{-dense with respect to } F_x];$$

this value is well-defined by (i) of the proposition and the function f defined in this way is recursive. To see that $f \in \mathsf{S}_l(B_0, \ldots, B_{l-1})$, let

$$T(x) = \{ \sigma \in {}^{n_x}k : (\forall y < n_x)(\forall i < k)[y \in A_i \implies \sigma(y) \neq i] \}.$$

Since $(A_0, \ldots, A_{k-1})$ is at most m-intersecting,

$$\overline{T}(x) := \{ \tau : (\exists \sigma \in T(x)) \tau \subseteq \sigma \}$$

is $(k - m)$-fat and for all x and $j < l$,

$$\begin{aligned} x \in B_j &\implies T(x) \subseteq \{ \sigma \in {}^{n_x}k : \Phi(\sigma)(x) \neq j \} \\ &\implies \overline{T}(x) \subseteq T_{l \setminus \{ j \}}^{F_x} \\ &\implies l \setminus \{ j \} \text{ is } (k-m)\text{-dense with respect to } F_x \\ &\implies \{ j \} \text{ is not } (m+1)\text{-dense with respect to } F_x \\ &\implies f(x) \neq j. \end{aligned}$$

□

Corollary 2.11 *For all $2 \leq l < k$, $\mathcal{S}_l^1 \cap \overline{\mathcal{S}}_k^1 = \{ \mathbf{0}_M \}$.*

Note that this provides a new proof of Proposition **??**, since $\mathsf{dg}_M(\mathsf{DNR}_k)$ is a non-0 element of $\mathcal{S}_k^1$ and hence is not a member of $\overline{\mathcal{S}}_{k+1}^1$.

Corollary 2.12

$$\begin{aligned} &\mathcal{S}_3^1 \subset \mathcal{S}_3^2 = \mathcal{S}_3; \\ &\mathcal{S}_4^1 \subset \mathcal{S}_4^2 = \mathcal{S}_4^3 = \mathcal{S}_4; \\ &\mathcal{S}_5^1 \subset \mathcal{S}_5^2 \subset \mathcal{S}_5^3 = \mathcal{S}_5^4 = \mathcal{S}_5; \\ &\mathcal{S}_6^1 \subset \mathcal{S}_6^2 \subset \mathcal{S}_6^3 = \mathcal{S}_6^4 = \mathcal{S}_6^5 = \mathcal{S}_6; \\ &\mathcal{S}_7^1 \subset \mathcal{S}_7^2 \subset \mathcal{S}_7^3 \subset \mathcal{S}_7^4 = \mathcal{S}_7^5 = \mathcal{S}_7^6 = \mathcal{S}_7. \end{aligned}$$

Proof For example, since $\lceil \frac{5}{2} \rceil = 3$, $\mathcal{S}_5^2 = \mathcal{S}_3^1 \vee \mathcal{S}_4^1 \vee \mathcal{S}_5^1$ and in particular $\mathcal{S}_3^1 \subseteq \mathcal{S}_5^2$, but $\mathcal{S}_3^1 \not\subseteq \mathcal{S}_5^1$ since $\mathcal{S}_3^1 \cap \overline{\mathcal{S}}_5^1 = \{\mathbf{0}_M\}$. □

Since, for example, $\mathcal{S}_4^2 = \mathcal{S}_3 \vee \mathcal{S}_4^1$ it is natural to ask if $\mathcal{S}_4^2$ really contains new degrees or whether simply $\mathcal{S}_4^2 = \mathcal{S}_3 \cup \mathcal{S}_4^1$. To see that that this latter equality does not hold, we give first a generalization of Theorem **??**.

Proposition 2.13 *For any $k, l \geq 2$ and $1 \leq m < k$, if $l < \lceil \frac{k}{m} \rceil$, then*

i. for any $\mathbf{p} \in \mathcal{S}_l$, $\mathbf{s} \in \mathcal{S}_k^m$ and $\mathbf{r}$, if $\mathbf{p} \leq \mathbf{r} \vee \mathbf{s}$, then $\mathbf{p} \leq \mathbf{r}$;
ii. for any class $\mathcal{E}$ of Π_1^0 Medvedev degrees, $\mathcal{S}_l \cap \overline{\mathcal{E} \vee \mathcal{S}_k^m} = \mathcal{S}_l \cap \overline{\mathcal{E}}$.

Proof With notation as in the proof of Theorem **??** and R any Π_1^0 class, suppose that Φ is a recursive functional such that

$$\Phi : R \vee \mathsf{S}_k(A_0, \ldots, A_{k-1}) \to \mathsf{S}_l(B_0, \ldots, B_{l-1}).$$

We shall define a recursive functional $\Psi : R \to \mathsf{S}_l(B_0, \ldots, B_{l-1})$. For each $x \in \omega$ and $h \in R$, there exists $n_{x,h}$ such that for all $\sigma \in {}^{n_{x,h}}k$, $F_{x,h}(\sigma) := \Phi(\sigma, h)(x)\downarrow$. Set

$$\Psi(h)(x) := \text{least } j < l \ [\{j\} \text{ is } (m+1)\text{-dense with respect to } F_{x,h}].$$

To see that $\Psi(h) \in \mathsf{S}_l(B_0, \ldots, B_{l-1})$, let

$$T(x,h) = \left\{ \sigma \in {}^{n_{x,h}}k : (\forall y < n_{x,h})(\forall i < k)[y \in A_i \implies \sigma(y) \neq i] \right\}.$$

Since $(A_0, \ldots, A_{k-1})$ is at most m-intersecting, $T(x,h)$ is $(k-m)$-fat and as before for all x and $j < l$,

$$x \in B_j \implies f(x) \neq j.$$

This establishes (i); (ii) is then immediate. □

Proposition 2.14 *For any r.e. Turing degree $\mathbf{c} > \mathbf{0}$ and any $q \geq 2$, there exist pairwise disjoint r.e. sets $A_0, \ldots, A_{q-1}$ of degree $\mathbf{c}$ such that $\mathsf{dg}_M\big(\mathsf{S}_q(A_0, \ldots, A_{q-1})\big) > \mathbf{0}_M$.*

Proof We adapt the proof of Shoenfield for the case $q = 2$ as given in Proposition III.6.22 of [**?**]. Fix an r.e. set C of degree $\mathbf{c}$ and a stage enumeration $\langle C_s : s \in \omega \rangle$ of C. For each $i < q$ set

$$A_i := \{ \langle a, x \rangle : \exists s \, (x \in C_{s+1} \setminus C_s \wedge \{a\}_s(\langle a, x \rangle) \simeq i) \}.$$

Clearly each $A_i \leq_T C$. To see that $C \leq_T A_i$, let a_i be an index for the function with constant value i and $g_i(x) :=$ least s $[\{a_i\}_s(\langle a_i, x \rangle) \simeq i]$. Then $x \in C \iff \langle a_i, x \rangle \in A_i \vee x \in C_{g_i(x)}$.

Finally, suppose towards a contradiction that $\mathsf{S}_q(A_0, \ldots, A_{q-1})$ has a recursive member f. Let a be an index for f and $g(x) :=$ least s $[\{a\}_s(\langle a, x \rangle)\downarrow]$. Then $x \in C \iff x \in C_{g(x)}$, since if $x \in C \setminus C_{g(x)}$, then for each $i < q$,

$$f(\langle a, x \rangle) = i \implies \{a\}(\langle a, x \rangle) \simeq i \implies \langle a, x \rangle \in A_i \implies f(\langle a, x \rangle) \neq i.$$

But then C is recursive, contrary to hypothesis. □

Proposition 2.15 *For any Π_1^0 Medvedev degree $\mathbf{r} > \mathbf{0}_M$ and any $q \geq 2$, there exists $\mathbf{s} \in \mathcal{S}_q^1 \setminus \{\mathbf{0}_M\}$ such that $\mathbf{r} \not\leq \mathbf{s}$.*

Proof Fix a Π_1^0 class R of Medevedev degree $\mathbf{r}$. By a result of Jockusch and Soare, Theorem 2 of [**?**], there is a non-$\mathbf{0}$ r.e. Turing degree $\mathbf{c}$ such that no member of R has Turing degree $\leq \mathbf{c}$. Let $S := \mathsf{S}_q(A_0, \ldots, A_{q-1})$ be as in the preceding proposition. Then $R \not\leq_M S$, since if some recursive $\Phi : S \to R$, then in particular for f the characteristic function of A_0, $\Phi(f)$ would be a member of R recursive in C. Hence $\mathbf{r} \not\leq \mathbf{s} := \mathsf{dg}_M(S)$. □

Theorem 2.16 *For all $2 \leq q < k$,*

$$\mathcal{S}_q^1 \vee \cdots \vee \mathcal{S}_k^1 \not\subseteq \mathcal{S}_{k-1} \cup (\mathcal{S}_{q+1}^1 \vee \cdots \vee \mathcal{S}_k^1).$$

Proof Given $2 \leq q < k$, let $\mathbf{r}$ be any non-$\mathbf{0}_M$ member of $\mathcal{S}_k^1$ and $\mathbf{s} \in \mathcal{S}_q^1 \setminus \{\mathbf{0}_M\}$ as in the preceding proposition such that $\mathbf{r} \not\leq \mathbf{s}$. Then $\mathbf{p} := \mathbf{r} \vee \mathbf{s}$ belongs to the left side of the displayed formula. Suppose first, towards a contradiction, that $\mathbf{p} \in \mathcal{S}_{k-1}$. Then by Proposition **??**, $\mathbf{p} \leq \mathbf{s}$, whence $\mathbf{r} \leq \mathbf{s}$, contrary to hypothesis. On the other hand, if $\mathbf{p} \in \mathcal{S}_{q+1}^1 \vee \cdots \vee \mathcal{S}_k^1$, then in particular $\mathbf{s} \in \overline{\mathcal{S}_{q+1}^1 \vee \cdots \vee \mathcal{S}_k^1}$, whence by repeated application of Proposition **??** followed by Proposition **??**, $\mathbf{s} = \mathbf{0}_M$, contrary to hypothesis. □

3 Density and splitting

In [**?**] we established that the structure $(\mathcal{D}_2, \leq)$ is a dense partial ordering. That proof can be modified to establish the density of each $(\mathcal{S}_k^m, \leq)$, but here we shall get a strengthened version of this result in a different and easier way.

Definition 3.1 i. For each $k \geq 2$ and $1 \leq m < k$, $\mathcal{L}_k^m$ is the sublattice of $\mathcal{D}_2$ generated by $\mathcal{S}_k^m$.

ii. An upper semi-lattice $(\mathcal{L}, \vee, <)$ has the *splitting property* iff for all $\mathbf{p}, \mathbf{q} \in \mathcal{L}$, if $\mathbf{p} < \mathbf{q}$, then there exist $\mathbf{q}^+, \mathbf{q}^- \in \mathcal{L}$ such that $\mathbf{p} < \mathbf{q}^+, \mathbf{q}^- < \mathbf{q}$ and $\mathbf{q}^+ \vee \mathbf{q}^- = \mathbf{q}$.

Remark 3.2 Because $\mathcal{D}_2$ is a distributive lattice and $\mathcal{S}_k^m$ is an upper semi-lattice, the members of $\mathcal{L}_k^m$ are exactly the finite meets of elements of $\mathcal{S}_k^m$.

In Theorem 8 of [**?**] Binns proved that $\mathcal{D}_2$ has the splitting property. Of course, this provides also an independent proof of density. His argument shows directly that $\mathcal{S}_2^1$ has the splitting property; below we extend this to to all $\mathcal{S}_k^m$ and $\mathcal{L}_k^m$. The main work lies in establishing the following technical

Proposition 3.3 *For each $k \geq 2$ and $1 \leq m < k$, any $\mathbf{q} \in \mathcal{S}_k^m$ and any $\mathbf{p}, \mathbf{r} \in \mathcal{D}_2$ such that $\mathbf{p} < \mathbf{q}$, $\mathbf{r} \leq \mathbf{q}$ but $\mathbf{r} \not\leq \mathbf{p}$, there exist $\mathbf{q}^0, \ldots, \mathbf{q}^{2^m-1} \in \mathcal{S}_k^m$ such that for all $i < 2^m$, $\mathbf{p} < \mathbf{q}^i < \mathbf{q}$, $\mathbf{r} \not\leq \mathbf{q}^i$ and $\mathbf{q}^0 \vee \cdots \vee \mathbf{q}^{2^m-1} = \mathbf{q}$.*

Proof To reduce indexical clutter, we do the proof first for $m = 2$ and $k = 3$ and afterwards indicate how to extend to the general case. Let P and R be Π^0_1 classes of Medvedev degree $\mathbf{p}$ and $\mathbf{r}$, respectively, T_P the canonical (co-r.e.) tree for $P = [T_P]$ described in Section 1, and U a recursive tree such that $R = [U]$. Let (A, B, C) be an at most 2-intersecting sequence of r.e. sets such that $\mathbf{q} = \mathsf{dg}_M\big(\mathsf{S}_3(A, B, C)\big)$. We shall construct r.e. sets A^i and B^j $(i, j < 2)$ which partition A and B respectively such that with $\mathbf{q}^{ij} := \mathbf{p} \vee \mathsf{dg}_M\big(\mathsf{S}_3(A^i, B^j, C)\big)$,

i. $\mathbf{r} \not\leq \mathbf{q}^{ij} < \mathbf{q}$;
ii. $\bigvee_{i,j<2} \mathbf{q}^{ij} = \mathbf{q}$.

We have $\mathbf{p} \leq \mathbf{q}^{ij}$ by construction, but we do not claim that always $\mathbf{p} < \mathbf{q}^{ij}$. However, it follows that this must hold for at least two pairs (i, j), and any $\mathbf{q}^{ij} = \mathbf{p}$ make no contribution to the join $\bigvee_{i,j<2} \mathbf{q}^{ij}$ so may be replaced by copies of one of the $\mathbf{q}^{ij} > \mathbf{p}$ to produce $\mathbf{q}^0, \ldots, \mathbf{q}^3$ satisfying the conclusion of the proposition.

The construction of A^i and B^j is in the style of the Sacks Splitting Theorem, Theorem VII.3.2 of [**?**]. For $i, j < 2$, let g^{ij} be the functions defined by

$$g^{ij}(x) := \begin{cases} 0, & \text{if } x \notin A^i; \\ 1, & \text{if } x \in A^i \text{ but } x \notin B^j; \\ 2, & \text{otherwise.} \end{cases}$$

For any $A^i \subseteq A$ and $B^j \subseteq B$, (A^i, B^j, C) is at most 2-intersecting and thus $g^{ij} \in \mathsf{S}_3(A^i, B^j, C)$ — if $x \in A^i \cap B^j$, then $x \notin C$. The construction is designed to satisfy the following requirements.

$$\begin{aligned} P_x : &\quad x \in A \implies x \in A^0 \quad \text{or} \quad x \in A^1 \quad \text{but not both;} \\ Q_x : &\quad x \in B \implies x \in B^0 \quad \text{or} \quad x \in B^1 \quad \text{but not both;} \\ N_{b,i,j} : &\quad \textbf{not} \quad \{b\} : P \vee \left\{g^{ij}\right\} \to R. \end{aligned}$$

Conditions P_x and Q_x ensure that A^i and B^j partition A and B respectively. Conditions $N_{b,i,j}$ ensure that $R \not\leq_M P \vee \left\{g^{ij}\right\}$ and hence that $\mathbf{r} \not\leq \mathbf{q}^{ij}$ so also $\mathbf{q} \not\leq \mathbf{q}^{ij}$. That $\mathbf{q}^{ij} \leq \mathbf{q}$ is immediate, so (i) is satisfied. For (ii), we describe an algorithm which from any four functions $f^{ij} \in \mathsf{S}_3(A^i, B^j, C)$ $(i, j < 2)$ computes a function $f \in \mathsf{S}_3(A, B, C)$:

$$f(x) := \begin{cases} 2, & \text{if } (\exists i < 2)(\exists j < 2)\, f^{ij}(x) = 2; \\ 1, & \text{else if } (\exists i < 2)(\forall j < 2)\, f^{ij}(x) = 1; \\ 0, & \text{otherwise.} \end{cases}$$

Towards the construction, we define the following length and restraint functions.

$$\ell(b,i,j) := \begin{cases} \infty, & \text{if } \{b\} : P \Vvee \{g^{ij}\} \to R; \\ \text{least } y\big[(\exists f \in P)\{b\}^{f \oplus g^{ij}} \restriction (y+1) \notin U\big], & \text{otherwise}; \end{cases}$$

$$\ell(b,i,j,s) := \text{least } y\big[(\exists \sigma \in T^s_{P,s})\{b\}^{\sigma \oplus g^{ij}_s}_s \restriction (y+1) \notin U\big];$$

$$r(b,i,j,s) := \max\left\{\, \mathsf{u}\big(g^{ij}_s; \sigma \oplus g^{ij}_s, b, z, s\big) : z < \ell(b,i,j,s) \wedge \sigma \in T^s_{P,s} \,\right\},$$

where $g^{i,j}_s$ is the initial segment of $g^{i,j}$ of length s. Here a condition of the form $F \restriction (y+1) \notin U$ is true if either $F(z)$ is undefined for some $z \leq y$ or $F \restriction (y+1)$ is defined but not in U. The *use* $\mathsf{u}(h;\ \dots)$ is $1+$the largest value of h used in the indicated computation. Since the sequence $\langle T_{P,s} : s \in \omega \rangle$ is recursive, so are the functions $\ell(b,i,j,s)$ and $r(b,i,j,s)$. Readers familiar with similar arguments in r.e. degree theory should note that because U is a fixed recursive tree we can simplify the argument below by using $z < \ell(b,i,j,s)$ instead of $z \leq \ell(b,i,j,s)$ in the definition of $r(b,i,j,s)$.

Choose recursive enumerations of the necessarily infinite sets A and B such that exactly one new element of A appears at each even stage, but none at odd stages and exactly one new element of B appears at each odd stage but none at even stages. Now at an even stage s, let x_s be the unique element of $A_{s+1} \setminus A_s$. Let (a_s, i_s, j_s) be minimal (in the lexicographic ordering) such that $x_s < r(a_s, i_s, j_s, s)$ if there is such a triple, and set, for $j < 2$,

$$A^{1-i_s}_{s+1} := A^{1-i_s}_s \cup \{x_s\}; \qquad A^{i_s}_{s+1} := A^{i_s}_s; \qquad B^j_{s+1} := B^j_s.$$

Otherwise, do the same with $i_s = 0$. At an odd stage s do the same with the roles of A and B reversed. This completes the construction.

A computation $\{b\}^{\sigma \oplus g^{ij}_s}_s(z)\downarrow$ is called *correct* if the values of g^{ij}_s used are correct — that is, $g^{ij}_s \restriction u = g^{ij} \restriction u$ for $u = \mathsf{u}\big(g^{ij}_s; \sigma \oplus g^{ij}_s, b, z, s\big)$. We say that $\ell(b,i,j,s) \geq y$ *correctly* iff $\ell(b,i,j,s) \geq y$ and all of the computations $\{b\}^{\sigma \oplus g^{ij}_s}_s(z)$ for $\sigma \in T^s_{P,s}$ and $z < y$ are correct. We say that a stage t is (b,i,j)-*safe* iff for all (a,i',j') which precede (b,i,j) lexicographically and all $s \geq t$,

iii. $r(a,i',j',s)$ has the same value denoted $r(a,i',j')$;
iv. $A_s \restriction r(a,i',j') = A \restriction r(a,i',j')$ and $B_s \restriction r(a,i',j') = B \restriction r(a,i',j')$.

We now establish that for all b, i, j, s and y,

1. if $\ell(b,i,j,s) \geq y$ correctly, then for all $t \geq s$, $\ell(b,i,j,t) \geq y$ and $\ell(b,i,j) \geq y$;
2. if $\ell(b,i,j) \geq y$, then $\exists t\,(\forall s \geq t)\,\ell(b,i,j,s) \geq y$;
3. if s is (b,i,j)-safe and $\ell(b,i,j,s) \geq y$, then $\ell(b,i,j,s) \geq y$ correctly;
4. $\ell(b,i,j) < \infty$ and $\lim_{s\to\infty} r(b,i,j,s)$ exists and is finite.

From (4) it follows that all requirements $N_{b,i,j}$ are satisfied, so this will complete the proof. (1) is immediate just because A^i and B^j are r.e. sets. For (2), assume that $\ell(b,i,j) \geq y$. Then

$$(\forall f \in P)\exists s \left[\{b\}_s^{f\restriction s \oplus g^{ij}} \restriction y \in U \quad \text{and} \quad g_s^{ij} \restriction u_f = g^{ij} \restriction u_f\right],$$

where

$$u_f := \max\{\, \mathsf{u}(g^{ij}; f \oplus g^{ij}, b, z) : z < y \,\}.$$

By König's Lemma (compactness),

$$\exists s\, (\forall f \in P) \left[\{b\}_s^{f\restriction s \oplus g_s^{ij}} \restriction y \in U\right].$$

Fix such an $\bar{s}$. Since T_P has no leaves, also

$$(\forall \sigma \in T_P^{\bar{s}}) \left[\{b\}_{\bar{s}}^{\sigma \oplus g_{\bar{s}}^{ij}} \restriction y \in U\right],$$

so for $s \geq \bar{s}$ large enough such that $T_{P,s}^{\bar{s}} = T_P^{\bar{s}}$ we have $\ell(b,i,j,s) \geq y$.

For (3), suppose that t is (b,i,j)-safe and $\ell(b,i,j,t) \geq y$. Then for any $s \geq t$, if $x_s < r(b,i,j,s)$, then $(b,i,j) = (a_s, i_s, j_s)$, so x_s is enumerated into either A^{1-i} or B^{1-j} and thus does not affect the value of $g^{ij}(x_s)$. Hence

$$g_s^{ij} \restriction r(b,i,j,s) = g^{ij} \restriction r(b,i,j,s),$$

so in particular for all $z < \ell(b,i,j,s)$ and all $\sigma \in T_{P,s}^s$, $\{b\}_s^{\sigma \oplus g_s^{ij}}(z)\downarrow$ correctly.

Finally we establish (4) by induction on the lexicographic ordering of the tuples (b,i,j). Assume as induction hypothesis that (4) holds for all (a,i',j') preceding (b,i,j). It follows that there exists a (least) (b,i,j)-safe stage $\bar{t}$. Suppose, towards a contradiction, that $\ell(b,i,j) = \infty$. By (2), for all y $(\exists s \geq \bar{t})\, \ell(b,i,j,s) \geq y$, and by (3), for such s, $\ell(b,i,j,s) \geq y$ correctly, so in particular,

$$(\forall \sigma \in T_{P,s}^s)\, \{b\}_s^{\sigma \oplus g_s^{ij}} \restriction y \simeq \{b\}_s^{\sigma \oplus g^{ij}} \restriction y \in U.$$

Let $h(y) \simeq$ least $s \geq \bar{t}\, [\ell(b,i,j,s) \geq y+1]$ and

$$\Phi(f)(y) \simeq \{b\}_{h(y)}^{f \oplus g_{h(y)}^{ij}}(y).$$

Then Φ is a partial recursive functional, and for all $f \in P$, $\forall y\, [\Phi(f) \restriction y \in U]$ — that is, $\Phi : P \to R$ contrary to the hypothesis that $R \not\leq_M P$. We conclude that $\ell(b,i,j) < \infty$. By (2) and (3),

$$\exists s\, (\forall t \geq s)\, \ell(b,i,j,t) \geq \ell(b,i,j),$$

but by (1) and (3),

$$\neg\, (\exists t \geq \bar{t})\, \ell(b,i,j,t) \geq \ell(b,i,j) + 1.$$

Hence for all sufficiently large $t \geq \bar{t}$, $\ell(b,i,j,t) = \ell(b,i,j)$ with correct computations and $r(b,i,j,t)$ has as its common value the maximum of the uses of all of these computations.

This completes the proof of the special case $m = 2$, $k = 3$ and we turn to the general case with

$$\mathbf{q} = \mathsf{dg}_M\big(\mathsf{S}_k(A_0, \dots, A_{m-1}, A_m, \dots, A_{k-1})\big),$$

where $(A_0, \dots, A_{m-1}, A_m, \dots, A_{k-1})$ is a sequence of at most m-intersecting r.e. sets. Here we need to construct r.e. sets A_n^i for $n < m$ and $i < 2$ such that (A_n^0, A_n^1) partitions A_n and for each $\varepsilon \in {}^m2$, if

$$\mathbf{q}^\varepsilon := \mathbf{p} \Vvee \mathsf{dg}_M\big(\mathsf{S}_k(A_0^{\varepsilon(0)}, \dots, A_{m-1}^{\varepsilon(m-1)}, A_m, \dots, A_{k-1})\big),$$

then

v. $\mathbf{r} \not\leq \mathbf{q}^\varepsilon$;
vi. $\bigvee\!\!\!\bigvee_{\varepsilon \in {}^m2} \mathbf{q}^\varepsilon = \mathbf{q}$.

To achieve (v), we use functions

$$g^\varepsilon(x) := \begin{cases} \text{least } i < m, & x \notin A_0^{\varepsilon(0)} \cap \dots \cap A_i^{\varepsilon(i)} \text{ if any;} \\ m, & \text{otherwise.} \end{cases}$$

Easily each $g^\varepsilon \in \mathsf{S}_k\big(A_0^{\varepsilon(0)}, \dots, A_{m-1}^{\varepsilon(m-1)}, A_m, \dots, A_{k-1}\big)$, and for (v) it will suffice to construct the sets A_n^i to satisfy conditions

$$\begin{aligned} P_{n,x}: &\quad x \in A_n \implies x \in A_n^0 \quad \text{or} \quad x \in A_n^1 \quad \text{but not both;} \\ N_{b,\varepsilon}: &\quad \textbf{not} \quad \{b\} : P \Vvee \{g^\varepsilon\} \to R. \end{aligned}$$

This construction is a straightforward extension of the one above and is omitted. Finally, for (vi) we describe an algorithm that from functions f^ε for each $\varepsilon \in {}^m2$ such that

$$f^\varepsilon \in \mathsf{S}_k\big(A_0^{\varepsilon(0)}, \dots, A_{m-1}^{\varepsilon(m-1)}, A_m, \dots, A_k\big)$$

computes a function $f \in \mathsf{S}_k(A_0, \dots, A_{m-1}, A_m, \dots, A_k)$. For any x and $i < l$, let

$$\phi(x,i) \text{ be } \begin{cases} (\exists \varepsilon \in {}^m2)\, f^\varepsilon(x) = i, & \text{if } m \leq i < k; \\ (\exists \delta \in {}^i2)(\exists \sigma, \tau \in {}^{m-i-1}2)\, \big[f^{\delta^\frown(0)^\frown\sigma}(x) = i = f^{\delta^\frown(1)^\frown\tau}(x)\big], & \text{otherwise.} \end{cases}$$

Easily $\phi(x,i) \implies x \notin A_i$, so it suffices to prove that $\forall x\, \exists i\, \phi(x,i)$ and set $f(x) :=$ least $i\ \phi(x,i)$. For $i \leq m$, let

$$\psi(x,i) \quad \text{be} \quad \big(\forall \delta \in {}^i2\big)\big(\exists \sigma \in {}^{m-i}2\big)\, f^{\delta\sigma}(x) < i.$$

We claim then that for all x,

vii. $(\forall i \geq m) \neg\, \phi(x,i) \implies \psi(x,m)$;
viii. for $0 < i \leq m$, $\psi(x,i) \implies \phi(x,i-1) \vee \psi(x,i-1)$;
ix. $\neg\, \psi(x,0)$.

Parts (vii) and (ix) are obvious. For (viii), assume $\psi(x,i)$ and $\neg\phi(x,i-1)$ — that is,

$$\big(\forall\delta\in {}^{i-1}2\big)\big(\forall\sigma,\tau\in {}^{m-i}2\big)\Big[f^{\delta^\frown(0)^\frown\sigma}(x)\neq i\vee f^{\delta^\frown(1)^\frown\tau}(x)\neq i\Big].$$

By $\psi(x,i)$,

$$\big(\forall\delta\in {}^{i-1}2\big)\big(\exists\sigma,\tau\in {}^{m-i}2\big)\Big[f^{\delta^\frown(0)^\frown\sigma}(x)< i\vee f^{\delta^\frown(1)^\frown\tau}(x)< i\Big].$$

Hence,

$$\big(\forall\delta\in {}^{i-1}2\big)\big(\exists\upsilon\in {}^{m-(i-1)}2\big)\, f^{\delta\upsilon}(x)< i,$$

which is exactly $\psi(x,i-1)$. □

Theorem 3.4 *For each $k\geq 2$ and $1\leq m<k$, $\mathcal{S}_k^m$ and $\mathcal{L}_k^m$ have the splitting property; in particular, they are densely ordered.*

Proof Consider first $\mathcal{S}_k^m$ and for $\mathbf{p}<\mathbf{q}$, let $\mathbf{q}^0,\dots,\mathbf{q}^{2^m-1}$ be as in the Proposition for $\mathbf{r}=\mathbf{q}$. Let

$$\mathbf{s}_0:=\mathbf{q}^0\Vee\cdots\Vee\mathbf{q}^{2^{m-1}-1}\qquad\text{and}\qquad \mathbf{s}_1:=\mathbf{q}^{2^{m-1}}\Vee\cdots\Vee\mathbf{q}^{2^m-1}.$$

If both $\mathbf{s}_0<\mathbf{q}$ and $\mathbf{s}_1<\mathbf{q}$, then we may use them as $\mathbf{q}^+$ and $\mathbf{q}^-$ to witness the splitting property. Otherwise, if (say) $\mathbf{s}_0=\mathbf{q}$, let

$$\mathbf{t}_0:=\mathbf{q}^0\Vee\cdots\Vee\mathbf{q}^{2^{m-2}-1}\qquad\text{and}\qquad \mathbf{t}_1:=\mathbf{q}^{2^{m-2}}\Vee\cdots\Vee\mathbf{q}^{2^{m-1}-1},$$

and make the same argument. After at most m such steps we must produce appropriate $\mathbf{q}^+$ and $\mathbf{q}^-$.

Now suppose that $\mathbf{p}<\mathbf{q}$ in $\mathcal{L}_k^m$. As noted above, $\mathbf{q}$ may be represented in the form $\mathbf{q}=\mathbf{s}_0\Wedge\cdots\Wedge\mathbf{s}_{n-1}$ for some $\mathbf{s}_i\in\mathcal{S}_k^m$. Apply the proposition to each $\mathbf{s}_i$ to find $\mathbf{s}_i^j$ for $i<n$ and $j<2^m$ such that

$$p<\mathbf{s}_i^j<s_i,\qquad \mathbf{q}\not\leq\mathbf{s}_i^j\qquad\text{and}\qquad \mathbf{s}_i^0\Vee\cdots\Vee\mathbf{s}_i^{2^m-1}=\mathbf{s}_i.$$

By distributivity,

$$\mathbf{q}=\bigwedge_{i<n}\ \bigvee_{j<2^m}\mathbf{s}_i^j=\bigvee_{\varepsilon\in{}^n(2^m)}\mathbf{s}^\varepsilon,\quad\text{where}\quad \mathbf{s}^\varepsilon:=\bigwedge_{i<n}\mathbf{s}_i^{\varepsilon(i)}.$$

Clearly $\mathbf{p}\leq\mathbf{s}^\varepsilon<\mathbf{q}$ and we may now proceed first as in the proof of the proposition to replace any $\mathbf{s}^\varepsilon=\mathbf{p}$ by others which satisfy $\mathbf{s}^\varepsilon<\mathbf{p}$ and then as in the first part of this proof to subdivide this sequence of 2^{mn} degrees to find after at most mn steps a pair $\mathbf{q}^+$ and $\mathbf{q}^-$ which witness the splitting of $\mathbf{q}$. □

In [**?**] Binns and Simpson prove that every finite distributive lattice can be embedded in $\mathcal{D}_2$ and hence in each $\mathcal{D}_k$. The proof does not seem to be easily adaptable to yield embeddings into the sublattices $\mathcal{L}_k^m$, and we only pose this as a question. However, it is easy to adapt the mechanism for embedding partial orderings in the r.e. Turing degrees to show

Theorem 3.5 *For each $k \geq 2$ and $1 \leq m < k$, every countable partial ordering is embeddable in $(\mathcal{S}_k^m, \leq)$.*

Proof We first observe that for any $k \geq 2$ and $1 \leq m < k$, there exists a u.r.e. sequence of r.e. sets $\langle A_i^n : i < k \wedge n \in \omega \rangle$ such that for all n, $A_0^n, \ldots, A_{k-1}^n$ is at most m-intersecting and any sequence $\langle f^n : n \in \omega \rangle$ such that for all n, $f^n \in P^n := \mathsf{S}_k(A_0^n, \ldots, A_{k-1}^n)$ is recursively independent. Hence $\langle P^n : n \in \omega \rangle$ is Medvedev independent. The first assertion is a simple extension of [**?**], Theorem 4.1, and the second follows immediately.

As in the case of r.e. Turing degrees it suffices to embed an arbitrary recursive partial ordering $\preceq$ of ω. With P^n as above, set

$$R^m := \bigvee_{i \preceq m} P^i.$$

Then easily $m \preceq n \implies R^m \leq_M R^n$. Suppose, towards a contradiction that $m \not\preceq n$ but $R^m \leq_M R^n$. Then if $Q^m := \bigvee_{i \neq m} P^i$ we have $R^n \leq_M Q^m$ and thus

$$P^m \leq_M R^m \leq_M R^n \leq_M Q^m,$$

contrary to the Medvedev independence of $\langle P^n : n \in \omega \rangle$. □

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