Special Year In Logic – Computability and Complexity in Analysis 2006

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May 13, 2008

The Third International Conference on Computability and Complexity in Analysis took place November 1-5, 2006 at the University of Florida in Gainesville, Florida as part of the National Science Foundation-sponsored Special Year in Logic. This special issue consists of selected papers from the conference.

Computability and complexity are two central areas of research in mathematical logic and theoretical computer science. Computability theory is the study of the limitations and abilities of computers in principle. Computational complexity provides a framework for understanding the cost of solving computational problems, as measured by the requirement for resources such as time and space. Classical computability and complexity theory consider algorithms as operating on finite strings of symbols from a finite alphabet, which may represent various discrete objects such as integers or algebraic expressions.

Computability theory over the real numbers and over more general continuous data structures is needed for most mathematical models in applied science. Computable analysis is the theory of the computability and complexity of real numbers and functions of real variables. This includes the analyzing the effective content of results from classical and modern analysis. The topics of interest include foundational work on various models and approaches for describing computability and complexity over the real numbers. The CCA 2006 conference focused in particular on effectively closed sets and on algorithmic randomness of reals, including randomness of real continuous functions.

Manuel Campagnolo and Kerry Ojakian [The elementary computable functions over the real numbers: Applying two new techniques] present two new techniques (*approximation* and *lifting*) to construct function algebras for the real (Kalmar or C^2) elementary computable functions in a new way.

Vasco Brattka [Borel Complexity and Computability of the Hahn-Banach Theorem analyzes the Hahn-Banach theorem, which states that any linear bounded functional defined on a linear subspace of a normed space admits a norm-preserving linear bounded extension to the whole space. He shows that the Hahn-Banach extension operator is upper semi-computable, whereas it is known that it cannot be lower semi-computable in general.

The paper of Josef Berger and Douglas Bridges [The anti-Specker property, a Heine–Borel property, and uniform continuity] is part of the ongoing program of constructive reverse mathematics. Working within Bishop's constructive framework, the authors examine the connection between a weak version of the Heine–Borel property, an antithetical property from Specker's theorem in recursive analysis, and the uniform continuity theorem for integer-valued functions.

Mathematical theorems frequently postulate the existence of a (real) solution to a certain problem. For example, the Intermediate Value Theorem states that a continuous function which takes on both positive and negative values must have a zero. The set of solutions to such a problem may be assigned a *degree of difficulty* in the sense of Medvedev, where a set P of reals is less difficult than a set Q if there is a computable function mapping Q into P, so that from any element of Q, we may compute an element of P. If P represents the set of solutions to problem A and Q represents the set of solutions to problem B, then from any solution of B we can compute some solution of A. For a computable problem, the set of solutions generally makes up an effectively closed set, or Π^0_1 class.

The study of Π^0_1 classes has been an important part of computability theory since its inception. Of particular interest are the complexity of the members of a Π^0_1 class and the structure of the family of Π^0_1 classes, including both the lattice of degrees of difficulty and the lattice under inclusion. A closed subset K of 2^ω may be represented as the set of infinite paths through a tree $T \subseteq 2^{<\omega}$ and K is a Π^0_1 class (effectively closed) if it can be represented by a computable tree. Since there is an effective one-to-one correspondence between ω and $2^{<\omega}$, a tree $T \subseteq 2^\omega$ may be encoded as a subset of ω and hence as a real number. The complexity (and also the randomness) o a closed set K may be measured by the complexity (respectively, randomness) of this parameter.

Cenzer and Peter G. Hinman [Degrees of difficulty of generalized r.e. separating classes] study the degrees of difficulties of certain effectively closed sets; that is, generalized computably enumerable separating classes. Given a k-tuple $A_1, \dots, A_k$ of c.e. sets, let $GS_k(A_1, \dots, A_k)$ be the set of functions $f \in \omega^k$ such that $x \in A_i$ implies $f(x) \neq i$. An example is the set of k-ary diagonally noncomputable functions. Results include density and the splitting property for this family.

Josh Cole [Embedding $FD(\omega)$ into $\mathcal{P}_s$ Densely] shows that the free distributive lattice on countably many generators may be embedded between any two Medvedev degrees, greatly improving the original density result of Cenzer and Hinman.

Another way to determine the complexity of a problem is via *index sets.* The Π^0_1 classes have a natural enumeration $P_0, P_1, \dots$ and the index set associated with a given property R is $\{e : R(P_e)\}$. For example, $\{e : P_e \text{ has a computable member}\}$ is a Σ^0_3-complete set. This significantly refines and improves the fact that a nonempty Π^0_1 class need not contain a computable member. Then if one similarly enumerates the computably continuous functions as $F_0, F_1, \dots$, it follows that $\{e : F_e \text{ has a computable zero}\}$ is Σ^0_3 complete.

Paul Brodhead and Cenzer [Effectively Closed Sets and Enumerations] examine the theory of enumerations, or numberings, of Π^0_1 classes. In particular, one numbering $(P_0, P_1, \dots)$ is reducible to another $(Q_0, Q_1, \dots)$ if there is a computable function f such that $P_e = Q_{f(e)}$. Many commonly used numberings of Π^0_1 classes are shown to be mutually reducible via a computable permutation. A computable injective numbering is given and numberings are also studied for various subfamilies, such as the decidable Π^0_1 classes and the thin classes.

The notion of algorithmic randomness is directly related to Π^0_1 classes by way of the Martin-Löftest. That is, a real number x is Martin-Löfrandom if it belongs to the union of any effective, increasing sequence of Π^0_1 classes with measure (effectively) approaching one. It turns out that any Martin-Löfrandom real belongs to a Π^0_1 class of positive measure which contains only random reals.

Since a closed set K of reals may be represented by a real parameter, we may say that K is Martin-Löfrandom if the parameter is Martin-Löfrandom. Similarly a continuous real function F is computable relative to some real parameter and F may be said to be Martin-Löfrandom if this parameter is Martin-Löfrandom. George Barmpalias, Cenzer, Jeffrey Remmel and Weber [Algorithmic Randomness of Continuous Functions] have extended the notion of algorithmic randomness to closed sets and to continuous functions. They show that random Δ^0_2 continuous functions exist, but no computable function can be random and no random function can map a computable real to a computable real. The set of zeroes of a random continuous function is always a random closed set, but the image of a random continuous function need not be a random closed set.

The notion of triviality and lowness for randomness has been of great interest. A real x is *low for randomness* if any random real y is random relative to x. Nies showed that this is equivalent to x being K-trivial; that is, having low Kolmogorov complexity.

Johanna Franklin [Schnorr Trivial Reals: A construction] examines an alternate notion of algorithmic randomness is that a real is (Schnorr) random if it belongs to the union of any increasing sequence of Π^0_1 classes Q_i which have measure exactly $1 - 2^{-i}$. A notion of Schnorr triviality is developed, and it is shown that although Schnorr randomness is closely related to Martin-Löfrandomness, the set of Turing degrees containing K-trivial reals has very different properties from the set of Turing degrees that contain Schnorr trivial reals. In particular, if $\mathbf{h}' \geq_T \mathbf{0}''$, then $\mathbf{h}$ contains a Schnorr trivial real.

The editors would like to thank the National Science Foundation for support under grant DMS 0532644. The first editor is also partially supported by NSF DMS 0554841 and DMS 0652732 and the second editor is supported by NSF DMS 0652326 Thanks also to the American Institute of Mathematics for support during the 2006 Effective Randomness Workshop.